# A crowded $Q$-point under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$

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**Abstract**

In this note we prove that the version $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ of the Covering Property Axiom, which holds in the iterated Sacks model, implies that there exists an $\omega_1$-generated crowded ultrafilter on $\mathbb{Q}$ which is also a $Q$-point. Since no crowded ultrafilter can be a $P$-point this constitutes an interesting example of a $Q$-point which is not a $P$-point.

## 1 Introduction

We will use standard set theoretic notation as in [5]. Let $\mathcal{U}$ be a non-principal ultrafilter on a countable set $X$. Then, $\mathcal{U}$ is a *P-point* if for every partition $\mathcal{P}$ of $X$ either $\mathcal{U} \cap \mathcal{P} \neq \emptyset$ or there exists an $X \in \mathcal{U}$ such that $X \cap P$ is finite for each $P \in \mathcal{P}$. $\mathcal{U}$ is called a *Q-point* if for every partition $\mathcal{P}$ of $X$ into finite pieces there exists an $X \in \mathcal{U}$ such that $|X \cap P| \leq 1$ for each $P \in \mathcal{P}$. Given a non-principal ultrafilter $\mathcal{U}$ on $X$ we say that $\mathcal{B} \subset \mathcal{U}$ is a basis for $\mathcal{U}$ if for every $U \in \mathcal{U}$ there exists a $B \in \mathcal{B}$ such that $B \subset U$. Then, we can define the *character* of $\mathcal{U}$ as $\chi(\mathcal{U}) = \min\{|\mathcal{B}| \colon \mathcal{B} \text{ is a basis for } \mathcal{U}\}$. We say that $\mathcal{U}$ is $\kappa$-generated if $\chi(\mathcal{U}) = \kappa$.

Consider $\mathbb{Q}$ with the subspace topology induced by the usual topology on $\mathbb{R}$ and denote by $\mathrm{Perf}(\mathbb{Q})$ the family of its perfect subsets. A non-principal

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filter $\mathcal{U}$ on $\mathbb{Q}$ is *crowded* if the family $\mathrm{Perf}(\mathbb{Q}) \cap \mathcal{U}$ forms a basis for $\mathcal{U}$. The crowded ultrafilters have been studied in connection with the remainder of the Stone-Čech compactification of $\mathbb{Q}$ and their existence follows from the Continuum Hypothesis, Martin's Axiom for countable posets [4], or from the equality $\mathfrak{b} = \mathfrak{c}$ [3].

In [1] Ciesielski and Pawlikowski showed that a version of their Covering Property Axiom called $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, which holds in the iterated Sacks model, implies that there exists an $\omega_1$-generated crowded ultrafilter on $\mathbb{Q}$ and they noted that no crowded ultrafilter can be a $P$-point. This result is interesting because CPA implies $\mathfrak{b} < \mathfrak{c}$.

The main result of this paper is that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies the existence of an $\omega_1$-generated crowded ultrafilter on $\mathbb{Q}$ which is also a $Q$-point [1]. Notice that this contradicts the remark by Ciesielski and Pawlikowski in [1, page 49] that crowded ultrafilters cannot be $Q$-points.

It is a result of A.Miller [7] that there are no $Q$-points in Laver's [6] model for Borel's Conjecture. Since the equality $\mathfrak{b} = \mathfrak{c}$ holds in Laver's model, it is consistent with ZFC that no crowded ultrafilter on $\mathbb{Q}$ is a $Q$-point.

# 2 Preliminaries on $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ and $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$

## 2.1 Cubes and Prisms.

The framework of CPA rests on the concepts of *cube* and *prism*. If $\mathfrak{C}$ denotes the space $2^\omega$ with its usual product topology and $\mathfrak{X}$ is a Polish space then we define

$$\mathrm{Perf}(\mathfrak{X}) = \{C \subset \mathfrak{X} \colon C \text{ is homeomorphic to } \mathfrak{C}\}.$$

A *perfect cube* in $\mathfrak{C}^\omega$ is any set $C = \prod_{i<\omega} C_i$ where $C_i \in \mathrm{Perf}(\mathfrak{C})$ for every $i < \omega$. If $\mathfrak{X}$ is a Polish space, then a *cube* in $\mathfrak{X}$ is a pair $\langle f, P\rangle$ where $f \colon C \to \mathfrak{X}$ is a continuous injection and $P = f[C]$ for some perfect cube $C$. The following theorem is one of the principal tools for using CPA, and it is a refinement of a theorem proved independently by H.G. Eggleston and M.L. Brodskiĭ.

**Proposition 1** (K.Ciesielski, J.Pawlikowski [2, claim 1.1.5]) *Consider $\mathfrak{C}^\omega$ with its usual topology and its usual product measure. If $G$ is a Borel subset*

[1] Recently the author has proven that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there is also a crowded $Q$-point of character $\mathfrak{c}$.

*of $\mathfrak{C}^\omega$ which is either of second category or of positive measure then $G$ contains a perfect cube.*

The notion of *prism* is a generalization of that of a cube. If $\alpha < \omega_1$ is a non-zero countable ordinal let $\Phi_{\text{prism}}(\alpha)$ be the set of all functions $f\colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ with the property that

$$f(x) \restriction \xi = f(y) \restriction \xi \Leftrightarrow x \restriction \xi = y \restriction \xi \qquad \text{for all } \xi < \alpha \text{ and } x, y \in \mathfrak{C}^\alpha.$$

Then we define $\mathbb{P}_\alpha = \{\text{range}(f)\colon f \in \Phi_{\text{prism}}(\alpha)\}$ and $\mathbb{P}_{\omega_1} = \bigcup_{0<\alpha<\omega_1} \mathbb{P}_\alpha$. The elements of $\mathbb{P}_{\omega_1}$ are called the *iterated perfect sets*. If $\mathfrak{X}$ is a Polish space, then a *prism* on $X$ is a pair $\langle f, P\rangle$ where $f\colon E \to \mathfrak{X}$ is injective and continuous, $E \in \mathbb{P}_{\omega_1}$, and $P = f[E]$.

It is also inmediate to observe that if the pair $\langle f, P\rangle$ and $f\colon E \to P$ and $E \in \mathbb{P}_\alpha$ then, we can assume that $f$ is defined on the entire $\mathfrak{C}^\alpha$.

It is important to note that the previous definitions imply that perfect cubes are, in particular, iterated perfect sets and therefore, that cubes are prisms. On the other hand, if $\langle g, P\rangle$ is a prism, where $g\colon E \to P$ and $E \in \mathbb{P}_\alpha$, then there exists an $f \in \Phi_{\text{prism}}(\alpha)$ with $E = \text{range}(f)$. In particular, $h = g \circ f\colon \mathfrak{C}^\alpha \to P$ is a continuous injection and the pair $\langle h, P\rangle$ is a cube. Thus, any prism can be thought as a cube with a different coordinate system imposed on it.

## 2.2 Subcubes and Subprisms.

If $\langle f, P\rangle$ is a cube, then we say that $Q$ is its subcube provided there exists a perfect cube $C \subset \text{dom}(f)$ such that $Q = f[C]$. Subprisms are defined similarly but replacing the perfect cube $C$ by an iterated perfect set $E$. Since in the games defined below we will need to consider singletons in the same position as cubes (or prism) as defined above, in what follows *singletons will be considered as cubes and prisms.* If $P$ is a singleton in $\mathfrak{X}$ then its only subcube is $P$ itself.

## 2.3 Games and Strategies.

For a Polish space $\mathfrak{X}$ consider the following game $\text{GAME}_{\text{cube}}(\mathfrak{X})$ of length $\omega_1$ played by two players, Player I and Player II. At each stage $\xi < \omega_1$ of the game Player I can play an arbitrary cube $P_\xi$ in $\mathfrak{X}$ (i.e., $P_\xi$ either belongs to $\mathrm{Perf}(\mathfrak{X})$ or is a singleton in $\mathfrak{X}$) and Player II must respond by playing a subcube $Q_\xi$ of $P_\xi$. The game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ is won by Player I provided

$$\mathfrak{X} = \bigcup_{\xi<\omega_1} Q_\xi;$$

otherwise Player II wins.

A strategy for Player II is any function $S$ such that $S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi)$ is a subcube of $P_\xi$ for every partial game $\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle$. We say that a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ is played according to a strategy $S$ for Player II provided $Q_\xi = S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi)$ for every $\xi < \omega_1$. A strategy $S$ for Player II is a *winning strategy* provided Player II wins any game played according the strategy $S$. The corresponding notions of games, strategies etc. for prisms are defined in a similar way.

### 2.4 The Axioms.

The following principles capture the combinatorial core of the iterated Sacks model.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$: $\mathfrak{c} = \omega_2$ and for any Polish space $\mathfrak{X}$ Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{cube}}(\mathfrak{X})$.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$: $\mathfrak{c} = \omega_2$ and for any Polish space $\mathfrak{X}$ Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{prism}}(\mathfrak{X})$.

These axioms are consequences of a more general principle, similar in spirit, called CPA [2]. Their importance comes from the following theorem.

**Proposition 2** (K.Ciesielski, J.Pawlikowski [1, 2]) CPA *holds in the iterated perfect set model. In particular,* CPA *is consistent with ZFC set theory.*

## 3 An $\omega_1$-generated crowded $Q$-point on $\mathbb{Q}$

If the set $X = [\omega]^{<\omega} \setminus \{\emptyset\}$ has the discrete topology then the product space $\mathfrak{X} = X^\omega$ is a Polish space and the sets $U_{\langle n,a\rangle} = \{x \in \mathfrak{X} \colon x(n) = a\}$, where $a \in [\omega]^{<\omega}$ and $n < \omega$, constitutes a subbasis for the product topology. Consider the set

$$\mathcal{P} = \{x \in \mathfrak{X} \colon \ \{x(k) \colon k < \omega\} \text{ is a partition of } \omega\}.$$

It is important to know that

- $\mathcal{P}$ is a $G_\delta$ subset of $\mathfrak{X}$. Therefore, $\mathcal{P}$ is a Polish space with the relative topology inherited from $\mathfrak{X}$.

**Lemma 1** *Let $P$ be a prism in $\mathcal{P}$ and let $\{A_n : n < \omega\} \subset [\mathbb{Q}]^\omega$ be arbitrary. Then, there exist a subprism $Q$ of $P$ and $B \in [\mathbb{Q}]^\omega$ such that $|B \cap A_n| = \omega$ for every $n < \omega$ and $|x(k) \cap B| \leq 1$ for every $x \in Q$ and $k < \omega$. Moreover, if $P$ is a cube then, $Q$ is a cube as well.*

Proof. Since $|\mathbb{Q}| = \omega$ we can suppose that $\{A_n : n < \omega\} \subset [\omega]^\omega$. Let $\langle R_n : n < \omega \rangle$ be an enumeration of $\{A_n : n < \omega\}$ where each set appears infinitely often.

Case (a): If $P = \{z\}$ then, define a sequence $\langle b_n \in \omega : n < \omega \rangle$ inductively such that $b_n \in R_n \setminus \bigcup\{z(k) : k < \omega \;\&\; z(k) \cap \{b_0, \ldots, b_{n-1}\} \neq \emptyset\}$ for every $n < \omega$. It is easy to see that $B = \{b_n : n < \omega\}$ works.

Case (b): If $P \in \mathrm{Perf}(\mathcal{P})$, let $f$ be a witness function for $P$. By our remarks in section 2, we can assume that $f$ acts from $\mathfrak{C}^\alpha$ onto $P$. Thus, $P$ is a cube. It is enough to find its subcube with the desired properties.

Let $\mu$ be the standard product probability measure on $\mathfrak{C}^\alpha$.

We construct, by induction on $n < \omega$, a sequence $\langle K_n : n < \omega \rangle$ of open subsets of $\mathfrak{C}^\alpha$ and two sequences, $\langle b_n \in R_n : n < \omega \rangle$ and $\langle B_n \in [\omega]^{<\omega} : n < \omega \rangle$, such that for every $n < \omega$:

(i) $b_n > \max\left(\{b_i : i < n\} \cup \bigcup_{j<n} B_j\right)$,

(ii) $\mu(K_n) \geq 1 - 2^{-(n+2)}$, and

(iii) $f(h)(k) \subseteq B_n$ for every $h \in K_n$ and $k < \omega$ for which $b_n \in f(h)(k)$.

If this construction is possible put $B = \{b_n : n < \omega\}$. Then, clearly $|B \cap A_n| = \omega$. Condition (ii) implies that $\mu\left(\bigcap_{n<\omega} K_n\right) \geq \frac{1}{2}$. Hence, by Proposition 1, there exists a perfect cube $C \subseteq \bigcap_{n<\omega} K_n$. Then $Q = f[C]$ is a subcube of $P$ and the pair $\langle Q, B \rangle$ is as required. To see this, it is enough to show that $|z(k) \cap B| \leq 1$ for every $z \in Q$ and $k < \omega$. Let $z = f(h)$ for some $h \in C$. By conditions (i) and (iii), for every $b_j \in z(k) = f(h)(k)$ and $n > j$ we have that $b_n \notin z(k)$. Therefore, no two elements of $B$ are in the same $z(k)$ or, in other words, $|z(k) \cap B| \leq 1$ for every $k < \omega$.

Next, we show that the inductive construction is possible. Let $n < \omega$ be such that the appropriate $b_i$, $K_i$, and $B_i$ are already constructed for every $i < n$. We will construct $b_n$, $K_n$, and $B_n$ satisfying (i)–(iii). We pick an $b_n$ as an arbitrary element of $R_n$ satisfying (i). If $L = \{a \in [\omega]^{<\omega} \colon b_n \in a\}$ then, $\left\{f^{-1}\left(U_{\langle m,a\rangle}\right) : \langle m,a\rangle \in \omega \times L\right\}$ is a partition of $\mathfrak{C}^\alpha$ into clopen sets. Thus, we can find a finite set $S \subseteq \omega \times L$ such that $K_n = \bigcup\left\{f^{-1}\left(U_{\langle m,a\rangle}\right) : \langle m,a\rangle \in S\right\}$ satisfies condition (ii). Let $B_n = \bigcup\{a \colon \langle m,a\rangle \in S \text{ for some } m < \omega\}$. Then clearly, $B_n$ is finite. To see that it satisfies (iii) take an $h \in K_n$. Then $f(h) \in U_{\langle m,a\rangle}$ for some $\langle m,a\rangle \in S$. Let $k < \omega$ be such that $b_n \in f(h)(k)$. Since we have also $b_n \in a = f(h)(m)$ we conclude that $k = m$. So, $f(h)(k) = f(h)(m) = a \subseteq B_n$. ■

Fix a $p \in \mathbb{R} \setminus \mathbb{Q}$. For $\mathcal{D} \subset [\mathbb{Q}]^\omega$ let $F(\mathcal{D}) = F(p,\mathcal{D})$ be the filter generated by the family $\mathcal{D} \cup \{I_n \colon n < \omega\}$, where $I_n = [p - 2^{-n}, p + 2^{-n}] \cap \mathbb{Q}$.

**Lemma 2** (K.Ciesielski, J.Pawlikowski [1, lemma 4.23]) *Let $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ be a countable family such that $F(\mathcal{D})$ is crowded. Then, for every prism $P$ in $[\mathbb{Q}]^\omega$ there exists a subprism $Q$ of $P$ and a $Z \in \mathrm{Perf}(\mathbb{Q})$ such that $F(\mathcal{D} \cup \{Z\})$ is crowded and either*

*(i) $Z \cap x = \emptyset$ for every $x \in Q$, or else*

*(ii) $Z \subset x$ for every $x \in Q$.*

We will need also the following easy fact.

**Lemma 3** (K.Ciesielski, J.Pawlikowski [1, Fact 4.21]) *Every non-scattered set $B \subset \mathbb{Q}$ contains a subset from* $\mathrm{Perf}(\mathbb{Q})$.

**Lemma 4** *Let $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ be a countable family such that $F(\mathcal{D})$ is crowded and let $P$ be prism in $\mathcal{P}$ then there exists a subprism $Q$ of $P$ and $Z \in \mathrm{Perf}(\mathbb{Q})$ such that $F(\mathcal{D} \cup \{Z\})$ is crowded and $|Z \cap x(k)| \leq 1$ for every $x \in Q$.*

PROOF. Observe that since $F(\mathcal{D})$ is crowded it is possible to find a sequence $\langle D_n \in \mathrm{Perf}(\mathbb{Q}) \colon n < \omega\rangle$ coinitial in $F(\mathcal{D})$ such that $D_{n+1} \subset D_n \subset I_n$ for every $n < \omega$. Note that

- there are sequences $\langle J_k \colon k < \omega\rangle$ of pairwise disjoint intervals in $\mathbb{Q}$ and $\langle S_k \subset J_k \colon k < \omega\rangle$ of perfect subsets of $\mathbb{Q}$ such that if $S = \bigcup_{k<\omega} S_k$ then for every $D \in F(\mathcal{D})$ there exists an $n < \omega$ such that $S \cap I_n \subset D$.

To see it, define two sequences $\langle n_k\colon k<\omega\rangle$ and $\langle S_k \in \mathrm{Perf}(\mathbb{Q})\colon k<\omega\rangle$ such that $S_k \subset D_k \cap I_{n_k} \cap J_k$ where $J_k$ is a clopen interval such that $p \notin \mathrm{cl}_{\mathbb{R}}(J_k)$. If $n_k$ and $S_k$ are already defined pick $n_{k+1} > n_k$ with $J_k \cap I_{n_{k+1}} = \emptyset$. Since $D_{k+1} \cap I_{n_{k+1}} \in F(\mathcal{D})$ and $F(\mathcal{D})$ is crowded we can find a clopen interval $J_{k+1}$ such that $p \notin \mathrm{cl}_{\mathbb{R}}(J_{k+1})$ and $J_{k+1} \cap D_{k+1} \cap I_{n_{k+1}} \neq \emptyset$. Define $S_{k+1} = J_{k+1} \cap D_{k+1} \cap I_{n_{k+1}}$. Then, $S_{k+1} \in \mathrm{Perf}(\mathbb{Q})$ and $S_{k+1} \subset D_{k+1} \cap I_{n_{k+1}}$. Now, put $S = \bigcup_{k<\omega} S_k$. Then, $S \in \mathrm{Perf}(\mathbb{Q})$ and $S \cap I_{n_k} = \bigcup_{i\geq k} S_i \cap I_{n_k} = \bigcup_{i\geq k} S_i \subset D_k$. This proves our claim.

Let $\mathcal{B}$ be a countable basis for the topology on $\mathbb{Q}$ consisting of clopen sets and consider the family $\mathcal{B}_0 = \{B \in \mathcal{B}\colon |B \cap S| = \omega\}$.

If $P \in \mathrm{Perf}(\mathcal{P})$ apply Lemma 1 to $P$ and the family $\{B \cap S\colon B \in \mathcal{B}_0\}$ to find a set $T \in [S]^\omega$ and a subprism $Q$ of $P$ such that

(a) $|T \cap (B \cap S)| = \omega$ for every $B \in \mathcal{B}_0$ and

(b) $|T \cap x(k)| \leq 1$ for every $x \in Q$ and $k \in \omega$.

If $P = \{x\}$ is a singleton we put $Q = P$ and apply Lemma 1 to the family $\{B \cap S\colon B \in \mathcal{B}_0\}$ and to $x$ to obtain a $T$ satisfying (a) and (b).

In both cases we obtain from (a) that $T$ is dense in $S$. Since $S_k \in \mathrm{Perf}(\mathbb{Q})$ for every $n<\omega$ we conclude that $T \cap S_k$ is non-scattered and contains a subset $Z_k$ from $\mathrm{Perf}(\mathbb{Q})$ for every $k<\omega$. Hence, if we put $Z = \bigcup_{k<\omega} Z_k$ then, $Z \in \mathrm{Perf}(\mathbb{Q})$, $Z \cap I_k \subset D_k$ for every $k<\omega$ and $|Z \cap x(k)| \leq 1$ for every $x \in Q$ and every $k<\omega$. To see that $F(\mathcal{D} \cup \{Z\})$ is crowded note that $Z \cap D_{n_k} \subset S \cap I_{n_k} \subset D_k$ for every $k<\omega$. ■

**Theorem 3** $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists an $\omega_1$-generated crowded $Q$-point on $\mathbb{Q}$.*

Proof. For $\mathcal{Y} = [\mathbb{Q}]^\omega \cup \mathcal{P}$ consider the topology $\tau$ on $\mathcal{Y}$ whose open sets are those $U \subset \mathcal{Y}$ such that $U \cap [\mathbb{Q}]^\omega$ and $U \cap \mathcal{P}$ are open in $[\mathbb{Q}]^\omega$ and $\mathcal{P}$ respectively. Then $\langle \mathcal{Y}, \tau\rangle$ is a Polish space. Note that $[\mathbb{Q}]^\omega$ and $\mathcal{P}$ are clopen in $\mathcal{Y}$ with this topology. Every prism $P \in \mathrm{Perf}(\mathcal{Y})$ must intersect either $[\mathbb{Q}]^\omega$ or $\mathcal{P}$. Since every non-empty clopen set in a prism is its subprism (see [2], or use Proposition 1) we can suppose without any loss of generality that either $P \in \mathrm{Perf}([\mathbb{Q}]^\omega)$ or $P \in \mathrm{Perf}(\mathcal{P})$. Of course, every singleton is in either $[\mathbb{Q}]^\omega$ or $\mathcal{P}$. Therefore, given a prism $P$ in $\mathcal{Y}$ and a countable family $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ such that $F(\mathcal{D})$ is crowded we denote by $Z(\mathcal{D}, P) \in \mathrm{Perf}(\mathbb{Q})$ and a subprism

$Q(\mathcal{D}, P)$ of $P$ as in Lemma 4 provided $P \subset [\mathbb{Q}]^\omega$ and as in Lemma 2 for $P \subset \mathcal{P}$ respectively.

Consider the following strategy $S$ for Player II:

$$S(\langle\langle P_{\eta,}, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(Z(\{Z_\eta \colon \eta < \xi\}), P_\xi),$$

where sets $Z_\eta$ are defined inductively by $Z_\eta = Z(\{Z_\zeta \colon \zeta < \eta\}, P_\eta)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ strategy $S$ is not a winning strategy for Player II. Hence, there is a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to $S$ for which Player II loses so, $\mathcal{Y} = \bigcup_{\xi<\omega_1} Q_\xi$.

Let $\mathcal{U} = F(\{Z_\xi \colon \xi < \omega_1\})$. To see it is an ultrafilter note that if $x \in [\mathbb{Q}]^\omega$ then there exists a $\xi < \omega_1$ such that $x \in Q_\xi$. But then, either $Z_\xi \subset x$ or $Z_\xi \cap x = \emptyset$. Therefore either $x$ or its complement is in $\mathcal{U}$. This proves that $\mathcal{U}$ is an ultrafilter and that $\langle Z_\xi \colon \xi < \omega_1\rangle \subset \mathrm{Perf}(\mathbb{Q})$ is basis for $\mathcal{U}$. So, $\mathcal{U}$ is crowded. Since, no crowded ultrafilter can be principal it follows that $\mathcal{U}$ is also non-principal. To see that $\mathcal{U}$ is a $Q$-point, pick an $x \in \mathcal{P}$. Then, there exists a $\xi < \omega_1$ such that $x \in Q_\xi$. Thus, $Z_\xi \in \mathcal{U}$ and $|Z_\xi \cap x(k)| \leq 1$ for every $k < \omega$. ■

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