$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ and ultrafilters on $\mathbb{Q}$ and ω

Andres Millán*

Department of Mathematics, West Virginia University,

Morgantown, WV 26506-6310, USA; e-mail: amillan@math.wvu.edu

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Abstract

In this paper we use the version $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ of the Covering Property Axiom, which has been formulated by Ciesielski and Pawlikowski and holds in the iterated perfect set model, to study the relations between different kinds of ultrafilters on ω and $\mathbb{Q}$. In particular, we will give a full account for the logical relations between the properties of being a selective ultrafilter, a P-point, a Q-point, and an ω_1-OK point.

1 Introduction

We use standard set theoretical notation and terminology as in [10]. In particular, if A is a set $|A|$ denotes its cardinality and $\mathcal{P}(A)$ the set of all its subsets. Lower case Greek letters denote ordinal numbers. The first infinite cardinal is ω and ω_1 is the first uncountable cardinal. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. We also use the letter κ to denote any unespecified uncountable cardinal. If A and B are arbitrary sets, then we write $A \subseteq^* B$ provided that $|A \setminus B| < \omega$.

Let $\mathcal{U}$ be a nonprincipal ultrafilter on an infinite countable set X. (We will use for X either ω or $\mathbb{Q}$.) We say that:

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- $\mathcal{U}$ is a *P-point* if for every partition $\mathcal{P}$ of X either $\mathcal{U} \cap \mathcal{P} \neq \emptyset$ or there exists a $U \in \mathcal{U}$ such that $U \cap P$ is finite for each $P \in \mathcal{P}$.

- $\mathcal{U}$ is a *Q-point* if for every partition $\mathcal{P}$ of X into finite sets there exists a $U \in \mathcal{U}$ such that $|U \cap P| \leq 1$ for each $P \in \mathcal{P}$.

- $\mathcal{U}$ is *selective* if for every partition $\mathcal{P}$ of X either $\mathcal{U} \cap \mathcal{P} \neq \emptyset$ or there exists a $U \in \mathcal{U}$ such that $|U \cap P| \leq 1$ for each $P \in \mathcal{P}$.

- $\mathcal{U}$ is a κ-OK point, where κ is an infinite cardinal number, provided for every $\langle V_n \in \mathcal{U} \colon n < \omega\rangle$ there exists a $\langle U_\alpha \in \mathcal{U} \colon \alpha < \kappa\rangle$ such that $\bigcap_{i=1}^{n} U_{\alpha_i} \subseteq^* V_n$ for every $n < \omega$ and $\alpha_0 < \cdots < \alpha_n < \kappa$. Sequence $\langle U_\alpha \in \mathcal{U} \colon \alpha < \kappa\rangle$ will be referred to as OK for $\langle V_n \in \mathcal{U} \colon n < \omega\rangle$.

It is obvious from the definitions that

Fact 1.1 *$\mathcal{U}$ is a selective ultrafilter if and only if $\mathcal{U}$ is simultaneously a P-point and a Q-point.*

P-points have been studied extensively by many people in connection with the remainder ω^* of the Čech-Stone compactification of the integers and the problem of its homogeneity. The existence of P-points cannot be proven or refuted in the usual framework of set theory ZFC (see, e.g., [15] or [2]) but they do exist under several additional set theoretical assumptions like the *Continuum Hypothesis* CH or *Martin's Axiom* MA.

Given a nonprincipal ultrafilter $\mathcal{U}$ on X we say that $\mathcal{B} \subseteq \mathcal{U}$ is a basis for $\mathcal{U}$ if for every $U \in \mathcal{U}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq U$. We define the *character* of $\mathcal{U}$ as $\chi(\mathcal{U}) = \min\{|\mathcal{B}| \colon \mathcal{B}$ is a basis for $\mathcal{U}\}$. If $\kappa = \chi(\mathcal{U})$ then we say that the ultrafilter $\mathcal{U}$ is *κ-generated.*

In [11], K. Kunen introduced κ-OK points to give a proof of the nonhomogeneity of ω^* without any extra assumption beyond ZFC. The following results are relevant to this paper.

Proposition 1.2 (Kunen [11]) Every P-point is κ-OK for every κ.

Proposition 1.3 (Kunen [11]) There are $2^{\mathfrak{c}}$ many distinct $\mathfrak{c}$-OK points on ω. Moreover, these ultrafilters can be made $\mathfrak{c}$-generated.

Consider $\mathbb{Q}$ with the subspace topology induced by the usual topology on $\mathbb{R}$ and denote by $\mathrm{Perf}(\mathbb{Q})$ the family of its perfect subsets (i.e., closed subsets with no isolated points).

- A nonprincipal filter $\mathcal{U}$ on $\mathbb{Q}$ is *crowded* if the family $\mathrm{Perf}(\mathbb{Q})\cap\mathcal{U}$ forms a basis for $\mathcal{U}$.

The crowded ultrafilters have been studied in connection with the remainder of the Čech-Stone compactification of $\mathbb{Q}$ and their existence follows from the Continuum Hypothesis, Martin's Axiom for countable posets [7], or from the equality $\mathfrak{b}=\mathfrak{c}$ [6].

In [4, thm. 4.8 and cor. 4.14] Ciesielski and Pawlikowski showed that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there exist ω_1-generated selective ultrafilters as well as ω_1-generated nonselective P-points. Since a nonselective P-point cannot be a Q-point (see Fact 1.1), this second result shows that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there exists a P-point which is not a Q-point. In the same paper, [4, thm. 4.22], the authors also established the existence of an ω_1-generated crowded ultrafilter on $\mathbb{Q}$ under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$. They also proved, [4, prop. 4.25], that a crowded ultrafilter cannot be a P-point.

In this paper we establish, under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, the existence of a nonselective Q-point (i.e., a Q-point which is not a P-point) by constructing an ω_1-generated crowded Q-point which is also an ω_1-OK point (Corollary 6.15). This improves our construction from [13] of an ω_1-generated crowded Q-point on $\mathbb{Q}$. We also prove, under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, that there exist crowded ω_1-generated Q-points that are not ω_1-OK points (Corollary 5.4), crowded ω_1-generated ω_1-OK points which are neither P-points nor Q-points (Theorem 6.13), and crowded ω_1-generated ultrafilters on ω that are neither Q-points nor ω_1-OK points (Theorem 4.3). These complete all the logical implications between being a P-point, a Q-point, or an ω_1-OK point as Table 1 shows.

Besides the properties explicitly listed in Table 1 we consider also two other properties: being ω_1-generated (with $\omega_1<\mathfrak{c}$) and being crowded.

As mentioned above, the first four examples from Table 1 are also crowded. On the other hand that no other example from Table 1 can be crowded, since a crowded ultrafilter cannot be a P-point [4, prop. 4.25]. It is also easy to see that we can destroy the property of being crowded without changing any of the remaining properties. To see this, note that if $\mathcal{U}$ is an ultrafilter on $\mathbb{Q}$ and f is a bijection between $\mathbb{Q}$ and a scattered subset S of $\mathbb{Q}$, then $\mathcal{V}=\{A\subseteq\mathbb{Q}\colon f^{-1}(A)\in\mathcal{U}\}$ is a noncrowded ultrafilter that has the remaining properties identical to that of $\mathcal{U}$.

One of the key features of our examples is that they are all ω_1-generated with $\omega_1<\mathfrak{c}$. This cannot be achieved in ZFC, since in many models of ZFC, for example under MA, every nonprincipal ultrafilter on a countable

P-point	Q-point	ω_1-OK point	Existence	Reference
$-$	$-$	$-$	under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$	Theorem 4.3
$-$	$-$	$+$	under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$	Theorem 6.13
$-$	$+$	$-$	under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$	Corollary 5.4
$-$	$+$	$+$	under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$	Corollary 6.15
$+$	$-$	$-$	No, in ZFC	Proposition 1.2
$+$	$-$	$+$	under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$	[4] or [5]
$+$	$+$	$-$	No, in ZFC	Proposition 1.2
$+$	$+$	$+$	under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$	[4] or [5]

Table 1: Existence of different ultrafilters. All constructed ultrafilters are nonprincipal and ω_1-generated. Moreover, the first four examples can be made also crowded.

set has character $\mathfrak{c}$. On the other hand, every example cited in Table 1 can be constructed under MA if we are willing to settle for $\mathfrak{c}$-generated filters. An interesting issue is whether under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ the examples from Table 1 must be ω_1-generated. The answer is positive for the last example from the table, since Ciesielski and Pawlikowski proved (see [4, cor. 2.7] or [5, cor. 1.5.4]) that under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ every selective ultrafilter is ω_1-generated. There is some indication suggesting that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies that every P-point is ω_1-generated. This would take care of the bottom half of the table. Recently, the autor have constructed, under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, a crowded Q-point of character $\mathfrak{c}$. This will appear in a forthcoming paper. This particular example is not a weak P-point so it cannot be an ω_1-OK point. (See [11, Lemma 1.3].) The existence of an example of character $\mathfrak{c}$ as in the fourth row in the table is left open. The first two examples from Table 1 do not need to be ω_1-generated. By Proposition 4.1 the Fubini product $\mathcal{U}\otimes\mathcal{U}$, where $\mathcal{U}$ is a Kunen's example from Proposition 1.3, is as the first ultrafilter from Table 1. The second of these is justified by a slight modification[1] of Kunen's example from Proposition 1.3.

Finally, let us address a question, whether any of the examples from Table 1 can be constructed in ZFC. The answer is clearly no for all but the first two examples, since there are models of ZFC with no P-points (see [15])

[1]Let $\mathcal{F}_0$ be the dual filter of the ideal $\mathcal{I}_0 = \{A \subseteq \omega\colon \overline{\lim}_{n\to\infty}|A\cap P_n| < +\infty\}$, where $\{P_n\colon n<\omega\}$ is the partition of ω such that $P_n = \{m<\omega\colon 2^n - 1 \le m < 2^{n+1}-1\}$. Construct a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family w.r.t $\mathcal{F}_0$ and follow the argument from [11].

as well as models of ZFC with no Q-points (see [14]). There are, however, a ZFC examples for the first two entries of Table 1 as mentioned above. These need not be ω_1-generated, as we already noted. Whether they can be crowded remains unclear, since it is an open problem if there exists a crowded ultrafilter in ZFC.

2 Axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ and other preliminaries

The framework of CPA rests on the concept of a *prism*. If $\mathfrak{C}$ denotes the space 2^ω with its usual product topology then we define for a Polish space $\mathfrak{X}$

$$\mathrm{Perf}(\mathfrak{X}) = \{C \subseteq \mathfrak{X} \colon C \text{ is homeomorphic to } \mathfrak{C}\}.$$

If $0 < \alpha < \omega_1$ is an ordinal let $\Phi_{\mathrm{prism}}(\alpha)$ be the set of all continuous injections $f \colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ with the property that

$$f(x) \restriction \xi = f(y) \restriction \xi \iff x \restriction \xi = y \restriction \xi \qquad \text{for all } \xi < \alpha \text{ and } x, y \in \mathfrak{C}^\alpha.$$

Then we define $\mathbb{P}_\alpha = \{\mathrm{range}(f) \colon f \in \Phi_{\mathrm{prism}}(\alpha)\}$ and $\mathbb{P}_{\omega_1} = \bigcup_{0<\alpha<\omega_1} \mathbb{P}_\alpha$. The elements of $\mathbb{P}_{\omega_1}$ are called the *iterated perfect sets*. The simplest elements of $\mathbb{P}_\alpha$ are of the form $C = \prod_{\xi<\alpha} C_\xi$, where $C_\xi \in \mathrm{Perf}(\mathfrak{C})$ for every $\xi < \alpha$. We refer to them as *perfect cubes*.

If $\mathfrak{X}$ is a Polish space, then a *prism* in $\mathfrak{X}$ is a pair $\langle f, P \rangle$ where $f \colon E \to \mathfrak{X}$ is injective and continuous, $E \in \mathbb{P}_{\omega_1}$, and $P = f[E]$. Function f can be considered as a coordinate system imposed on P. We will usually abuse this terminology and refer to P itself as a prism. In this case function f, given only implicitly, will be referred to as a *witness function* for P. If the domain of the witness function of a prism P happens to be a perfect cube, we will sometimes refer also to P as a *cube* in $\mathfrak{X}$.

If $\langle f, P \rangle$ is a prism, then we say that Q is its *subprism* provided there exists an iterated perfect set $E \subseteq \mathrm{dom}(f)$ such that $Q = f[E]$. We will refer to Q as a *subcube* of P when E is a perfect cube. Notice that

Remark 2.1 *If we need to prove that a prism P contains a subprism Q with some "nice property," we can always assume that the witness function f for P is defined on the entire set $\mathfrak{C}^\alpha$.*

Proof. Indeed, assume that we can find a desired subprism Q of a prism P as long as its witness function f is defined on the entire set $\mathfrak{C}^\alpha$.

Next, take an arbitrary witness function g from $E \in \mathbb{P}_\alpha$ onto P and let $h \in \Phi_{\mathrm{prism}}(\alpha)$ be onto E. Then $f = g \circ h$ is a continuous injection from $\mathfrak{C}^\alpha$ onto P, so by the above assumption we can find a subprism Q of $\langle f, P\rangle$ with the "nice property" we are after. To finish the argument it is enough to note that Q is also a subprism of $\langle g, P\rangle$. Indeed, since $Q = f[E_0]$ for some $E_0 \in \mathbb{P}_\alpha$, there exists an $h_0 \in \Phi_{\mathrm{prism}}(\alpha)$ onto E_0. But then $h \circ h_0 \in \Phi_{\mathrm{prism}}(\alpha)$ and $Q = f[E_0] = (g \circ h)[h_0[\mathfrak{C}^\alpha]] = g[h \circ h_0[\mathfrak{C}^\alpha]]$ is a subprism of $\langle g, P\rangle$ as $h \circ h_0[\mathfrak{C}^\alpha] \in \mathbb{P}_\alpha$. ■

Since in the game defined below we will need to consider singletons in the same position as prisms as defined above, in what follows *singletons will be considered as prisms.* If P is a singleton in $\mathfrak{X}$ then its only subprism is P itself.

The following theorem is one of the principal tools for finding subprism of a prism, so also for using CPA. This result is a refinement of a theorem proved independently by H.G. Eggleston [8] and M.L. Brodskiĭ [3].

Proposition 2.2 (K. Ciesielski and J. Pawlikowski, [5, claim 1.1.5]) *Let $0 < \alpha < \omega_1$ and consider $\mathfrak{C}^\alpha$ with its usual topology and its usual product measure. If G is a Borel subset of $\mathfrak{C}^\alpha$ which is either of second category or of positive measure, then G contains a perfect cube E. In particular $E \in \mathbb{P}_\alpha$.*

Strictly speaking, in [5, claim 1.1.5] (see also [4, claim 2.3]) the result is proved only for $\alpha = \omega$. But this easily implies the above version.

We will need also the following fusion lemma, which is an easy compilation of Lammas 3.1.1 and 3.1.2 from [5]. The proof of the compilation is identical to that of [5, cor. 3.1.3].

Proposition 2.3 (K. Ciesielski and J. Pawlikowski [5]) *Let $0 < \alpha < \omega_1$ and for every $n < \omega$ let $\mathcal{D}_n \subseteq [\mathbb{P}_\alpha]^{<\omega}$ be a family of pairwise disjoint sets such that $\emptyset \in \mathcal{D}_n$, $\mathcal{D}_n$ is closed under refinements, and*

(†) *for every $\mathcal{E} \in \mathcal{D}_n$ and $E \in \mathbb{P}_\alpha$ which is disjoint with $\bigcup \mathcal{E}$ there exists an $E' \in \mathbb{P}_\alpha \cap \mathcal{P}(E)$ such that $\{E'\} \cup \mathcal{E} \in \mathcal{D}_n$.*

Then for every $n < \omega$ there is a family $\mathcal{E}_n = \{E_k \colon k < 2^n\} \in \mathcal{D}_n$ of pairwise disjoint sets such that $E = \bigcap_{n<\omega} \bigcup \mathcal{E}_n \in \mathbb{P}_\alpha$.

For a Polish space $\mathfrak{X}$ consider the following game $\mathrm{GAME}_{\mathrm{prism}}(\mathfrak{X})$ of length ω_1 played by two players, Player I and Player II. At each stage $\xi < \omega_1$ of the game Player I can play an arbitrary prism P_ξ in $\mathfrak{X}$ (i.e., P_ξ either is a singleton in $\mathfrak{X}$ or it belongs to $\mathrm{Perf}(\mathfrak{X})$ and comes with a witness function) and Player II must respond by playing a subprism Q_ξ of P_ξ. The game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi<\omega_1\rangle$ is won by Player I provided

$$\mathfrak{X} = \bigcup_{\xi<\omega_1} Q_\xi;$$

otherwise Player II wins. A strategy for Player II is any function S such that $S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle, P_\xi)$ is a subprism of P_ξ for every partial game $\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle$. We say that a game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi<\omega_1\rangle$ is played according to a strategy S for Player II provided $Q_\xi = S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle, P_\xi)$ for every $\xi<\omega_1$. A strategy S for Player II is a *winning strategy* provided Player II wins any game played according the strategy S.

The following principle captures a combinatorial core of the iterated Sacks model.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$**:** $\mathfrak{c}=\omega_2$ and for any Polish space $\mathfrak{X}$ Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{prism}}(\mathfrak{X})$.

The axiom is consequence of a slightly more general principle, similar in spirit, called CPA, see [5]. Its importance comes from the following theorem.

Proposition 2.4 (K. Ciesielski and J. Pawlikowski [5, thm. 7.2.1]) CPA *holds in the iterated perfect set model. In particular,* CPA *is consistent with ZFC set theory.*

The proof of the consistency of $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ can be also found in [4, thm. 5.3].

A set $B\subseteq\mathbb{Q}$ is *scattered* if every nonempty subset of B has isolated points. It is easy to see that the scattered subsets of $\mathbb{Q}$ form an ideal, which we will denote by $\mathcal{I}_S$. The following facts will be used in what follows. For the proofs see [4] or [5, Fact 5.5.1].

Fact 2.5 *Every nonscattered set* $B\subseteq\mathbb{Q}$ *contains a subset from* $\mathrm{Perf}(\mathbb{Q})$.

Let $\mathcal{J}$ be an ideal on a countable set X. Then we define $\mathcal{J}^+=\mathcal{P}(X)\setminus\mathcal{J}$. We say that $\mathcal{J}$ is *weakly selective* if for every $A\in\mathcal{J}^+$ and $f\colon A\to X$ there exists a $B\in\mathcal{P}(A)\cap\mathcal{J}^+$ such that $f\restriction B$ is either one-to-one or constant.

Fact 2.6 *The ideals $[\omega]^{<\omega}$ and $\mathcal{I}_S$ are weakly selective.*

The proof of the following result can be found in [4, lem. 4.9(b)] or in [5, lem. 5.3.4(b)], where $[X]^\omega$ comes with a subspace topology of $\mathcal{P}(X)$, with $\mathcal{P}(X)$ being identified with 2^X via characteristic function.

Proposition 2.7 (K. Ciesielski, J. Pawlikowski [4, 5]) *Let X be countably infinite and let $\mathcal{J} \subseteq \mathcal{P}(X)$ be a weakly selective ideal. For every prism $P \subseteq [X]^\omega$ and every $A \in \mathcal{J}^+$ there exist a subprism Q of P, a $B \in \mathcal{P}(A) \cap \mathcal{J}^+$, and an $i < 2$ such that $g \restriction B$ is constant equal to i for every $g \in Q$.*

3 Some important lemmas.

Let X be a countably infinite set. If $\mathcal{F} \subseteq [X]^\omega$ is nonempty, we say that $\mathcal{F}$ has the *strong finite intersection property, SFIP*, provided that $|\bigcap F| = \omega$ for every nonempty $F \in [\mathcal{F}]^{<\omega}$. The following is a very well known and easy fact.

Lemma 3.1 *If $\mathcal{F} \subseteq [X]^\omega$ is nonempty, countable, and has the SFIP, then there exists a $C(\mathcal{F}) \in [X]^\omega$ such that $C(\mathcal{F}) \subseteq^* B$ for every $B \in \mathcal{F}$.*

PROOF. If $\mathcal{F}$ is finite, we can put $C(\mathcal{F}) = \bigcap \mathcal{F}$; otherwise $\mathcal{F} = \{B_n \colon n < \omega\}$ and we can pick inductively $b_n \in \bigcap_{k \le n} B_k$ such that $b_n \notin \{b_k \colon k < n\}$. The set $C(\mathcal{F}) = \{b_n \colon n < \omega\}$ works. ■

Let X be a countably infinite set. If the set $Z_X = [X]^{<\omega} \setminus \{\emptyset\}$ has the discrete topology then the product space $\mathcal{Z}_X = (Z_X)^\omega$ is a Polish space and the sets $U_{\langle n,a\rangle} = \{z \in \mathcal{Z} \colon z(n) = a\}$, where $a \in [\omega]^{<\omega}$ and $n < \omega$, constitute a subbasis for the product topology. Consider the set

$$\boldsymbol{\mathcal{P}}_X = \{z \in \mathcal{Z}_X \colon \ \{z(k) \colon k < \omega\} \text{ is a partition of } \omega\}.$$

If $X = \omega$ we will drop the indexes, that is, $\mathcal{Z} = \mathcal{Z}_\omega$ and $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_\omega$.

Lemma 3.2 *$\boldsymbol{\mathcal{P}}_X$ is a G_δ subset of $\mathcal{Z}_X$. Therefore $\boldsymbol{\mathcal{P}}_X$ is a Polish space with the relative topology inherited from $\mathcal{Z}_X$.*

PROOF. We can assume that $X=\omega$. If $A=\{z\in\mathcal{Z}\colon \bigcup_{n<\omega} z(n)=\omega\}$ and $B=\{z\in\mathcal{Z}\colon \{z(n)\colon n<\omega\}$ is pairwise disjoint$\}$ then $\boldsymbol{\mathcal{P}}=A\cap B$. The set A is G_δ because we have $A=\bigcap_{k\in\omega}\bigcup_{n<\omega}\bigcup\{U_{\langle n,a\rangle}\colon a\in[\omega]^{<\omega}\ \&\ k\in a\}$. The set B is G_δ since it can be written as $\bigcap_{m<n<\omega}\bigcup\{U_{\langle m,a\rangle}\cap U_{\langle n,b\rangle}\colon a\cap b=\emptyset\}$. Thus, $\boldsymbol{\mathcal{P}}$ is G_δ in $\mathcal{Z}$. ■

Definition 1 Let X be a countably infinite set and let $\mathcal{J}\subseteq\mathcal{P}(X)$ be an ideal on X containing all the singletons. We say that $\mathcal{J}$ is *Q-like* provided that for every $A\in\mathcal{J}^+$ there exists a countable indexed family $\{A_n\in[A]^\omega\colon n<\omega\}$ such that no set $\{b_n\colon n<\omega\}$ belongs to $\mathcal{J}$ provided $b_n\in A_n$ for every $n<\omega$.

Lemma 3.3 *Let X be a countably infinite set, let $\mathcal{J}$ be a Q-like ideal on X and let $A\in\mathcal{J}^+$ be arbitrary. If P is a prism on $\boldsymbol{\mathcal{P}}_X$, then there exist a subprism Q of P and a $B\in\mathcal{P}(A)\cap\mathcal{J}^+$ such that $|z(k)\cap B|\le 1$ for every $z\in Q$ and $k<\omega$. Moreover, if P is a cube than Q can be chosen as a subcube of P.*

PROOF. We can suppose that $X=\omega$. Let $\langle A_n\in[A]^\omega\colon n<\omega\rangle$ be the sequence associated to A in the definition of Q-like.

Case (a): If $P=\{z\}$ then, define a sequence $\langle b_n\in\omega\colon n<\omega\rangle$ inductively such that $b_n\in A_n\setminus\bigcup\{z(k)\colon k<\omega\ \&\ z(k)\cap\{b_0,\ldots,b_{n-1}\}\neq\emptyset\}$ for every $n<\omega$. It is easy to see that $B=\{b_n\colon n<\omega\}$ works.

Case (b): If $P\in\mathrm{Perf}(\boldsymbol{\mathcal{P}}_\omega)$, let f be a witness function for P. By Remark 2.1 we can assume that f acts from $\mathfrak{C}^\alpha$ onto P. Thus, P is a cube. It is enough to find its subcube with the desired properties.

Let μ be the standard product probability measure on $\mathfrak{C}^\alpha$. We construct, by induction on $n<\omega$, a sequence $\langle K_n\colon n<\omega\rangle$ of open subsets of $\mathfrak{C}^\alpha$ and two sequences, $\langle b_n\in A_n\colon n<\omega\rangle$ and $\langle B_n\in[\omega]^{<\omega}\colon n<\omega\rangle$, such that for every $n<\omega$:

(i) $b_n>\max\left(\{b_i\colon i<n\}\cup\bigcup_{j<n}B_j\right)$,

(ii) $\mu(K_n)\ge 1-2^{-(n+2)}$, and

(iii) $f(h)(k)\subseteq B_n$ for every $h\in K_n$ and $k<\omega$ for which $b_n\in f(h)(k)$.

If this construction is possible, put $B=\{b_n\colon n<\omega\}$. Then, clearly $B\in\mathcal{P}(A)\cap\mathcal{J}^+$ since that $\mathcal{J}$ is Q-like and $b_n\in A_n$ for every $n<\omega$. Condition (ii) implies that $\mu\left(\bigcap_{n<\omega}K_n\right)\ge\frac{1}{2}$. Hence, by Proposition 2.2, there exists a perfect cube $C \subseteq \bigcap_{n<\omega} K_n$. Then $Q = f[C]$ is a subcube of P and the pair $\langle Q, B\rangle$ is as required. To see this, it is enough to show that $|z(k) \cap B| \leq 1$ for every $z \in Q$ and $k < \omega$. Let $z = f(h)$ for some $h \in C$. By conditions (i) and (iii), for every $b_j \in z(k) = f(h)(k)$ and $n > j$ we have that $b_n \notin z(k)$. Therefore, no two elements of B are in the same $z(k)$ or, in other words, $|z(k) \cap B| \leq 1$ for every $k < \omega$.

Next, we show that the inductive construction is possible. Let $n < \omega$ be such that the appropriate b_i, K_i, and B_i are already constructed for every $i < n$. We will construct b_n, K_n, and B_n satisfying (i)–(iii). We pick an b_n as an arbitrary element of A_n satisfying condition (i). Next, we define $L = \{a \in [\omega]^{<\omega} \colon b_n \in a\}$ and note that $\left\{f^{-1}\left(U_{\langle m,a\rangle}\right) : \langle m, a\rangle \in \omega \times L\right\}$ is a partition of $\mathfrak{C}^\alpha$ into clopen sets. Thus, we can find a finite set $S \subseteq \omega \times L$ such that $K_n = \bigcup\left\{f^{-1}\left(U_{\langle m,a\rangle}\right) : \langle m, a\rangle \in S\right\}$ satisfies condition (ii). Let $B_n = \bigcup\{a \colon \langle m, a\rangle \in S \text{ for some } m < \omega\}$. Then clearly, B_n is finite. To see that it satisfies (iii), take an $h \in K_n$. Then $f(h) \in U_{\langle m,a\rangle}$ for some $\langle m, a\rangle \in S$. Let $k < \omega$ be such that $b_n \in f(h)(k)$. Since we have also $b_n \in a = f(h)(m)$, we conclude that $k = m$. So, $f(h)(k) = f(h)(m) = a \subseteq B_n$. ■

Definition 2 Let X be a countably infinite set. We say that an ideal $\mathcal{J}$ on X is *prism-friendly* provided that it contains all singletons and

(•) given a prism P in 2^X and an $A \in \mathcal{J}^+$ there exists a subprism Q of P, a $B \in \mathcal{P}(A) \cap \mathcal{J}^+$, and an $i < 2$ such that $g \restriction B$ is constant equal i for every $g \in Q$.

Definition 3 Let X be a countably infinite set. We say that an ideal $\mathcal{J}$ on X is *rich* if it is prism-friendly and

(#) given an $A \in \mathcal{J}^+$ there exists a family $\mathcal{A} \subseteq \mathcal{P}(A) \cap \mathcal{J}^+$ of cardinality $\mathfrak{c}$ which is almost disjoint, that is, such that $|A \cap B| < \omega$ for every distinct $A, B \in \mathcal{A}$.

Also, notice that, in ZFC, condition (•) does not imply condition (#). Indeed, if $\mathcal{U}$ is a selective ultrafilter, then its dual ideal $\mathcal{I}_\mathcal{U}$ is weakly selective. So, see [5], $\mathcal{I}_\mathcal{U}$ is prism-friendly. However, $\mathcal{I}_\mathcal{U}^+ = \mathcal{U}$ and no two members in $\mathcal{U}$ can be almost disjoint.

Lemma 3.4 *The ideals* $[\omega]^{<\omega}$ *and* $\mathcal{I}_S$ *are Q-like and rich.*

PROOF. It is easy to see that $[\omega]^{<\omega}$ is Q-like. To see that $\mathcal{I}_S$ is also Q-like pick any $A \in \mathcal{I}_S^+$. By Fact 2.6 we can assume that $A \in \mathrm{Perf}(\mathbb{Q})$. Let $\mathcal{B}$ be a countable basis for the topology on $\mathbb{Q}$ and let $\{A_n \colon n < \omega\}$ be an enumeration of the set $\{S \cap A \colon S \in \mathcal{B} \ \& \ |S \cap A| = \omega\}$. If $b_n \in A_n$ for every $n < \omega$ then $B = \{b_n \colon n < \omega\}$ is dense in A and in consequence, it is in $\mathcal{I}_S^+$.

By Fact 2.6, the ideals $[\omega]^{<\omega}$ and $\mathcal{I}_S$ are weakly selective so, by Proposition 2.7, they are prism-friendly. Thus, we need only to check that each of these ideals satisfies the condition (#) from the definition of rich ideal.

It is well known that (#) holds for $[\omega]^{<\omega}$. To check that (#) also holds for $\mathcal{I}_S$, fix a countable basis $\mathcal{B}$ for the topology on $\mathbb{Q}$ and pick an $A \in \mathcal{I}_S^+$. By Fact 2.5, we can assume that $A \in \mathrm{Perf}(\mathbb{Q})$. Let $\{B_n \colon n < \omega\}$ be an enumeration of $\mathcal{B}_A = \{B \in \mathcal{B} \colon |B \cap A| = \omega\}$ and construct $\{a_s \colon s \in 2^{<\omega}\}$ by induction on the length of s in such a way that $\{a_s \colon s \in 2^n\} \in [A \cap B_n]^{2^n}$ and that $\{a_s \colon s \in 2^n\} \cap \bigcup \{a_t \colon t \in 2^{<n}\} = \emptyset$ for every $n < \omega$. If for $x \in 2^\omega$ we put $A_x = \{a_{x \restriction n} \colon n < \omega\}$, then $A_x \in \mathcal{I}_S^+$ for every $x \in 2^\omega$, since A_x is dense in A. Then $\mathcal{A} = \{A_x \colon x \in 2^\omega\}$ is almost disjoint and satisfies (#). ∎

Definition 4 Let X be a countably infinite set and let $\mathcal{J} \subseteq \mathcal{P}(X)$ be an ideal on X containing all singletons. The *Fubini product* of the ideals $[\omega]^{<\omega}$ and $\mathcal{J}$ is the ideal $\mathcal{K}$ on $\omega \times X$ denoted $[\omega]^{<\omega} \otimes \mathcal{J}$ and defined as the family of all subsets A of $\omega \times X$ such that

$$\mathrm{supp}(A) \stackrel{\mathrm{def}}{=} \{n < \omega \colon (A)_n \in \mathcal{J}^+\} \text{ is finite,}$$

where $(A)_n = \{x \in X \colon \langle n, x \rangle \in A\}$.

Lemma 3.5 *If $\mathcal{J}$ is a Q-like ideal, then $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$ is also Q-like.*

PROOF. Let $A \in \mathcal{K}^+$. For each $n \in \mathrm{supp}(A)$ let $\{A_n^m \in [(A)_n]^\omega \colon m < \omega\}$ be a family from the definition of Q-like for $(A)_n \in \mathcal{J}^+$. Then the family $\{\{n\} \times A_n^m \colon n \in \mathrm{supp}(A) \ \& \ m < \omega\}$ satisfies the definition of Q-like for the set A. ∎

Lemma 3.6 *Let X be a countably infinite set, $\mathcal{J}$ a prism-friendly ideal on X, P a prism in $2^{\omega \times X}$, $I \in [\omega]^\omega$, and let $\langle A_n \in \mathcal{J}^+ \colon n \in I \rangle$ be arbitrary. Then, there exist a subprism Q of P, a set $J \in [I]^\omega$, a sequence $\langle B_n \in \mathcal{P}(A_n) \cap \mathcal{J}^+ \colon n \in J \rangle$, and an $i < 2$, such that $g \restriction B$ is constant equal i for every $g \in Q$ provided that $B = \bigcup \{\{n\} \times B_n \colon n \in J\}$.*

In particular, if $\mathcal{J}$ is prism-friendly, then so is $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$.

Proof. We can suppose that $I = \omega$. If P is a singleton the lemma follows easily from the fact that $\mathcal{J}$ is an ideal containing the singletons and the pigeon hole principle. So, suppose that $P \in \mathrm{Perf}(2^{\omega\times X})$. Let f be a function witnessing that P is a prism. By Remark 2.1 we can assume that f is defined on $\mathfrak{C}^\alpha$ for some $0 < \alpha < \omega_1$. We will construct a subprism Q_0 of P and a sequence $\langle B_n \in [A_n]^\omega \cap \mathcal{J}^+ \colon n < \omega\rangle$ such that for every $n < \omega$

$$g \restriction \{n\} \times B_n \text{ is constant for every } g \in Q_0. \tag{1}$$

This will be done using Proposition 2.3.

For each $n < \omega$ let $\mathcal{D}_n$ be the collection of all pairwise disjoint families $\mathcal{E} \in [\mathbb{P}_\alpha]^{<\omega}$ such that there exists an $A_{\langle\mathcal{E},n\rangle} \in [A_n]^\omega \cap \mathcal{I}^+$ with the property that for every $E \in \mathcal{E}$

$$f(h) \restriction \{n\} \times A_{\langle\mathcal{E},n\rangle} = f(h') \restriction \{n\} \times A_{\langle\mathcal{E},n\rangle} \text{ for all } h, h' \in E. \tag{2}$$

Clearly, each $\mathcal{D}_n$ is closed under refinaments. To see that $\mathcal{D}_n$ satisfies the condition (†) from Proposition 2.3 pick $\mathcal{E} \in \mathcal{D}_n$ and $E \in \mathbb{P}_\alpha$ such that $E \cap \bigcup \mathcal{E} = \emptyset$. Decreasing $A_{\langle\mathcal{E},n\rangle}$, if necessary, we can assume that $X \setminus A_{\langle\mathcal{E},n\rangle}$ is infinite. Let $b_n \colon \omega \times X \to X$ be any bijection such that $b_n(n, a) = a$ for every $a \in A_{\langle\mathcal{E},n\rangle}$. This bijection induces a homeomorphism $f_n \colon 2^{\omega\times X} \to 2^X$ defined by $f_n(g)(x) = g(b_n^{-1}(x))$ for every $g \in 2^{\omega\times X}$ and $x \in X$. Clearly, f_n is continuous and injective. Hence, $Q^* = (f_n \circ f)[E]$ is a prism in 2^X. Since $\mathcal{J}$ is prism-friendly, we can find a subprism Q^{**} of Q^*, an $A' \in [A_{\langle\mathcal{E},n\rangle}]^\omega \cap \mathcal{J}^+$, and an $i < 2$ such that $g[A'] = \{i\}$ for every $g \in Q^{**}$. But $Q^{**} = f_n[E']$ for some $E' \in \mathbb{P}_\alpha \cap \mathcal{P}(E)$. So, if we put $\mathcal{E}' = \mathcal{E} \cup \{E'\}$ and $A_{\langle\mathcal{E}',n\rangle} = A'$ we get that $\mathcal{E}' \in \mathcal{D}_n$ and the condition (†) is satisfied. Thus, by Proposition 2.3, for every $n < \omega$ there exists a family $\mathcal{E}_n = \{E_k \colon k < 2^n\} \in \mathcal{D}_n$ of pairwise disjoint sets with $E^0 = \bigcap_{n<\omega} \bigcup \mathcal{E}_n \in \mathbb{P}_\alpha$. We will prove that $Q_0 = f[E^0]$ satisfies (1) with some sequence $\langle B_n \colon n < \omega\rangle$.

To see this fix an $n < \omega$, for each $k < 2^n$ pick an $h_k \in E_k$, and define $\varphi_n \colon A_{\langle\mathcal{E}_n,n\rangle} \to 2^{2^n}$ by $\varphi_n(p)(k) = f(h_k)(n, p)$. Since $A_{\langle\mathcal{E}_n,n\rangle} \in \mathcal{I}^+$ and $\mathcal{J}$ is an ideal, we can find an $s_n \in 2^{2^n}$ such that $B_n = \varphi_n^{-1}(s_n) \in \mathcal{J}^+$. To see that B_n satisfies (1), pick a $g \in Q_0$. Then there exists a $k < 2^n$ and an $h \in E_k$ such that $g = f(h)$. Since $B_n \subseteq A_{\langle\mathcal{E}_n,n\rangle}$, by (2) we have that $g \restriction \{n\} \times B_n = f(h_k) \restriction \{n\} \times B_n$. In particular, $g(p) = f(h_k)(n, p) = \varphi_n(p)(k) = s_n(k)$ for every $p \in B_n$. So, $g \restriction \{n\} \times B_n$ is constant equal to $s_n(k)$ and (1) holds.

To finish the proof of the lemma pick a $b_n \in B_n$ for each $n < \omega$. Then, the set $S = \{\langle n, b_n\rangle \in \{n\} \times B_n \colon n < \omega\}$ is a selector for $\{\{n\} \times B_n \colon n < \omega\}$.

Let $\mathcal{I} = [\omega \times X]^{<\omega}$. Then $\mathcal{I}$ is weakly selective and $S \in \mathcal{I}^+$. If we identify $2^{\omega\times X}$ with $\mathcal{P}(\omega \times X)$, then Q_0 can be treated as a prism in $\mathcal{P}(\omega \times X)$. Since $[\omega \times X]^\omega$ is residual in $\mathcal{P}(\omega \times X)$, by Proposition 2.2 we can assume that Q_0 is a prism in $[\omega \times X]^\omega$. So, by Proposition 2.7, there exist a subprism Q of Q_0, a set $S_0 \in [S]^\omega$, and an $i < 2$ such that $g[S_0] = \{i\}$ for every $g \in Q$. Define $J = \{n < \omega \colon \langle n, b_n\rangle \in S_0\}$.

To see that the conclusion of the lemma holds take a $g \in Q$ and an $\langle n, b\rangle \in B$. Then $n \in J$ and $b \in B_n$. So, by (1), $g(n, b) = g(n, b_n) = i$, since $\langle n, b_n\rangle \in S_0$. ■

Lemma 3.7 *Let X be a countably infinite set and let $\mathcal{J} \subseteq \mathcal{P}(X)$ be an ideal containing all singletons and satisfying condition (#) from the definition of a rich ideal. Then the ideal $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$ also satisfies (#).*

In particular, if $\mathcal{J}$ is rich, then so is $\mathcal{K}$.

PROOF. Let $A \in \mathcal{K}^+$. Then $\mathrm{supp}(A)$ is infinite. Let $\mathcal{A} = \{A_\xi \colon \xi < \mathfrak{c}\} \subseteq [\mathrm{supp}(A)]^\omega$ be an almost disjoint family. Since $\mathcal{J}$ satisfies (#), for every $n < \omega$ there exists an almost disjoint family $\mathcal{B}_n = \{B^n_\xi \colon \xi < \mathfrak{c}\} \subseteq \mathcal{P}((A)_n) \cap \mathcal{J}^+$. If for every $\xi < \mathfrak{c}$ we define

$$U_\xi = \bigcup \{\{n\} \times B^n_\xi \colon n \in A_\xi\},$$

then the family $\{U_\xi \colon \xi < \mathfrak{c}\} \subseteq \mathcal{P}(A) \cap \mathcal{K}^+$ works. The other part of the lemma is consequence of this and of Lemma 3.6. ■

4 An ω_1-generated crowded bad point.

Definition 5 If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters then, the *Fubini product* of $\mathcal{U}$ and $\mathcal{V}$ is defined as

$$\mathcal{U} \otimes \mathcal{V} = \{A \subseteq \omega \times \omega \colon \{n \colon (A)_n \in \mathcal{V}\} \in \mathcal{U}\}.$$

Proposition 4.1 (Folklore) *If $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters in ω then $\mathcal{U} \otimes \mathcal{V}$ is a nonprincipal ultrafilter which is not a P-point, a Q-point, or even an ω_1-OK point.*

PROOF. It is easy to see that $\mathcal{U}\otimes\mathcal{V}$ is a nonprincipal ultrafilter. To see that $\mathcal{U}\otimes\mathcal{V}$ cannot be a P-point observe that the set $\{L_m : m<\omega\}$ of all sections $L_m=\{\langle m,n\rangle : n\in\omega\}$ is a partition of $\omega\times\omega$ into infinite pieces not in $\mathcal{U}\otimes\mathcal{V}$ and that every $X\in\mathcal{U}\otimes\mathcal{V}$ intersects infiniteley many L_m's on an infinite set.

To see that $\mathcal{U}\otimes\mathcal{V}$ cannot be a Q-point consider the *partial partition* $\{P_n : n<\omega\}$ of $\omega\times\omega$ where $P_n=\{\langle m,n\rangle : m\le n\}$ for every $n<\omega$. Notice that $\bigcup_{n<\omega}P_n\in\mathcal{U}\otimes\mathcal{V}$. Let $\mathcal{P}\subseteq[\omega\times\omega]^{<\omega}$ be a partition of $\omega\times\omega$ such that $\{P_n : n<\omega\}\subseteq\mathcal{P}$. It is easy to see that there is no $X\in\mathcal{U}\otimes\mathcal{V}$ such that $|X\cap P|\le 1$ for every $P\in\mathcal{P}$.

To see that $\mathcal{U}\otimes\mathcal{V}$ is not an ω_1-OK point consider $\{V_n : n<\omega\}\subseteq\mathcal{U}\otimes\mathcal{V}$, where $V_n=\bigcup_{m>n}L_m$. By the way of contradiction, suppose that the sequence $\bar{\mathcal{U}}=\langle U_\xi\in\mathcal{U}\otimes\mathcal{V} : \xi<\omega_1\rangle$ is OK for $\{V_n : n<\omega\}$. Then, by the pigeon hole principle, there exist an $m<\omega$ and an $X\in[\omega_1]^{\omega_1}$ such that $(U_\xi)_m\in\mathcal{V}$ for every $\xi\in X$. Pick ordinals $\xi_1<\xi_2<\cdots<\xi_m$ in X. Since $\bar{\mathcal{U}}$ is OK for $\{V_n : n<\omega\}$ we have that $\bigcap_{i=1}^m U_{\xi_i}\subseteq^* V_m\subseteq\omega\times\omega\setminus L_m$. Therefore, $|\bigcap_{i=1}^m U_{\xi_i}\cap L_m|<\omega$. But also, $(\bigcap_{i=1}^m U_{\xi_i})_m=\bigcap_{i=1}^m(U_{\xi_i})_m\in\mathcal{V}$. This implies that $|(\bigcap_{i=1}^m U_{\xi_i})\cap L_m|=\omega$, which is a contradiction. ■

Given $f,g\in\omega^\omega$ we write $g\le^* f$ provided that $g(n)\le f(n)$ for all but finitely many $n<\omega$. We say that an $F\subseteq\omega^\omega$ is *dominating* provided that for every $g\in\omega^\omega$ there exists an $f\in F$ such that $g\le^* f$. The *dominating number* $\mathfrak{d}$ is defined as the minimum cardinality of a dominating family in ω^ω. This and other cardinal invariants have been studied extensively in the literature. See for example [1] or [2]. It is easy to show that $\omega_1\le\mathfrak{d}\le\mathfrak{c}$ and that this is all that can be said in ZFC about the value of $\mathfrak{d}$. For instance, the continuum hipothesis implies that $\mathfrak{d}=\omega_1=\mathfrak{c}$, while Martin's Axiom + $\mathfrak{c}>\omega_1$ imply that $\mathfrak{d}=\mathfrak{c}>\omega_1$. See, for example [10].

In [5, sec. 1.3] Ciesielski and Pawlikowski proved that a weak version of $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, called $\mathrm{CPA}_{\mathrm{cube}}$, implies that $\mathrm{cof}(\mathcal{N})=\omega_1<\mathfrak{c}$.[2] It is known that this fact implies that $\mathfrak{d}=\omega_1$.

It is not difficult to prove that $\mathfrak{d}=\omega_1$ implies that for every countable infinite set X there is an $F\subseteq([\omega_1]^{<\omega})^X$ of cardinality ω_1 which is $\subseteq$-dominant, that is, such that

$$\text{for every } g\in([\omega_1]^{<\omega})^X \text{ there is an } f\in F \text{ with } g(x)\subseteq f(x) \text{ for all } x\in X.$$

This follows from the fact that $([\omega_1]^{<\omega})^X=\bigcup_{\alpha<\omega_1}([\alpha]^{<\omega})^X$. This is the form of $\mathfrak{d}=\omega_1$ which we will use in the next proposition.

[2] $\mathrm{cof}(\mathcal{N})=\min\{|\mathcal{A}| : \mathcal{A}\subseteq\mathcal{N}\ \forall \mathrm{X}\in\mathcal{N}\ \exists \mathrm{Y}\in\mathcal{A}\ (\mathrm{X}\subseteq\mathrm{Y})\}$, where $\mathcal{N}$ is the null ideal on $\mathfrak{C}$.

Proposition 4.2 *Assume* $\mathfrak{d} = \omega_1$ *and let* X *and* Y *be countably infinte sets. If* $\mathcal{U}$ *and* $\mathcal{V}$ *are* ω_1*-generated ultrafilters on* X *and* Y*, respectively, then their Fubini product* $\mathcal{U} \otimes \mathcal{V}$ *is also* ω_1*-generated.*

Proof. Let $\{U_\alpha \colon \alpha < \omega_1\}$ and $\{V_\beta \colon \beta < \omega_1\}$ be the bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. Since $\mathfrak{d} = \omega_1$, there exists a $\subseteq$-dominant family $\langle f_\gamma \colon \gamma < \omega_1 \rangle \subseteq ([\omega_1]^{<\omega})^X$. We claim that the family $\{W_{\alpha,\gamma} \colon \langle \alpha, \gamma \rangle \in \omega_1 \times \omega_1\} \subseteq \mathcal{U} \otimes \mathcal{V}$, where $W_{\alpha,\gamma} = \bigcup \{\{x\} \times \bigcap_{\beta \in f_\gamma(x)} V_\beta \colon x \in U_\alpha\}$, is a basis for $\mathcal{U} \otimes \mathcal{V}$. To check this, pick an $A \in \mathcal{U} \otimes \mathcal{V}$. Then, $\{x \in X \colon \{y \colon \langle x, y \rangle \in A\} \in \mathcal{V}\} \in \mathcal{U}$. Pick an $\alpha < \omega_1$ such that $U_\alpha \subseteq \{x \in X \colon \{y \colon \langle x, y \rangle \in A\} \in \mathcal{V}\}$. Then, given an $x \in U_\alpha$ there exists a $\beta_x < \omega_1$ such that $V_{\beta_x} \subseteq \{y \in Y \colon \langle x, y \rangle \in A\}$. This implies that $\{x\} \times V_{\beta_x} \subseteq A$ for every $x \in U_\alpha$.

Consider the function $g \colon X \to [\omega_1]^{<\omega}$ defined as

$$g(x) = \begin{cases} \{\beta_x\} & \text{if } x \in U_\alpha \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $\langle f_\gamma \colon \gamma < \omega_1 \rangle$ is a $\subseteq$-dominant family, there exists a $\gamma < \omega_1$ such that $g(x) \subseteq f_\gamma(x)$ for every $x \in U_\alpha$. This implies that $\beta_x \in f_\gamma(x)$ and that $\{x\} \times \bigcap_{\beta \in f_\gamma(x)} V_\beta \subseteq A$ for every $x \in U_\alpha$. Hence, $W_{\alpha,\gamma} \subseteq A$. ■

Theorem 4.3 $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists an* ω_1*-generated crowded ultrafilter which is not a P-point, a Q-point, or even an* ω_1*-OK point.*

Proof. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies the existence of an ω_1-generated crowded ultrafilter $\mathcal{U}$ on $\mathbb{Q}$, see [4, prop. 4.25]. We will show that $\mathcal{U} \otimes \mathcal{U}$ is as desired.

By Proposition 4.1, it is not a P-point, a Q-point, or an ω_1-OK point. Also, since $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies $\mathfrak{d} = \omega_1$, by Proposition 4.2 the ultrafilter $\mathcal{U} \otimes \mathcal{U}$ is ω_1-generated by some family $\mathcal{B}$.

To see that $\mathcal{U} \otimes \mathcal{U}$ can be treated as crowded, consider $\mathbb{Q} \times \mathbb{Q}$ as the product of $\langle \mathbb{Q}, \tau_d \rangle$ and $\langle \mathbb{Q}, \tau_s \rangle$, where τ_d is the discrete topology and τ_s is the standard topology. Then $\mathbb{Q} \times \mathbb{Q}$ is homemorphic to $\mathbb{Q}$.

For $B \in \mathcal{B}$ let $\bar{B} = \{x \colon (B)_x \in B\} \in \mathcal{U}\}$. Using Fact 2.5, for every $x \in \bar{B}$ we can choose a subset $B^x \in \mathrm{Perf}(\mathbb{Q})$ of $(B)_x$. Let $B^* = \bigcup \{\{x\} \times B^x \colon x \in \bar{B}\}$. Then B^* is a perfect subset of $\mathbb{Q} \times \mathbb{Q}$. Thus, $\{B^* \colon B \in \mathcal{B}\}$ is a basis of $\mathcal{U} \otimes \mathcal{U}$ of cardinality ω_1 formed with perfect subsets of $\mathbb{Q} \times \mathbb{Q}$. ■

5 A crowded Q-point which is not an ω_1-OK point.

Definition 6 Let X be a countably infinite set and let $\mathcal{J} \subseteq \mathcal{P}(X)$ be an ideal on X. If $A, B \in \mathcal{J}^+$ we write $A \preceq^{\mathcal{J}} B$ if and only if $A \setminus B \in \mathcal{J}$.

Definition 7 Let X be a countably infinite set and let $\mathcal{J} \subseteq \mathcal{P}(X)$ be an ideal on X. We say that $\mathcal{J}$ has the *extension property* provided that for every $\preceq^{\mathcal{J}}$-decreasing sequence $\langle A_n \in \mathcal{J}^+ \colon n < \omega\rangle$ there exists an $A \in \mathcal{J}^+$ such that $A \preceq^{\mathcal{J}} A_n$ for every $n < \omega$.

Let X and $\mathcal{J}$ be as above and let $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$. We will consider a relation $\sqsubseteq$ defined on $\mathcal{K}^+$ as

$$A \sqsubseteq B \Leftrightarrow \mathrm{supp}(A) \subseteq^* \mathrm{supp}(B) \ \ \& \ \ (A)_n \preceq^{\mathcal{J}} (B)_n \ \ \forall n \in \mathrm{supp}(A) \cap \mathrm{supp}(B).$$

Note that for $A, B \in \mathcal{K}^+$

$$A \subseteq B \Longrightarrow A \sqsubseteq B \Longrightarrow A \preceq^{\mathcal{K}} B$$

but none of these implications can be reversed. Also, it is not difficult to see that the relation $\sqsubseteq$ is not transitive. Nevertheless, we say that for $\xi < \omega_1$ a sequence $\langle U_\eta \in \mathcal{K}^+ \colon \eta < \xi\rangle$ is $\sqsubseteq$-decreasing provided $U_\eta \sqsubseteq U_\zeta$ for every $\zeta < \eta < \xi$.

Lemma 5.1 *Let X be a countably infinite set, let $\mathcal{J} \subseteq \mathcal{P}(X)$ be an ideal on X with the extension property, and let $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$. Then, for every $\xi < \omega_1$ and every $\sqsubseteq$-decreasing sequence $\langle U_\eta \in \mathcal{K}^+ \colon \eta < \xi\rangle$ there exists a $C \in \mathcal{K}^+$ such that $C \sqsubseteq U_\eta$ for every $\eta < \xi$. Moreover, the sequence $\langle U_\eta \colon \eta \leq \xi\rangle$ is $\sqsubseteq$-decreasing for every $U_\xi \in \mathcal{P}(C) \cap \mathcal{K}^+$.*

Proof. sequence. Since the sequence $\langle \mathrm{supp}(U_\eta) \colon \eta < \xi\rangle$ is $\subseteq^*$-decreasing, we can find an $S \in [\omega]^\omega$ such that $S \subseteq^* U_\eta$ for every $\eta < \xi$. For each $m \in S$ consider the set $I_m = \{\eta < \xi \colon m \in \mathrm{supp}(U_\eta)\}$. Then, since $\mathcal{J}$ has the extension property, we can find a $C_m \in \mathcal{J}^+$ such that $C_m \preceq^{\mathcal{J}} (U_\eta)_m$ for every $\eta \in I_m$. Put $C = \bigcup\{\{m\} \times C_m \colon m \in S\}$. Then clearly $C \sqsubseteq U_\eta$ for every $\eta < \xi$. The additional part follows from the fact that $U \subset C \sqsubseteq V$ implies $U \sqsubseteq V$. ∎

Lemma 5.2 *Let X, $\mathcal{J}$, and $\mathcal{K}$ be as above. Let $\langle U_\xi \in \mathcal{K}^+ \colon \xi < \omega_1\rangle$ be a $\sqsubseteq$-decreasing sequence in $\mathcal{K}^+$ such that for every $g \in \omega \times X$ there exists a $\xi < \omega_1$ such that $g \restriction U_\xi$ is constant. Then, the family $\{U_\xi \colon \xi < \omega_1\}$ forms a base for a nonprincipal ultrafilter on $\omega \times X$ which is not an ω_1-OK point.*

Proof. We check first that the family $\{U_\xi \colon \xi < \omega_1\}$ has $SFIP$. So, choose $\xi_0 < \cdots < \xi_n < \omega_1$. Since $U_{\xi_n} \sqsubseteq \cdots \sqsubseteq U_{\xi_1} \sqsubseteq U_{\xi_0}$ we can pick an $m \in \bigcap_{i\le n} \operatorname{supp}(U_{\xi_i})$. If $I_m = \{\xi < \omega_1 \colon m \in \operatorname{supp}(A_\xi)\}$, then $\{\xi_i \colon i \le n\} \subseteq I_m$. Therefore, $(U_{\xi_n})_m \preceq^{\mathcal{J}} \cdots \preceq^{\mathcal{J}} (U_{\xi_0})_m$. This implies that $(\bigcap_{i\le n} U_{\xi_i})_m \in \mathcal{J}^+$. In particular, $(\bigcap_{i\le n} U_{\xi_i})_m$ is infinite and so is $\bigcap_{i\le n} U_{\xi_i}$. Let $\mathcal{U}$ be a filter generated $\{U_\xi \colon \xi < \omega_1\}$.

To see that $\mathcal{U}$ is actually an ultrafilter, pick any $A \subseteq \omega \times X$. Then, there exists a $\xi < \omega_1$ and an $i < 2$ such that $\chi_A \restriction U_\xi$ is constant equal i. If $i = 0$ then $U_\xi \subseteq (\omega \times X) \setminus A$ and $(\omega\times X)\setminus A \in \mathcal{U}$. If $i = 1$ then $U_\xi \subseteq A$ and $A \in \mathcal{U}$. Therefore, $\mathcal{U}$ is an ultrafilter and $\{U_\xi \colon \xi < \omega_1\}$ is a base for $\mathcal{U}$. Observe that $\mathcal{U}$ is nonprincipal because each set in $\mathcal{U}$ contains an infinite set U_ξ.

To see that $\mathcal{U}$ is not an ω_1-OK point consider a sequence $\langle V_n \in \mathcal{U} \colon n < \omega\rangle$, where $V_n = \bigcup_{i>n}(\{i\} \times X)$. Suppose that there exists a $\langle W_\xi \in \mathcal{U} \colon \xi < \omega_1\rangle$ which is OK for $\langle V_n \in \mathcal{U} \colon n < \omega\rangle$. Since $\{U_\xi \colon \xi < \omega_1\}$ is a basis for $\mathcal{U}$, for every for every $\xi < \omega_1$ there exists a $U_{\alpha_\xi} \subseteq W_\xi$. This implies that

$$\langle U_{\alpha_\xi} \colon \xi < \omega_1\rangle \quad \text{is OK for} \quad \langle V_n \in \mathcal{U} \colon n < \omega\rangle.$$

By the pigeon hole principle, there exist a $T \in [\omega_1]^{\omega_1}$ and an $m < \omega$ such that $m = \min(\operatorname{supp}(U_{\alpha_\xi}))$ for every $\xi \in T$. Hence, $T \subseteq I_m$. Pick any ordinals $\alpha_{\xi_0} < \cdots < \alpha_{\xi_m}$ in T. Since $\langle U_{\alpha_\xi} \colon \xi < \omega_1\rangle$ is OK for $\langle V_n \in \mathcal{U} \colon\ < \omega\rangle$ we have that $\bigcap_{i\le m} U_{\alpha_{\xi_i}} \subseteq^* V_m$. Hence, $|(\bigcap_{i\le m} U_{\alpha_{\xi_i}}) \cap (\{m\} \times X)| < \omega$ by the definition of V_m. On the other hand, $\{\alpha_{\xi_i} \colon i \le m\} \subseteq I_m$. Therefore, $(U_{\alpha_{\xi_0}})_m \preceq^{\mathcal{J}} \cdots \preceq^{\mathcal{J}} (U_{\alpha_{\xi_m}})_m$. This implies that $|(\bigcap_{i\le m} U_{\alpha_{\xi_i}})_m| = \omega$. So, $|(\bigcap_{i\le m} U_{\alpha_{\xi_i}}) \cap (\{m\} \times X)| = \omega$. This contradiction indicates that $\mathcal{U}$ cannot be an ω_1-OK point. ∎

Let X, $\mathcal{J}$, and $\mathcal{K}$ be as before and let $\mathcal{D} \subseteq \mathcal{J}^+$ be dense in the sense that for every $A \in \mathcal{J}^+$ there exists a $D \in \mathcal{D}$ such that $D \subseteq A$. Then, the family $\mathcal{D}^* \subseteq \mathcal{K}^+$ consisting of the sets of the form $\bigcup\{\{n\} \times D_n \colon n \in I\}$ is dense in $\mathcal{K}^+$, where $I \in [\omega]^\omega$ and $D_n \in \mathcal{D}$ for every $n \in I$. Recall also that $\boldsymbol{\mathcal{P}}_{\omega\times X}$ is the space of all partitions of $\omega \times X$ into finite pieces, as defined in Section 3.

Theorem 5.3 *Let X be a countably infinite set, let $\mathcal{J} \subseteq \mathcal{P}(X)$ be an ideal with the extension property, and let $\mathcal{D} \subseteq \mathcal{J}^+$ be dense. If $\mathcal{J}$ is prism-friendly* *and Q-like and $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$, then $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there exists an ω_1-generated Q-point $\mathcal{U}$ on $\omega \times X$ which is not an ω_1-OK point and such that $\mathcal{U} \cap \mathcal{D}^*$ is a basis for $\mathcal{U}$.*

Proof. We construct a $\sqsubseteq$-decreasing sequence $\langle U_\xi \in \mathcal{K}^+ \cap \mathcal{D}^* \colon \xi < \omega_1\rangle$ such that:

(i) For every $g \in 2^{\omega\times X}$ there exists a $\xi < \omega_1$ such that $g \restriction U_\xi$ is constant.

(ii) For every $z \in \boldsymbol{\mathcal{P}}_{\omega\times X}$ there exists a $\xi < \omega_1$ such that $|z(k) \cap U_\xi| \le 1$ for every $k \in \omega$.

If this construction is possible, then, by Lemma 5.1, $\{U_\xi \in \mathcal{D}^* \colon \xi < \omega_1\}$ is a basis for a nonprincipal ultrafilter $\mathcal{U}$ on $\omega \times X$ which is not an ω_1-OK point. To see that $\mathcal{U}$ is a Q-point pick an arbitrary $z \in \boldsymbol{\mathcal{P}}_{\omega\times X}$. Then, by condition (ii), there exists a $\xi < \omega_1$ such that $|z(k) \cap U_\xi| \le 1$ for every $k < \omega$. Therefore, $\mathcal{U}$ is an ω_1-generated Q-point.

Let $\mathcal{Y} = 2^{\omega\times X} \cup \boldsymbol{\mathcal{P}}_{\omega\times X}$ and consider it with the topology τ formed with all sets $A \subseteq \mathcal{Y}$ such that $A \cap 2^{\omega\times X}$ and $A \cap \boldsymbol{\mathcal{P}}_{\omega\times X}$ are open in $2^{\omega\times X}$ and $\boldsymbol{\mathcal{P}}_{\omega\times X}$, respectively. Then $\langle \mathcal{Y}, \tau\rangle$ is a Polish space. Note that, by Lemmas 3.5 and 3.6, the ideal $\mathcal{K}$ is Q-like and prism-friendly. For a prism P in $\mathcal{Y}$ and $U \in \mathcal{K}^+$ we choose a subprism $Q(U,P)$ of P and $B(U,P) \in \mathcal{P}(U) \cap \mathcal{D}^*$ as follows.

- If $U \cap 2^{\omega\times X} \neq \emptyset$, then we can choose a subprism $P_0 \subseteq 2^{\omega\times X}$ of P. The choice of P_0 is obvious if P is a singleton; otherwise it follows from Proposition 2.2. Then $Q(U,P)$ is a subprism of P_0 such that $Q(U,P)$ and $B(U,P) \in \mathcal{P}(U) \cap \mathcal{K}^+$ satisfy condition ($\bullet$) from the definition of the prism-friendly ideal.

- If $U \cap 2^{\omega\times X} = \emptyset$, then P is a prism in $\boldsymbol{\mathcal{P}}_{\omega\times X}$. Then, by Lemma 3.3, there exist a subprism $Q(U,P)$ of P and a $B(U,P) \in \mathcal{P}(U) \cap \mathcal{K}^+$ such that $|z(k) \cap B(U,P)| \le 1$ for every $z \in Q(U,P)$ and $k < \omega$.

We can also assume that $B(U,P) \in \mathcal{D}^*$, since $\mathcal{D}^*$ is dense in $\mathcal{K}^+$.

Also, for $\xi < \omega_1$ and a $\sqsubseteq$-decreasing sequence $\langle U_\eta \in \mathcal{K}^+ \colon \eta < \xi\rangle$ let $C_\xi = C(\langle U_\eta \colon \eta < \xi\rangle)$ be such that $C_\xi \sqsubseteq U_\eta$ for every $\eta < \xi$. Its existence follows from Lemma 5.1. Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(C(\langle U_\eta \colon \eta < \xi\rangle), P_\xi),$$

where the U_η's are defined inductively by $U_\eta = B(C(\langle U_\zeta \colon \zeta < \eta\rangle), P_\eta)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, the strategy S is not a winning strategy for Player II. So, there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S in which Player II loses. Thus, $\mathcal{Y} = \bigcup_{\xi<\omega_1} Q_\xi$. Let $\langle U_\xi \in \mathcal{D}^* \colon \xi < \omega_1\rangle \subseteq \mathcal{K}^+$ be the sequence created in this game. This sequence is $\sqsubseteq$-decreasing by construction and Lemma 5.1. By the observations made before we only need to check that $\langle U_\xi \colon \xi < \omega_1\rangle$ satisfy conditions (i) and (ii).

If $g \in 2^{\omega\times X}$, then there exists a $\xi < \omega_1$ such that $g \in Q_\xi$. So, $Q_\xi \subseteq 2^{\omega\times X}$ and, by the construction, $g \restriction U_\xi$ is constant. This proves (i). Similarly, if $z \in \boldsymbol{\mathcal{P}}_{\omega\times X}$, then there exists a $\xi < \omega_1$ such that $z \in Q_\xi$. Hence, $Q_\xi \subseteq \boldsymbol{\mathcal{P}}_{\omega\times X}$ and, by the construction, $|z(k) \cap U_\xi| \le 1$ for every $k < \omega$. This proves (ii). ∎

Corollary 5.4 $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists an* ω_1*-generated crowded* Q*-point which is not an* ω_1*-OK point.*

Proof. Consider $X = \omega \times \mathbb{Q}$ with the product topology, where ω has the discrete topology and $\mathbb{Q}$ has the subspace topology inherited from $\mathbb{R}$. Then X is homeomorphic to $\mathbb{Q}$. We will find an ideal $\mathcal{J} \subseteq \mathcal{P}(X)$ to which we will apply Theorem 5.3.

Let $\mathcal{J} = [\omega]^{<\omega} \otimes \mathcal{I}_S$. It is clear that $\mathcal{J}$ contains all singletons. Also, $\mathcal{J}$ is prism-friendly by Lemmas 3.4 and 3.6 and Q-like by Lemmas 3.4 and 3.5. To see that $\mathcal{J}$ has the extension property pick a $\preceq^{\mathcal{J}}$-decreasing sequence $\langle A_n \in \mathcal{J}^+ \colon n < \omega\rangle$. By induction construct an increasing sequence $\langle n_k \colon k < \omega\rangle$ such that $n_k \in \mathrm{supp}(A_k) \setminus \mathrm{supp}\left(\bigcup_{i<k}(A_k \setminus A_i)\right)$. The choice can be made, since the set $\mathrm{supp}\left(\bigcup_{i<k}(A_k \setminus A_i)\right)$ is finite, as $\bigcup_{i<k}(A_k \setminus A_i) \in \mathcal{J}$. The choice of n_k gives also $\left(\bigcup_{i<k}(A_k \setminus A_i)\right)_{n_k} \in \mathcal{I}_S$. Thus, $\left(\bigcap_{i\le k} A_i\right)_{n_k} \notin \mathcal{I}_S$. Put $B = \bigcup\left\{\{n_k\} \times \left(\bigcap_{i\le k} A_i\right)_{n_k} \colon k < \omega\right\}$. Then $B \in \mathcal{J}^+$ and $B \preceq^{\mathcal{J}} A_n$ for every $n < \omega$.

Since $\bar{\mathcal{D}} = \mathrm{Perf}(\mathbb{Q})$ is dense in $(\mathcal{I}_S)^+$, the family $\mathcal{D} = \bar{\mathcal{D}}^*$ is dense in $\mathcal{J}^+$. Applying Theorem 5.3 to $\mathcal{J}$ and $\mathcal{D}$, we can find an ω_1-generated Q-point $\mathcal{U}$ on $\omega \times X$ which is not an ω_1-OK point and such that $\mathcal{U} \cap \mathcal{D}^*$ contains a basis for $\mathcal{U}$. Since $\omega \times X$ is homeomorphic to $\mathbb{Q}$ and $\mathcal{D}^*$ consists of perfect set in $\omega \times X$, it follows that $\mathcal{U}$ is crowded. ∎

6 Crowded ω_1-generated ω_1-OK points which are not P-points.

In this section we prove that the axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies the existence of an ω_1-OK point which is not a P-point. For this, we follow the schema used in [9] for the construction of such an ultrafilter in the model of ZFC obtained by adding Sacks reals side-by-side. Since that proof uses CH in the ground model, we have to modify things a bit to make it work in the context of $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$. One possiblity for avoiding the use of CH is to replace it with some weaker principle consistent with $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ like, for instance, $\mathfrak{d} = \omega_1$. Let Γ denote the set of all nonzero limit ordinals below ω_1. The following fact is a simple generalization of the remark above Proposition 4.2.

Fact 6.1 ($\mathfrak{d} = \omega_1$) *There exist a sequence $\langle g_\delta \colon \delta < \omega_1\rangle$ of functions from ω into $[\omega_1]^{<\omega}$ and a partition $\{S_\delta \in [\omega_1]^{\omega_1} \colon \delta < \omega_1\}$ of Γ such that:*

- *For every $h\colon \omega \to \omega_1$ there is a $\delta < \omega_1$ such that $h(n) \in g_\delta(n)$ for every $n < \omega$.*
- $\bigcup \mathrm{rang}(g_\delta) = \min(S_\delta)$ *for every $\delta < \omega_1$.*

Fix a countably infinite set X and put $\mathcal{P} = \{\{m\} \times X \colon m < \omega\}$. Then, $\mathcal{P}$ is a partition of $\omega \times X$ into infinitely many infinite pieces. The idea of the proof is to find a sequence $\langle U_\alpha \colon \alpha < \omega_1\rangle$ that forms a base for a nonprincipal ultrafilter $\mathcal{U}$ on $\omega \times X$ such that every U_α has infinite intersection with infinitely many members of $\mathcal{P}$ and, for each $\delta < \omega_1$,

$$\langle U_\alpha \colon \alpha \in S_\delta\rangle \text{ is OK for } \left\{ \bigcap_{\eta \in g_\delta(n)} U_\eta \colon n < \omega \right\}.$$

To see that such an $\mathcal{U}$ is an ω_1-OK point pick $\langle V_n \colon n < \omega\rangle \in (\mathcal{U})^\omega$. Since the sequence $\langle U_\alpha \colon \alpha < \omega_1\rangle$ is a basis for $\mathcal{U}$, for every $n < \omega$ there is a $\xi_n < \omega_1$ such that $U_{\xi_n} \subseteq V_n$. Therefore, there exists a $\delta < \omega_1$ such that $\xi_n \in g_\delta(n)$ for every $n < \omega$. Then $\langle U_\alpha \colon \alpha \in S_\delta\rangle$ is OK for $\langle V_n \colon n < \omega\rangle$ since for any sequence $\alpha_0 < \cdots < \alpha_n$ of elements in S_δ we have:

$$\bigcap_{i \le n} U_{\alpha_i} \subseteq^* \bigcap_{\eta \in g_\delta(n)} U_\eta \subseteq U_{\xi_n} \subseteq V_n.$$

Observe that $\mathcal{U}$ cannot be a P-point because each U_α intersects infinitely many members of $\mathcal{P}$ on an infinite set.

Let us start with fixing a rich ideal $\mathcal{J} \subseteq \mathcal{P}(X)$ and a dense $\mathcal{D} \subseteq \mathcal{J}^+$. We will consider the ideal $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$ on $\omega \times X$ and the set $\mathcal{D}^* \subseteq \mathcal{K}^+$ as defined in Section 5. We also fix, for each $\xi \in \Gamma$, an enumeration $\{\xi_i \colon i < \omega\}$ of ξ.

Let $\mathcal{T}$ be the set of triples $\langle I, f, B\rangle$ satisfying the following requirements:

- I is an infinite subset of ω,
- $f \in \prod_{m<\omega}(\mathcal{P}(X) \cap \mathcal{D})$, and
- $B \in \prod_{m\in\omega}(\mathcal{P}(f(m)) \cap \mathcal{D})^{\omega_1}$ such that every $B(m)$ is a sequence of almost disjoint sets.

If $\xi \leq \omega_1$ and $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} \colon \eta < \xi\rangle$, the sequence $\langle U_\eta \colon \eta < \xi\rangle$ associated with it is defined by

$$U_\eta = \bigcup \{\{m\} \times f_\eta(m) \colon m \in I_\eta\}.$$

Note that each U_η is in $\mathcal{D}^*$.

To prove that the resulting ultrafilter $\mathcal{U}$ in our construction is in fact an ω_1-OK point we will consider for every $\delta < \omega_1$, $\eta < \xi$, and $m < \omega$ the sets $K(\eta, m) = \{\zeta < \eta \colon f_\eta(m) \subseteq f_\zeta(m)\}$, the numbers $k_\delta(\eta, m) = |K(\eta, m) \cap S_\delta|$, and the functions l_δ defined by:

$$l_\delta(\eta, m) = \begin{cases} \infty & \text{if } \bigcup \operatorname{rang}(g_\delta) \subseteq K(\eta, m) \\ -1 & \text{if } g_\delta(0) \not\subseteq K(\eta, m) \\ \max\{l < \omega \colon \bigcup g_\delta[l+1] \subseteq K(\eta, m)\} & \text{otherwise.} \end{cases}$$

Definition 8 For $\xi \leq \omega_1$ a sequence $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} \colon \eta < \xi\rangle$ is *good* if:

(a) For every $\zeta < \eta < \xi$ and $m < \omega$, either $f_\zeta(m) \cap f_\eta(m)$ is finite, or there exists a $\gamma \leq \eta$ such that $f_\eta(m) \subseteq B_\zeta(m)(\gamma) \subset f_\zeta(m)$.

(b) For every $0 < \eta < \xi$ and $m < \omega$ there exists a $\zeta < \eta$ such that $f_\eta(m) \subseteq B_\zeta(m)(\eta)$.

(c) If $\eta < \xi$ is limit and $\{m_i : i < \omega\}$ is the increasing enumeration of I_η, then

$$m_i \in \bigcap_{j \leq i} I_{\eta_j} \qquad \text{and} \qquad f_\eta(m_i) \subseteq \bigcap_{j \leq i} f_{\eta_j}(m_i),$$

where $\{\eta_j : j < \omega\}$ is our fixed enumeration of η.

(d) $f_\eta(m) = B_0(m)(\eta)$ for every $m \in \omega \setminus I_\eta$.

(e) If $\eta + 1 < \xi$, then $I_{\eta+1} \subseteq I_\eta$ and $f_{\eta+1}(m) \subseteq B_\eta(m)(\eta+1) \subseteq f_\eta(m)$ for every $m \in I_{\eta+1}$.

(f) If $\delta < \omega_1$, $\eta < \xi$, and $\eta \in S_\delta$, then $l_\delta(\eta, m) > k_\delta(\eta, m)$ for every $m \in I_\eta$.

(g) If $\delta < \omega_1$, $\eta < \xi$, and $\bigcup \mathrm{rang}(g_\delta) \subseteq \eta$, then

$$\lim_{\substack{m \in I_\eta \\ m \to \infty}} (l_\delta(\eta, m) - k_\delta(\eta, m)) = \infty.$$

Remark 6.2 *It follows from (c) and (e) that if $\zeta < \eta < \xi$, then $I_\eta \subseteq^* I_\zeta$.*

Remark 6.3 *It is also easy to check that if $\xi \leq \omega_1$ is a limit ordinal, then the sequence $\langle \langle I_\zeta, f_\zeta, B_\zeta \rangle \in \mathcal{T} : \zeta < \xi \rangle$ is good if and only if the sequence $\langle \langle I_\zeta, f_\zeta, B_\zeta \rangle \in \mathcal{T} : \zeta < \eta \rangle$ is good for every $\eta < \xi$.*

Remark 6.4 *It is not difficult to see that*

$$\text{if } \alpha < \beta < \xi, \text{ then } f_\beta(m) \subseteq f_\alpha(m) \text{ for all but finitely many } m \in I_\beta. \tag{3}$$

If $\beta \in \Gamma$ this follows from (c). If $\Gamma \cap (\alpha, \beta] = \emptyset$, then it follows from (e). If $\Gamma \cap (\alpha, \beta] \neq \emptyset$ then there exist a maximal $\gamma \in \Gamma \cap (\alpha, \beta]$ and, by the above two cases, $f_\beta(m) \subseteq f_\gamma(m) \subseteq f_\alpha(m)$ for all but finitely many $m \in I_\beta$.

Remark 6.5 *For every $0 < \eta < \xi$ and $m < \omega$ there exists a $\gamma \leq \eta$ such that $f_\eta(m) \subseteq B_0(m)(\gamma)$. This follows from condition (b), since every strictly decreasing sequence of ordinals is finite.*

The dual filter of an ideal $\mathcal{K}$ on a set $\omega \times X$ is the family $\mathcal{F}_\mathcal{K}$ defined as $\mathcal{F}_\mathcal{K} = \{(\omega \times X) \setminus A : A \in \mathcal{K}\}$. The importance of the definition of a good sequence derives from the following lemma.

Lemma 6.6 *Let X be a countably infinite set, $\mathcal{J} \subseteq \mathcal{P}(X)$ a rich ideal in X, $\mathcal{D} \subseteq \mathcal{J}^+$ a dense family, and let $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$. If $\langle\langle I_\xi, f_\xi, B_\xi\rangle \in \mathcal{T} \colon \xi < \omega_1\rangle$ is a good sequence such that for every $g \in 2^{\omega\times X}$ there exists a $\xi < \omega_1$ such that $g \restriction U_\xi$ is constant, then $\langle U_\xi \in \mathcal{D}^* \colon \xi < \omega_1\rangle$ forms a base for a nonprincipal ultrafilter on $\omega \times X$ extending $\mathcal{F}_\mathcal{K}$ which is an ω_1-OK point but not a P-point.*

PROOF. The fact that $\{U_\xi \colon \xi < \omega_1\} \subseteq \mathcal{D}^*$ follows immediately from the definition of U_η and $\mathcal{D}^*$.

Next we prove that $\{U_\xi \colon \xi < \omega_1\}$ forms a base for a nonprincipal ultrafilter $\mathcal{U}$ on $\omega \times X$ extending the filter $\mathcal{F}_\mathcal{K}$. Given $\xi_0 < \cdots < \xi_n$ pick a $\gamma \in \Gamma$ with $\gamma > \xi_n$. By (c), we have that almost every $m \in I_\gamma$ is in $\bigcap_{i\le n} I_{\xi_i}$ and that $f_\gamma(m) \subseteq \bigcap_{i\le n} f_{\xi_i}(m)$. Therefore, $\bigcap_{i\le n} U_{\xi_i} \in \mathcal{K}^+$ and $\{U_\xi \colon \xi < \omega_1\}$ can be extended to a proper filter $\mathcal{U}$ on $\omega \times X$. If $A \subseteq \omega \times X$, then $\chi_A \in 2^{\omega\times X}$ and there exist a $\xi < \omega_1$ and an $i < 2$ such that $\chi_A \restriction U_\xi$ is constant equal i. If $i = 1$ then $U_\xi \subseteq A$ and $A \in \mathcal{U}$. If $i = 0$ then $U_\xi \subseteq (\omega \times X) \setminus A$ so $(\omega \times X) \setminus A \in \mathcal{U}$. This proves that $\mathcal{U}$ is an ultrafilter and that $\{U_\xi \colon \xi < \omega_1\}$ is a base for $\mathcal{U}$. Since no $A \in \mathcal{K}$ contains any U_ξ, it follows that $\mathcal{U}$ extends $\mathcal{F}_\mathcal{K}$. In particular, $\mathcal{U}$ is nonprincipal.

To see that $\mathcal{U}$ is not a P-point notice that every U_ξ intersects infinitely many pieces of the partition $\mathcal{P} = \{\{m\} \times X \colon m < \omega\}$ on an infinite set and so does every $V \in \mathcal{U}$.

To prove that $\mathcal{U}$ is an ω_1-OK point it is enough to prove that for every $\delta < \omega_1$

$$\langle U_\alpha \colon \alpha \in S_\delta\rangle \text{ is OK for } \left\{ \bigcap_{\eta\in g_\delta(n)} U_\eta \colon n < \omega \right\}. \tag{4}$$

Pick $\delta < \omega_1$ and $\xi_0 < \cdots < \xi_n$ in S_δ. First, we prove that for every $m \in I_{\xi_n}$

$$\text{either } \bigcap_{i\le n} f_{\xi_i}(m) \text{ is finite, or } \bigcap_{i\le n} f_{\xi_i}(m) \subseteq \bigcap_{\eta\in g_\delta(n)} f_\eta(m). \tag{5}$$

Indeed, assume that $\bigcap_{i\le n} f_{\xi_i}(m)$ is infinite. Then, by part (a) of Definition 8, $\xi_i \in K(\xi_n, m) \cap S_\delta$ for each $i \le n-1$. Therefore, $k_\delta(\xi_n, m) \ge n$ and $l_\delta(\xi_n, m) \ge n+1$ by Definition 8(f). In particular, $g_\delta(n) \subseteq K(\xi_n, m)$. Hence, by the definition of $K(\eta, m)$,

$$\bigcap_{i\le n} f_{\xi_i}(m) \subseteq f_{\xi_n}(m) \subseteq \bigcap_{\eta\in K(\xi_n,m)} f_\eta(m) \subseteq \bigcap_{\eta\in g_\delta(n)} f_\eta(m).$$

Also, Definition 8(c) implies that $f_{\xi_n}(m) \subseteq \bigcap_{\eta\in g_\delta(n)} f_\eta(m)$ for all but finitely many $m \in I_{\xi_n}$. Thus, the set $s = \left\{m \in I_{\xi_n} \colon \bigcap_{i\le n} f_{\xi_i}(m) \not\subseteq \bigcap_{\eta\in g_\delta(n)} f_\eta(m)\right\}$ is finite. Moreover, by (5), $\bigcap_{i\le n} f_{\xi_i}(m)$ is finite for every $m \in s$. So,

$$
\begin{aligned}
\bigcap_{i\le n} U_{\xi_i} &= \bigcup_{m\in\bigcap_{i\le n} I_{\xi_i}} \left(\{m\}\times\bigcap_{i\le qn} f_{\xi_i}(m)\right)\\
&\subseteq \bigcup_{m\in I_{\xi_n}} \left(\{m\}\times\bigcap_{i\le n} f_{\xi_i}(m)\right)\\
&= \bigcup_{m\in s} \left(\{m\}\times\bigcap_{i\le n} f_{\xi_i}(m)\right) \cup \bigcup_{m\in I_{\xi_n}\setminus s} \left(\{m\}\times\bigcap_{i\le n} f_{\xi_i}(m)\right)\\
&\subseteq^* \bigcup_{m\in I_{\xi_n}\setminus s} \left(\{m\}\times\bigcap_{\eta\in g_\delta(n)} f_{\xi_i}(m)\right)\\
&\subseteq \bigcup_{m\in I_{\xi_n}} \left(\{m\}\times\bigcap_{\eta\in g_\delta(n)} f_{\xi_i}(m)\right) = \bigcup_{\eta\in g_\delta(n)} U_\eta,
\end{aligned}
$$

which proves (4). So $\mathcal{U}$ is an ω_1-OK point. ■

Lemma 6.7 *Let $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} \colon \eta < \xi\rangle$ be a sequence satisfying condition (a) from Definition 8 and let $\alpha < \beta < \xi$ and $m < \omega$ be such that $f_\beta(m) \subseteq B_\alpha(m)(\beta)$. Then $K(\beta, m) = K(\alpha, m) \cup \{\alpha\}$. In particular, if the sequence satisfies conditions (a) and (b) from Definition 8, then the set $K(\eta, m)$ is finite for every $\eta < \xi$ and $m < \omega$.*

Proof. If $\eta \in K(\beta, m)$, then $\eta < \beta$ and $f_\beta(m) \subseteq f_\eta(m)$. Also, since $f_\beta(m) \subseteq B_\alpha(m)(\beta) \subseteq f_\alpha(m)$ we have that $|f_\eta(m) \cap f_\alpha(m)| = \omega$. If $\alpha < \eta$, then, by condition (a), there exists a $\gamma \le \eta$ such that $f_\eta(m) \subseteq B_\alpha(m)(\gamma)$; therefore $f_\eta(m) \subseteq B_\alpha(m)(\gamma) \cap B_\alpha(m)(\beta)$, which is impossible. Thus, $\eta \le \alpha$. If $\eta < \alpha$, then, again by (a), there is a $\gamma \le \alpha$ such that $f_\alpha(m) \subseteq B_\eta(m)(\gamma)$. Since $B_\eta(m)(\gamma) \subseteq f_\eta(m)$, we conclude that $f_\alpha(m) \subseteq f_\eta(m)$. Therefore, $\eta \in K(\alpha, m)$ and this proves that $K(\beta, m) \subseteq K(\alpha, m) \cup \{\alpha\}$.

Since $f_\beta(m) \subseteq B_\alpha(m)(\beta) \subseteq f_\alpha(m)$ we have that $\alpha \in K(\beta, m)$. If $\eta \in K(\alpha, m)$, then $f_\alpha(m) \subseteq f_\eta(m)$. But since $f_\beta(m) \subseteq B_\alpha(m)(\beta) \subseteq f_\alpha(m)$ we have that $f_\beta(m) \subseteq f_\eta(m)$. Therefore, $\eta \in K(\beta, m)$ and this proves that $K(\alpha, m) \cup \{\alpha\} \subseteq K(\beta, m)$. Thus, $K(\beta, m) = K(\alpha, m) \cup \{\alpha\}$.

Since condition (b) implies that for every $0 < \eta < \xi$ and $m < \omega$ there exists a $\zeta < \eta$ such that $f_\eta(m) \subseteq B_\zeta(m)(\eta)$, we have that for every $0 < \eta < \xi$ there exists a $\zeta < \eta$ such that $K(\eta, m) = K(\zeta, m) \cup \{\zeta\}$. Since $K(0, m) = \emptyset$ for every $m < \omega$, we can prove, by induction on η, that $K(\eta, m)$ is finite for every $\eta < \xi$ and $m < \omega$. ■

Lemma 6.8 *If $\xi \in \Gamma$ and $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} : \eta < \xi\rangle$ is good, then there exists an $\langle I_\xi, f_\xi, B_\xi\rangle \in \mathcal{T}$ such that the sequence $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} : \eta \leq \xi\rangle$ is good.*

Proof. Let $\{\xi_j : j < \omega\}$ be the fixed enumeration of ξ. Since S_δ's are pairwise disjoint, the set $\{\delta < \omega_1 : \min(S_\delta) < \xi\}$ is countable and it can be enumerated as $\{\delta_i : i < \omega\}$. Let $\delta^* < \omega_1$ be such that $\xi \in S_{\delta^*}$. We define $I_\xi = \{m_i : i < \omega\}$ inductively. Suppose that m_j has already been defined for every $j < i$. Put

$$\varepsilon_i = \max(g_{\delta^*}(0) \cup g_{\delta^*}(0) \cup \{\xi_j : j \leq i\} \cup \{\min(S_{\delta_j}) : j \leq i\}) + 1 < \xi.$$

Note that $\varepsilon_i < \xi$ since Remark 6.2 implies that $I_{\varepsilon_i} \subseteq^* \bigcap_{j \leq i} I_{\xi_j}$. Thus, we can pick an $m_i \in I_{\varepsilon_i} \cap \bigcap_{j \leq i} I_{\xi_j}$ so that:

(i) $m_i > m_j$ for every $j < i$,

(ii) $l_{\delta_j}(\varepsilon_i, m_i) - k_{\delta_j}(\varepsilon_i, m_i) > i$ for every $j \leq i$, and

(iii) $f_{\varepsilon_i}(m_i) \subseteq \bigcap_{j \leq i} f_{\xi_j}(m_i) \cap \bigcap\{f_\eta(m_i) : \eta \in g_{\delta^*}(0) \cup g_{\delta^*}(1)\}$.

Condition (ii) can be achieved since $\bigcup \mathrm{rang}(g_{\delta_j}) \subseteq \min(S_{\delta_j}) < \varepsilon_i < \xi$ and $\{\langle I_\eta, f_\eta, B_\eta\rangle : \eta < \xi\}$ is good, so, by (g),

$$\lim_{\substack{m \in I_{\varepsilon_i} \\ m \to \infty}} (l_{\delta_j}(\varepsilon_i, m) - k_{\delta_j}(\varepsilon_i, m)) = \infty$$

for every $j \leq i$. Condition (iii) can be ensured by Remark 6.4, since $\xi_j < \varepsilon_i$ for $j \leq i$ and $\eta < \varepsilon_i forall \eta \in g_{\delta^*}(0) \cup g_{\delta^*}(1)$. This completes the inductive definition of I_ξ. Define $f_\xi : \omega \to \mathcal{D}$ as

$$f_\xi(m) = \begin{cases} B_{\varepsilon_i}(m_i)(\xi) & \text{if } m = m_i \in I_\xi \\ \\ B_0(m)(\xi) & \text{otherwise.} \end{cases}$$

The B_ξ can be defined by taking for each $m < \omega$ an arbitrary ω_1-sequence of almost disjoint sets in $\mathcal{P}(f_\xi(m)) \cap \mathcal{D}$. This completes the definition of $\langle I_\xi, f_\xi, B_\xi \rangle$.

To make sure that (a) holds it is enough to check it only for the pair $\langle \eta, \xi \rangle$ in place of $\langle \zeta, \eta \rangle$. So, choose an $\eta < \xi$ and $m < \omega$. We need to show that either $f_\xi(m) \cap f_\eta(m)$ is finite, or there exists a $\gamma \leq \xi$ such that $f_\xi(m) \subseteq B_\eta(m)(\gamma)$. We will consider several cases.

$m \notin I_\xi$**:** We will consider here two subcases.

$\eta = 0$: Then $f_\xi(m) = B_0(m)(\xi) = B_\eta(m)(\gamma)$ for $\gamma = \xi$.

$\eta > 0$: Apply Remark 6.5 to find $\gamma \leq \eta$ such that $f_\eta(m) \subseteq B_0(m)(\gamma)$. Since $f_\xi(m) = B_0(m)(\xi)$, we have that $|f_\xi(m) \cap f_\eta(m)| < \omega$.

$m = m_i \in I_\xi$**:** We will consider here three subcases.

$\varepsilon_i < \eta$: We will show that $|f_\xi(m) \cap f_\eta(m)| < \omega$. So, by way of contradiction, assume that $|f_\xi(m) \cap f_\eta(m)| = \omega$. Therefore, the sets $f_\xi(m) \cap f_\eta(m) = B_{\varepsilon_i}(m)(\xi) \cap f_\eta(m) \subseteq f_{\varepsilon_i}(m) \cap f_\eta(m)$ are infinite. So, by (a), there exists a $\gamma \leq \eta$ such that $f_\eta(m) \subseteq B_{\varepsilon_i}(m)(\gamma)$. Thus, $B_{\varepsilon_i}(m)(\xi) \cap B_{\varepsilon_i}(m)(\gamma) = f_\xi(m) \cap B_{\varepsilon_i}(m)(\gamma) \supseteq f_\xi(m) \cap f_\eta(m)$ is infinite, which is impossible, as $\gamma \leq \eta < \xi$. This implies that $|f_\xi(m) \cap f_\eta(m)| < \omega$.

$\varepsilon_i > \eta$: If $f_{\varepsilon_i}(m) \cap f_\eta(m)$ is finite then so is $f_\xi(m) \cap f_\eta(m) \subseteq f_{\varepsilon_i}(m) \cap f_\eta(m)$. Otherwise, by (a), there is a $\gamma \leq \varepsilon_i$ such that $f_{\varepsilon_i}(m) \subseteq B_\eta(m)(\gamma)$. So, $f_\xi(m) = B_{\varepsilon_i}(m)(\xi) \subseteq f_{\varepsilon_i}(m) \subseteq B_\eta(m)(\gamma)$.

$\varepsilon_i = \eta$: Clearly $f_\xi(m) = B_{\varepsilon_i}(m)(\xi) = B_\eta(m)(\gamma)$ for $\gamma = \xi$.

Conditions (b), (c), and (d) are immediate from the definition of f_ξ. Condition (e) holds because $\langle \langle I_\eta, f_\eta, B_\eta \rangle \colon \eta < \xi \rangle$ is good and ξ is a limit ordinal.

To prove (f) and (g) first observe that, by Lemma 6.7, for every $i < \omega$ we have $K(\xi, m_i) = K(\varepsilon_i, m_i) \cup \{\varepsilon_i\}$. This implies that $l_{\delta_j}(\varepsilon_i, m_i) \leq l_{\delta_j}(\xi, m_i)$ and $k_{\delta_j}(\xi, m_i) = k_{\delta_j}(\varepsilon_i, m_i)$ for every $j \leq i$, because ε_i is a succesor ordinal. In particular, for every $j \leq i$ we have

$$l_{\delta_j}(\xi, m_i) - k_{\delta_j}(\xi, m_i) \geq l_{\delta_j}(\varepsilon_i, m_i) - k_{\delta_j}(\varepsilon_i, m_i). \tag{6}$$

To see (f) fix an $m = m_i \in I_\xi$. We need to show that $l_{\delta^*}(\xi, m) > k_{\delta^*}(\xi, m)$. First assume that $\xi = \min(S_{\delta^*})$. Then, $K(\xi, m) \subseteq \xi$ is disjoint with S_{δ^*}, so $k_{\delta^*}(\xi, m) = |K(\xi, m) \cap S_{\delta^*}| = 0$. On the other hand, condition (iii) implies that $\bigcup g_{\delta^*}[2] \subseteq K(\xi, m)$. So, $l_{\delta^*}(\xi, m) \geq 1 > 0 = k_{\delta^*}(\xi, m)$. Next, consider the case when $\xi > \min(S_{\delta^*})$. Then, $\delta^* = \delta_i$ for some $i < \omega$. Therefore, (ii) and (6) imply that

$$l_{\delta^*}(\xi, m_i) - k_{\delta^*}(\xi, m_i) \geq l_{\delta_i}(\varepsilon_i, m_i) - k_{\delta_i}(\varepsilon_i, m_i) > i \geq 0.$$

Thus (f) holds.

To see (g) fix a $\delta < \omega_1$ such that $\bigcup \mathrm{rang}(g_\delta) \subseteq \xi$. We need to show that $\lim_{i\to\infty}(l_\delta(\xi, m_i) - k_\delta(\xi, m_i)) = \infty$. First assume that $\xi > \min(S_\delta)$. Then $\delta = \delta_j$ for some $j < \omega$. So, by (ii) and (6), we have that for all $i \geq j$

$$l_\delta(\xi, m_i) - k_\delta(\xi, m_i) = l_{\delta_j}(\xi, m_i) - k_{\delta_j}(\xi, m_i) \geq l_{\delta_j}(\varepsilon_i, m_i) - k_{\delta_j}(\varepsilon_i, m_i) \geq i.$$

This ensures that (g) holds. Finally, assume that $\xi \leq \min(S_\delta)$. Then, for every $m < \omega$, we have $K(\xi, m) \cap S_\delta = \emptyset$ and so, $k_\delta(\xi, m) = 0$. Thus, in this case it is enough to show that $\lim_{i\to\infty} l_\delta(\xi, m_i) = \infty$. But for every $l < \omega$ we have $\bigcup g_\delta[l+1] \subseteq \xi = \{\xi_j \colon j < \omega\}$. Thus, there exists an $i_0 < \omega$ such that $\bigcup g_\delta[l+1] \subseteq \xi = \{\xi_j \colon j \leq i_0\}$. Since, by (iii), for every $i \geq i_0$ we have $\{\xi_j \colon j \leq i\} \subseteq K(\xi, m_i)$ we conclude that $\bigcup g_\delta[l+1] \subseteq K(\xi, m_i)$ for every $i \geq i_0$. Thus, $l_\delta(\xi, m_i) \geq l$ for every $i \geq i_0$ and so $\lim_{i\to\infty} l_\delta(\xi, m_i) = \infty$. ■

Lemma 6.9 *Let $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} \colon \eta \leq \xi\rangle$ be a good sequence, $I \in [I_\xi]^\omega$, and let $\langle D_m \in \mathcal{P}(B_\xi(m)(\xi+1)) \cap \mathcal{D} \colon m \in I\rangle$ be arbitrary. Then, the sequence $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} \colon \eta \leq \xi+1\rangle$ is good, where $\langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle \in \mathcal{T}$ is defined as*

(i) $I_{\xi+1} = I$,

(ii) $$f_{\xi+1}(m) = \begin{cases} D_m & \text{if } m \in I_{\xi+1} \\ B_0(m)(\xi+1) & \text{otherwise,} \end{cases}$$

(iii) *$B_{\xi+1}(m) \in (\mathcal{P}(f_{\xi+1}(m)) \cap \mathcal{D})^{\omega_1}$ is any almost disjoint sequence for every $m < \omega$.*

PROOF. To show that (a) holds it is enough to check it only for the pair $\langle \eta, \xi+1\rangle$ in place of $\langle \zeta, \eta\rangle$. So, choose an $\eta < \xi+1$ and $m < \omega$. We need to show that either $f_{\xi+1}(m) \cap f_\eta(m)$ is finite, or there exists a $\gamma \leq \xi+1$ such that $f_{\xi+1}(m) \subseteq B_\eta(m)(\gamma)$. We consider several cases.

$m \notin I_{\xi+1}$**:** We will consider two subcases.

$\eta = 0$: Then $f_{\xi+1}(m) = B_0(m)(\xi+1) = B_\eta(m)(\gamma)$ for $\gamma = \xi+1$.

$\eta > 0$: Apply Remark 6.5 to find a $\gamma \le \eta$ such that $f_\eta(m) \subseteq^* B_0(m)(\gamma)$. Since $f_{\xi+1}(m) = B_0(m)(\xi+1)$, we have that $|f_{\xi+1}(m) \cap f_\eta(m)| < \omega$.

$m \in I_{\xi+1}$**:** We compare η with ξ.

$\eta < \xi$: By (a) either $|f_\xi(m) \cap f_\eta(m)| < \omega$ or there exists a $\gamma \le \xi$ such that $f_\xi(m) \subseteq^* B_\eta(m)(\gamma)$. Since $f_{\xi+1}(m) \subseteq B_\xi(m)(\xi+1) \subseteq f_\xi(m)$ we have that $|f_{\xi+1}(m) \cap f_\eta(m)| < \omega$ or $f_{\xi+1}(m) \subseteq B_\eta(m)(\gamma)$.

$\eta = \xi$: Clearly $f_{\xi+1}(m) = D_m \subseteq B_\xi(m)(\xi+1) = B_\eta(m)(\gamma)$ for $\gamma = \xi+1$.

This proves that (a) holds.

Conditions (b), (d), and (e) are obvious by the definition of $f_{\xi+1}$. Conditions (c) and (f) hold, since there are no new limit ordinals $\eta < \xi+1$.

To see that (g) holds take a $\delta < \omega_1$ such that $\bigcup \mathrm{rang}(g_\delta) \subseteq \xi+1$. Then also $\bigcup \mathrm{rang}(g_\delta) \subseteq \xi$ since, by Fact 6.1, $\bigcup \mathrm{rang}(g_\delta) = \min(S_\delta)$ is a limit ordinal. Thus, $\lim_{\substack{m\in I_\xi \\ m\to\infty}} (l_\delta(\xi,m) - k_\delta(\xi,m)) = \infty$. Also, by Lemma 6.7 and the definition of $f_{\xi+1}(m)$ we have $K(\xi+1,m) = K(\xi,m) \cup \{\xi\}$ for every $m \in I_{\xi+1}$. This implies that $k_\delta(\xi+1,m) \le k_\delta(\xi,m)+1$ and $l_\delta(\xi,m) \le l_\delta(\xi+1,m)$ for every $m \in I_{\xi+1}$. So, $l_\delta(\xi+1,m) - k_\delta(\xi+1,m) \ge l_\delta(\xi,m) - k_\delta(\xi,m) - 1$. Since $I_{\xi+1} \subseteq I_\xi$ is infinite, we have

$$\lim_{\substack{m\in I_{\xi+1} \\ m\to\infty}} (l_\delta(\xi+1,m) - k_\delta(\xi+1,m)) \ge \lim_{\substack{m\in I_\xi \\ m\to\infty}} (l_\delta(\xi,m) - k_\delta(xi,m) - 1) = \infty.$$

So, (h) holds. ■

Corollary 6.10 *Let $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T} \colon \eta \le \xi\rangle$ be good. If P is a prism in $2^{\omega\times X}$, then there exists an $\langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle \in \mathcal{T}$, a suprism Q of P, and an $i < 2$ such that*

(i) *$\langle\langle I_\eta, f_\eta, B_\eta\rangle \colon \eta \le \xi+1\rangle$ is good and*

(ii) *$g \restriction U_{\xi+1}$ is constant equal to i for every $g \in Q$.*

PROOF. Apply Lemma 3.6 to the prism P, the set I_ξ, and the family $\{B_\xi(m)(\xi+1)\colon m \in I_\xi\}$ to find a subprism Q of P, a set $I_{\xi+1} \in [I_\xi]^\omega$, a sequence $\langle B_m \in \mathcal{P}(B(m)(\xi+1)) \cap \mathcal{J}^+\colon m \in I_{\xi+1}\rangle$, and an $i<2$ such that $g \restriction B$ is constant equal to i, where $B = \bigcup\{\{m\} \times B_m\colon m \in I_{\xi+1}\}$. For every $m \in I_{\xi+1}$ choose $D_m \in \mathcal{P}(B_m) \cap \mathcal{D}$. Then, if we define $f_{\xi+1}$ and $B_{\xi+1}$ as in Lemma 6.9, $\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi+1\rangle$ is good and $g \restriction U_{\xi+1}$ is constant equal to i. ■

Corollary 6.11 *Let X be a countably infinite set, $\mathcal{J} \subseteq \mathcal{P}(X)$ a Q-like ideal on X, $\langle\langle I_\eta, f_\eta, B_\eta\rangle \in \mathcal{T}\colon \eta \le \xi\rangle$ be good, and P be a prism on $\boldsymbol{\mathcal{P}}_{\omega\times X}$. Then, there exists a $\langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle \in \mathcal{T}$ and a subprism Q of P such that*

(i) $\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi+1\rangle$ *is good and*

(ii) $|z(k) \cap U_{\xi+1}| \le 1$ *for every $z \in Q$ and $k<\omega$.*

PROOF. Let $A = \bigcup\{\{m\} \times B_\xi(m)(\xi+1)\colon m \in I_\xi\}$. Then $A \in \mathcal{K}^+$ and, by Lemma 3.5, $\mathcal{K}$ is Q-like. So, by Lemma 3.3, there is a subprism Q of P and a $B \in \mathcal{P}(A) \cap \mathcal{K}^+$ such that $|z(k) \cap B| \le 1$ for every $z \in Q$ and $k<\omega$. Let $I_{\xi+1} = \mathrm{supp}(B) \subseteq I_\xi$ and for every $m \in I_{\xi+1}$ choose $D_m \in \mathcal{P}((B)_m) \cap \mathcal{D}$. Then, if we define $f_{\xi+1}$ and $B_{\xi+1}$ as in Lemma 6.9, $\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi+1\rangle$ is good and $|z(k) \cap U_{\xi+1}| \le 1$ for every $z \in Q$ and $k<\omega$. ■

Theorem 6.12 *Let X be a countably infinite set, $\mathcal{J} \subseteq \mathcal{P}(X)$ be a rich ideal, let $\mathcal{D} \subseteq \mathcal{J}^+$ be a dense family, and put $\mathcal{K} = [\omega]^{<\omega} \otimes \mathcal{J}$. Then, $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies that there exists an ω_1-generated ω_1-OK point extending $\mathcal{F}_\mathcal{K}$ with a basis $\{U_\xi\colon \xi < \omega_1\} \subseteq \mathcal{D}^*$ which is not a P-point.*

PROOF. To define a triple $\langle I_0, f_0, B_0\rangle$ put $I_0 = \omega$, for every $m<\omega$ define $f_0(m) = X$, and let $B_0(m) = \langle B_0(m)(\gamma)\colon \gamma < \omega_1\rangle$ be an arbitrary ω_1-sequence of almost disjoint sets in $\mathcal{P}(f_0(m)) \cap \mathcal{D}$.

For a good sequence $\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi\rangle$ and a prism P in $2^{\omega\times X}$ let us define a subprism $Q(\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi\rangle, P) = Q$ of P and the triple $T(\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi\rangle, P) = \langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle \in \mathcal{T}$ as in Corollary 6.10. We define a strategy S for Player II in the game $\mathrm{GAME}_{\mathrm{prism}}(2^{\omega\times X})$ as:

$$S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta < \xi\rangle, P_\xi) = Q(\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi\rangle, P_\xi),$$

where $\langle\langle I_\eta, f_\eta, B_\eta\rangle\colon \eta \le \xi\rangle$ is a good sequence defined by induction on $\eta \le \xi$ as follows. Assume that $\langle\langle I_\zeta, f_\zeta, B_\zeta\rangle\colon \zeta < \eta\rangle$ is already defined.

If $\eta = 0$, then $\langle I_\eta, f_\eta, B_\eta\rangle = \langle I_0, f_0, B_0\rangle$ is defined as above.

If $\eta = \zeta + 1$, then we put $\langle I_\eta, f_\eta, B_\eta\rangle = T(\langle\langle I_\delta, f_\delta, B_\delta\rangle \colon \delta \leq \zeta\rangle, P_\zeta)$.

If $\eta \in \Gamma$, then $\langle I_\eta, f_\eta, B_\eta\rangle$ is found using Lemma 6.8.

Notice that the sequence $\langle\langle I_\zeta, f_\zeta, B_\zeta\rangle \colon \zeta < \eta\rangle$ is good by the inductive hypothesis and Remark 6.3.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ strategy S is not a winning strategy for Player II. So, there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S for which Player II loses, this is, $2^{\omega\times X} = \bigcup_{\xi<\omega_1} Q_\xi$. If $\langle\langle I_\xi, f_\xi, B_\xi\rangle \in \mathcal{T} \colon \xi < \omega_1\rangle$ is the sequence created when Player II uses strategy S, then this sequence is good by construction. Application of Lemma 6.6 to this sequence finishes the proof. ■

Theorem 6.13 $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists an* ω_1*-generated, crowded* ω_1*-OK point on* $\mathbb{Q}$ *which is neither a* P*-point nor a* Q*-point.*

PROOF. The idea is to apply Theorem 6.12 to an apropriate ideal to get a crowded ultrafilter which is not a Q-point. Consider $X = \mathbb{Q} \times \omega$ with a natural product topology. Then, X is homeomorphic to $\mathbb{Q}$. For every $m < \omega$ put $P_m = \{n < \omega \colon 2^m - 1 \leq n < 2^{m+1} - 1\}$. Then $\{P_m \colon m < \omega\}$ is a partition of ω and $|P_m| = 2^m$. For $A \subset \mathbb{Q} \times \omega$ put

$$N_A(m) = \max\{k < \omega \colon \exists\, U \in \mathcal{I}_S^+ \ \exists\, P \in [P_m]^k \ \ U \times P \subseteq A\}$$

and define $\mathcal{J} \subseteq \mathcal{P}(\mathbb{Q} \times \omega)$ as

$$\mathcal{J} = \{A \subseteq \mathbb{Q} \times \omega \colon \overline{\lim}_{m\to\infty} N_A(m) < \infty\}.$$

To see that $\mathcal{J}$ is closed under finite unions notice first that

$$N_{A\cup B}(m) \leq N_A(m) + N_B(m) \quad \text{for every } m < \omega \text{ and } A, B \subseteq \mathbb{Q} \times \omega.$$

Indeed, take a $P \subseteq P_m$ of cardinality $N_{A\cup B}(m)$ and $U \in \mathcal{I}_S^+$ such that $U \times P \subseteq A \cup B$. Let $h \colon U \times P \to 2$ be a characteristic function of $A \cap (U \times P)$ and let $\varphi \colon U \to 2^P$ be defined by $\varphi(u)(p) = h(u, p)$. Since 2^P is finite, there exists a $g \in 2^P$ such that $V = \varphi^{-1}(g)$ belongs to $\mathcal{I}_S^+$. Let $P_A = g^{-1}(1)$ and $P_B = g^{-1}(0)$. Then $V \times P_A \subseteq A$ and $V \times P_B \subseteq B$. Therefore, $N_A(m) \geq |P_A|$ and $N_B(m) \geq |P_B|$. So, $N_{A\cup B}(m) = |P| = |P_A| + |P_B| \leq N_A(m) + N_B(m)$.

The above proved inequality easily implies that

$$\overline{\lim}_{m\to\infty} N_{A\cup B}(m) \leq \overline{\lim}_{m\to\infty} N_A(m) + \overline{\lim}_{m\to\infty} N_B(m)$$

for every and $A,B\subseteq\mathbb{Q}\times\omega$. Thus, $\mathcal{J}$ is closed under finite unions. Since it clearly is closed also under subsets, we can conclude that $\mathcal{J}$ is an ideal on $\mathbb{Q}\times\omega$ containing all the singletons. We will prove that

$$\text{the ideal } \mathcal{J} \text{ is rich.} \tag{7}$$

First notice how (7) implies the theorem. Since $\mathrm{Perf}(\mathbb{Q})$ is dense in $\mathcal{I}_S^+$, it is easy to see that $\mathcal{D}=\mathrm{Perf}(\mathbb{Q}\times\omega)$ is dense in $\mathcal{J}^+$. Let $\mathcal{U}$ be an ultrafilter on $\omega\times X$ from Theorem 6.12 applied to $\mathcal{J}$ and $\mathcal{D}$. Since $X=\mathbb{Q}\times\omega$ is homeomorphic to $\mathbb{Q}$, so is $\omega\times X$ and $\mathcal{D}^*$ contains only its perfect subsets. Therefore, $\mathcal{U}$ can be considered as crowded. Moreover, a partition $\mathcal{P}=\{\{n\}\times(\{q\}\times P_m)\colon q\in\mathbb{Q}\ \&\ n,m<\omega\}$ of $\omega\times X$ into finite sets does not admit partial selector in $\mathcal{U}$, since each such partial selector belongs to $\mathcal{K}=[\omega]^{<\omega}\times\mathcal{J}$. Thus, $\mathcal{U}$ is not a Q-point.

To prove property (7) fix an $A\in\mathcal{J}^+$. Then there exist $\langle m_k\in\omega\colon k<\omega\rangle$, $\{U_k\in\mathcal{I}_S^+\colon k<\omega\}$, and $\langle Q_k\subseteq P_{m_k}\colon k<\omega\rangle$ such that $U_k\times Q_k\subseteq A$ and $|Q_k|>k\cdot 2^{2^k}$ for every $k<\omega$.

First we prove condition (#) from Definition 3. Since, by Lemma 3.4, the ideal $\mathcal{I}_S$ on $\mathbb{Q}$ is rich, for every $k<\omega$ there exists an almost disjoint family $\{U^k_f\colon f\in 2^\omega\}\subseteq\mathcal{P}(U_k)\cap\mathcal{I}_S^+$. Also, for every $k<\omega$ there exists a pairwise disjoint family $\{A_s\colon s\in 2^k\}\subseteq[Q_k]^k$. For $f\in 2^\omega$ define $A_f=\bigcup\{U^k_f\times A_{f\restriction k}\colon k<\omega\}$. Then, $\{A_f\colon f\in 2^\omega\}\subseteq\mathcal{P}(A)\cap\mathcal{J}^+$ is almost disjoint, proving (#).

To prove that $\mathcal{J}$ is prism-friendly let P be a prism in 2^X. If P is singleton then condition ($\bullet$) is clearly satisfied. So, assume that $P\in\mathrm{Perf}(2^X)$ and let f be a witness function for it. By Remark 2.1 we can assume that f is defined on $\mathfrak{C}^\alpha$ for some $0<\alpha<\omega_1$. Our first goal is to find a subprism Q' of P and two sequences $\{V_k\subseteq U_k\colon k<\omega\}\subseteq\mathcal{I}_S^+$ and $\{A_k\in[P_{m_k}]^k\colon k<\omega\}$ such that

$$g\restriction V_k\times A_k \ \text{ is constant for every } \ g\in Q'. \tag{8}$$

For every $k<\omega$ define $\mathcal{D}_k$ as the set of all disjoint collections $\mathcal{E}\in[\mathbb{P}_\alpha]^{<\omega}$ such that there exists a $V_{\langle\mathcal{E},k\rangle}\in\mathcal{P}(U_k)\cap\mathcal{I}_S^+$ such that for every $q\in Q_k$, $E\in\mathcal{E}$, and $h,h'\in E$, each $f(h)$ is constant on $V_{\langle\mathcal{E},k\rangle}\times\{q\}$ and

$$f(h)\restriction V_{\langle\mathcal{E},k\rangle}\times\{q\}=f(h')\restriction V_{\langle\mathcal{E},k\rangle}\times\{q\}. \tag{9}$$

It is immediate that $\mathcal{D}_k$ is closed under refinaments. To prove that $\mathcal{D}_k$ satisfies the condition (†) from Proposition 2.3 let $\mathcal{E}\in\mathcal{D}_k$ and $E\in\mathbb{P}_\alpha$ be such that $E\cap\bigcup\mathcal{E}=\emptyset$. Let $\{q_i\colon i\leq r\}$ be an enumeration of Q_k. Using Proposition 2.7, construct inductively decreasing sequences $\langle E_i\in\mathbb{P}_\alpha\cap\mathcal{P}(E)\colon i\leq r\rangle$, $\langle V_i\in\mathcal{P}(V_{\langle\mathcal{E},k\rangle})\cap\mathcal{I}^+_S\colon i\leq r\rangle$, and a sequence $\langle j_i<2\colon i\leq r\rangle$ such that for every $i\leq r$

$$f(h)\restriction V_i\times\{q_i\}\ \text{ is constant equal to } j_i \text{ for every }\ h\in E_i. \tag{10}$$

Therefore, if we put $E'=E_r$ and $V_{\langle\mathcal{E}\cup\{E'\},k\rangle}=V_r$, then $\mathcal{E}\cup\{E'\}\in\mathcal{D}_k$ and condition (†) is satisfied. Thus, by Proposition 2.3, for every $k<\omega$ there exists a family $\mathcal{E}_k=\{E_i\colon i<2^k\}\in\mathcal{D}_k$ of pairwise disjoint sets with $E^0=\bigcap_{k<\omega}\bigcup\mathcal{E}_k\in\mathbb{P}_\alpha$. We will prove that $Q'=f[E^0]$ satisfies (8) with $V_k=V_{\langle\mathcal{E}_k,k\rangle}$ and some sequence $\langle A_k\in[Q_k]^k\colon k<\omega\rangle$.

To see this fix $k<\omega$ and $v_0\in V_k=V_{\langle\mathcal{E}_k,k\rangle}$, and for each $i<2^k$ pick an $h_i\in E_i\in\mathcal{E}_k$. Define $\varphi_k\colon Q_k\to 2^{2^k}$ by $\varphi_k(p)(i)=f(h_i)(v_0,p)$. Since $|Q_k|>k\cdot 2^{2^k}$, there exists an $s_k\in 2^{2^k}$ such that $|\varphi_k^{-1}\{s_k\}|\geq k$. Pick an $A_k\in[\varphi_k^{-1}\{s_k\}]^k$. To see that the pair $\langle V_k,A_k\rangle$ satisfies (8), pick a $g\in Q'$. Then there exists an $i<2^k$ and an $h\in E_i\in\mathcal{E}_k$ such that $g=f(h)$. We will show that $g[V_k\times A_k]=\{s_k(i)\}$.

Let $\langle v,q\rangle\in V_k\times A_k$. Since, by (9), $f(h)$ is constant on $V_k\times\{q\}$, we have $f(h)(v,q)=f(h)(v_0,h)$. Also, (9) gives $f(h)(v_0,q)=f(h_i)(v_0,q)$. Hence, $g(v,q)=f(h_i)(v_0,q)=\varphi_k(q)(i)=s_k(i)$. So, $g\restriction V_k\times A_k$ is constant equal to $s_k(i)$ and (8) holds.

To finish the proof for every $k<\omega$ pick $\langle v_k,a_k\rangle\in V_k\times A_k$ and put $S=\{\langle v_k,a_k\rangle\colon k<\omega\}$. Let $\mathcal{I}=[X]^{<\omega}$. Then $\mathcal{I}$ is weakly selective and $S\in\mathcal{I}^+$. If we identify 2^X with $\mathcal{P}(X)$, then Q' can be treated as a prism in $\mathcal{P}(X)$. Since $[X]^\omega$ is residual in $\mathcal{P}(X)$, by Proposition 2.2 we can assume that Q' is a prism in $[X]^\omega$. So, by Proposition 2.7, there exist a subprism Q of Q', a set $S_0\in[S]^\omega$, and an $i<2$ such that $g[S_0]=\{i\}$ for every $g\in Q$. Put $B=\bigcup\{V_k\times A_k\colon\langle v_k,a_k\rangle\in S_0\}$. Then, $g\restriction B$ is constant equal i for every $g\in Q$. It is clear that $B\subseteq A$. Since $V_k\times A_k\subseteq B$ and $A_k\in[P_{m_k}]^k$ we have $N_B(m_k)\geq k$. This implies that $\overline{\lim}_{m\to\infty}N_B(m)=\infty$ and that $B\in\mathcal{J}^+$. So, Q and B satisfy (•). ■

Theorem 6.14 *Let X be a countably infinite set, $\mathcal{J}\subseteq\mathcal{P}(X)$ a rich and Q-like ideal on X, and let $\mathcal{D}\subseteq\mathcal{J}^+$ be dense. Then, $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there exists an ω_1-generated, crowded ω_1-OK point on $\omega\times X$ which is also a Q-point but not a P-point.*

PROOF. This proof combines the elements of the proofs of Theorems 5.3 and 6.12. Let $\mathcal{Y} = \boldsymbol{\mathcal{P}}_{\omega\times X} \cup 2^{\omega\times X}$ be as in Theorem 5.3.

For a good sequence $\bar{G} = \langle\langle I_\eta, f_\eta, B_\eta\rangle \colon \eta \leq \xi\rangle$ and a prism P in $\mathcal{Y}$ let us define a subprism $Q(\bar{G}, P)$ of P and a triple $T(\bar{G}, P) \in \mathcal{T}$ as follows.

- If $U \cap 2^{\omega\times X} \neq \emptyset$, then we can choose a subprism $P_0 \subseteq 2^{\omega\times X}$ of P. The choice of P_0 is obvious if P is a singleton, and it follows from Proposition 2.2, otherwise. Then we apply Corollary 6.10 to $\bar{G}$ and P_0 to find appropriate subprism $Q(\bar{G}, P)$ of P_0 and $\langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle \in \mathcal{T}$. We put $T(\bar{G}, P) = \langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle$.

- If $U \cap 2^{\omega\times X} = \emptyset$, then P is a prism in $\boldsymbol{\mathcal{P}}_{\omega\times X}$. Then, we can use Corollary 6.11 to find appropriate $\langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle \in \mathcal{T}$ and a subprism $Q(\bar{G}, P)$ of P_0. We put $T(\bar{G}, P) = \langle I_{\xi+1}, f_{\xi+1}, B_{\xi+1}\rangle$.

We define a strategy S for Player II in the game $\mathrm{GAME}_{\mathrm{prism}}(\mathcal{Y})$ as:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(\langle\langle I_\eta, f_\eta, B_\eta\rangle \colon \eta \leq \xi\rangle, P_\xi),$$

where the sequence $\langle\langle I_\eta, f_\eta, B_\eta\rangle \colon \eta \leq \xi\rangle$ is defined as in Theorem 6.12.

By $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ strategy S is not a winning strategy for Player II. So, there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S for which Player II loses, this is, $\mathcal{Y} = \bigcup_{\xi<\omega_1} Q_\xi$. If $\langle\langle I_\xi, f_\xi, B_\xi\rangle \in \mathcal{T} \colon \xi < \omega_1\rangle$ is the sequence created when Player II uses strategy S, then, by Remark 6.3, this sequence is good.

If $g \in 2^{\omega\times X}$, then there exists a $\xi < \omega_1$ such that $g \in Q_\xi$. Therefore, $Q_\xi \subseteq 2^{\omega\times X}$ and $g \restriction U_{\xi+1}$ is constant. Thus, by Lemma 6.6, the family $\{U_\xi \colon \xi < \omega_1\}$ forms a base for a nonprincipal ultrafilter $\mathcal{U}$ on $\omega \times X$ which is an ω_1-OK point but not a P-point. Note that $\{U_\xi \colon \xi < \omega_1\} \subseteq \mathcal{D}^*$. To see that $\mathcal{U}$ is a Q-point, take a $z \in \boldsymbol{\mathcal{P}}_{\omega\times X}$. Then, there exists a $\xi < \omega_1$ such that $z \in Q_\xi$. This means that $Q_\xi \subseteq \boldsymbol{\mathcal{P}}_{\omega\times X}$ and that $|z(k) \cap U_{\xi+1}| \leq 1$ for every $k < \omega$. Hence, $\mathcal{U}$ is also a Q-point. ∎

Corollary 6.15 $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ *implies that there is an ω_1-generated, crowded ω_1-OK point on $\omega \times X$ which is also a Q-point but not a P-point.*

PROOF. Apply Theorem 6.14 with $X = \mathbb{Q}$, $\mathcal{J} = \mathcal{I}_S$, and $\mathcal{D} = \mathrm{Perf}(\mathbb{Q})$. ∎

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