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ON SOME PROPERTIES OF SETS BLOCKING ALMOST CONTINUOUS FUNCTIONS

Abstract

We define operators $\mathcal{E}(\,\cdot\,)$ and $\mathcal{N}(\,\cdot\,)$ for blocking sets for almost continuous functions. Using language of these operators we give new proofs for some classical theorems and prove some new theorems. Finally, we make some remarks regarding uniform limits of almost continuous functions.

1   Preliminaries

Let $\mathbb{I} = [0,1]$. We will consider the AC subclass of Darboux real functions defined on the interval $\mathbb{I}$ (a function $f\colon A \to \mathbb{R}$ is AC (almost continuous) if whenever $U \subset A \times \mathbb{R}$ is an open set containing the graph of f, then U contains the graph of a continuous function $g\colon A \to \mathbb{R}$). For properties of this and other Darboux-like classes of functions see e. g. the survey [GN]. In particular it is known, that if $f\colon \mathbb{I} \to \mathbb{R} \in$ AC then $f \restriction J$ is almost continuous as a function from J for every interval $J \subset \mathbb{I}$. It is also known, that for every finite set $F \subset \mathbb{I}$ and open neighbourhood G of f there exists continuous function $g \subset G$ such that $g \restriction F = f \restriction F$.

For every set $A \subset \mathbb{I} \times \mathbb{R}$ and $x \in \mathbb{I}$, by A_x we will denote $\{y \in \mathbb{R} \mid \langle x, y \rangle \in A\}$. By $\mathrm{bd}(A)$ we will denote border of A. We will say that set A is left-open (right-open) iff for every $\langle x, y \rangle \in A$ there exists open neighbourhood U of $\langle x, y \rangle$ such that $([0,x] \times \mathbb{R}) \cap U \subset A$ $(([x,1] \times \mathbb{R}) \cap U \subset A)$.

For an $a \in \mathbb{R}$ and a set $A \subset \mathbb{R}$ by $|a - A|$ we will denote distance between a and A.

We will use the symbol $\overline{A}$ for denoting the topological closure of set A. We will also use the symbol $\overline{\mathcal{F}}$ for denoting family of all limits of uniformly convergent sequences of functions from $\mathcal{F} \subset \mathbb{R}^{\mathbb{I}}$.

A closed set $B \subset \mathbb{I} \times \mathbb{R}$ is blocking if $f \cap B = \emptyset$ for at least one function $f \colon \mathbb{I} \to \mathbb{R}$ and $g \cap B \neq \emptyset$ for every continuous function $g \colon \mathbb{I} \to \mathbb{R}$. Obviously, f is almost continuous iff $f \cap B \neq \emptyset$ for every blocking set B (see e. g. [KG]).

2   Properties of blocking sets

There are known examples of sets blocking almost continuous functions which have some pathological properties (see e. g. [JHR]). To investigate general properties of such sets we define two operators.

Definition 1. *Let $B \subset \mathbb{I} \times \mathbb{R}$ be a blocking set. Define:*

- $\mathcal{E}(B) = \{\langle a, b \rangle \in \mathbb{I} \times \mathbb{R} \mid (\exists h \colon [0, a] \to \mathbb{R}) \ (h(a) = b \ \& \ h \cap B = \emptyset \ \& \ h$ is continuous$)\}$;

- $\mathcal{N}(B) = (\mathbb{I} \times \mathbb{R}) \backslash (B \cup \mathcal{E}(B))$.

Note that $B, \mathcal{E}(B), \mathcal{N}(B)$ are pairwise disjoint and $B \cup \mathcal{E}(B) \cup \mathcal{N}(B) = \mathbb{I} \times \mathbb{R}$. Thus every blocking set B divides the plane into three parts B, $\mathcal{E}(B)$ and $\mathcal{N}(B)$—we will show below that these parts have some nice properties.

Definition 2. *Let B be a blocking set. Define also:*

- $\mathrm{A_{EE}}(B) = \{\langle a, b \rangle \in \mathbb{I} \times \mathbb{R} \mid$ there exists an open set G such that $\langle a, b \rangle \in G \ \& \ G \subset \mathcal{E}(B)\}$;

- $\mathrm{A_{NN}}(B) = \{\langle a, b \rangle \in \mathbb{I} \times \mathbb{R} \mid$ there exists an open set G such that $\langle a, b \rangle \in G \ \& \ G \subset \mathcal{N}(B)\}$;

- $\mathrm{A_{NE}}(B) = \{\langle a, b \rangle \in \mathbb{I} \times \mathbb{R} \mid$ there exists an open set G such that $\langle a, b \rangle \in G \ \& \ ([0, a] \times \mathbb{R}) \cap G \subset \mathcal{N}(B) \ \& \ ((a, 1] \times \mathbb{R}) \cap G \subset \mathcal{E}(B)\}$.

Theorem 1. *Let $B \subset \mathbb{I} \times \mathbb{R}$ be a blocking set. Then:*

1. $\{0\} \times \mathbb{R} \subset B \cup \mathcal{E}(B)$;

2. $\{1\} \times \mathbb{R} \subset B \cup \mathcal{N}(B)$;

3. $\mathcal{E}(B)$ *is open;*

4. $\mathcal{N}(B)$ *is left-open;*

5. *if for every open neighbourhood G of $\langle x, y \rangle$ there exist $\langle x_1, y_1 \rangle \in G \cap \mathcal{E}(B)$, $\langle x_2, y_2 \rangle \in G \cap \mathcal{N}(B)$ such that $x_1 < x_2$, then $\langle x, y \rangle \in B$;*

6. *if $\langle x, y \rangle \notin B$ then $\langle x, y \rangle \in \mathrm{A_{EE}}(B) \cup \mathrm{A_{NN}}(B) \cup \mathrm{A_{NE}}(B)$;*

7. *if $f \colon \mathbb{I} \to \mathbb{R}$, $f \in \mathrm{AC}$, $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B)$, $\langle x_2, f(x_2) \rangle \in \mathcal{N}(B)$ and $x_1 < x_2$, then there exists $x \in (x_1, x_2)$ such that $\langle x, f(x) \rangle \in B$.[1]*

Proof. (1), (2) These facts are easy consequences of definitions.

(3) Let $\langle p, q \rangle \in \mathcal{E}(B)$. Since $\langle p, q \rangle \notin B$, there exists a rectangular open neighbourhood G of $\langle p, q \rangle$ such that $G \cap B = \emptyset$. Let $h \colon [0, p] \to \mathbb{R}$ be continuous, $h \cap B = \emptyset$ and $h(p) = q$. There exists $\langle s, t \rangle \in \mathrm{bd}(G)$ such that $s < p$ and $h(s) = t$. For every point $\langle a, b \rangle$ from $((s, 1] \times \mathbb{R}) \cap G$ we can extend h by linear segment $[\langle s, t \rangle, \langle a, b \rangle] \subset G$ to the continuous function $h' \colon [0, a] \to \mathbb{R}$. Hence $\langle a, b \rangle \in \mathcal{E}(N)$, so $((s, 1] \times \mathbb{R}) \cap G \subset \mathcal{E}(B)$.

(4) Suppose $\langle p, q \rangle \in \mathcal{N}(B)$. Since $\langle p, q \rangle \notin B$, there exists a rectangular open neighbourhood G of $\langle p, q \rangle$ such that $G \cap B = \emptyset$. If $([0, p] \times \mathbb{R}) \cap G \not\subset \mathcal{N}(B)$ then there exists $\langle s, t \rangle \in G \cap \mathcal{E}(B)$ such that $s \leq p$. Because $\mathcal{E}(B)$ is open, we can assume that $s < p$. Let $h \colon [0, s] \to \mathbb{R}$ be a continuous function such that $h \cap B = \emptyset$ and $h(s) = t$. We can extend h by the segment $[\langle s, t \rangle, \langle p, q \rangle] \subset G$ to the continuous function $h' \colon [0, p] \to \mathbb{R}$ such that $h' \cap B = \emptyset$, contrary to $\langle p, q \rangle \in \mathcal{N}(B)$.

(5) Suppose $\langle x, y \rangle \notin B$. Then there exists a rectangular open neighbourhood G of $\langle x, y \rangle$ such that $G \cap B = \emptyset$. Take $\langle x_1, y_1 \rangle \in G \cap \mathcal{E}(B)$, $\langle x_2, y_2 \rangle \in G \cap \mathcal{N}(B)$, $x_1 < x_2$. There exists continuous $h \colon [0, x_1] \to \mathbb{R}$ such that $h(x_1) = y_1$ and $h \cap B = \emptyset$. If we define $h' \colon [0, x_2] \to \mathbb{R}$ as extension of h by linear segment $[\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle] \subset G$ then h' is continuous. Since $\langle x_2, y_2 \rangle \in \mathcal{N}(B)$, then $h' \cap B \neq \emptyset$, so $[\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle] \cap B \neq \emptyset$ and $G \cap B \neq \emptyset$. This is the contradiction.

(6) Suppose $\langle x, y \rangle \notin B$. Then either $\langle x, y \rangle \in \mathcal{E}(B)$ or $\langle x, y \rangle \in \mathcal{N}(B)$. If $\langle x, y \rangle \in \mathcal{E}(B)$ then, since $\mathcal{E}(B)$ is open, $\langle x, y \rangle \in \mathrm{A_{EE}}(B)$. If $\langle x, y \rangle \in \mathcal{N}(B)$ then, since $\mathcal{N}(B)$ is left-open, there exists a rectangular open neighbourhood G of $\langle x, y \rangle$ such that $([0, x] \times \mathbb{R}) \cap G \subset \mathcal{N}(B)$ and $G \cap B = \emptyset$. Now suppose $\langle x, y \rangle \notin \mathrm{A_{NN}}(B) \cup \mathrm{A_{NE}}(B)$. This means that in every open neighbourhood of $\langle x, y \rangle$ we can find points from $\mathcal{E}(B)$ and $\mathcal{N}(B)$ to the right of x. Take $\langle x_2, y_2 \rangle \in G \cap \mathcal{N}(B)$, $x_2 > x$ and $\langle x_1, y_1 \rangle \in ([0, x_2) \times \mathbb{R}) \cap G \cap \mathcal{E}(B)$. Now, analogously to (4), the linear segment $[\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle]$ intersects B, contrary to $G \cap B = \emptyset$.

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[1] As an easy corollary we have that $f \in \mathrm{AC}$ iff for every blocking set B and $\langle x_1, y_1 \rangle \in \mathcal{E}(B)$, $\langle x_2, y_2 \rangle \in \mathcal{N}(B)$ such that $x_1 < x_2$ there exists $x \in (x_1, x_2)$ such that $\langle x, f(x) \rangle \in B$.

(7) Let $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B)$, $\langle x_2, f(x_2) \rangle \in \mathcal{N}(B)$, $x_1 < x_2$. Let $h \colon [0, x_1] \to \mathbb{R}$ be a continuous function such that $h \cap B = \emptyset$ and $h(x_1) = f(x_1)$. Suppose that for every $x \in (x_1, x_2)$, $\langle x, f(x) \rangle \notin B$. So, for every $x \in [x_1, x_2]$, there exists open neighbourhood $G_{\langle x, f(x) \rangle}$ of $\langle x, f(x) \rangle$ such that $G_{\langle x, f(x) \rangle} \cap B = \emptyset$. Let $G = \bigcup_{x \in [x_1, x_2]} G_{\langle x, f(x) \rangle}$. Since $f \restriction [x_1, x_2]$ is almost continuous, there exists continuous function $g \colon [x_1, x_2] \to \mathbb{R}$ such that $g(x_1) = f(x_1)$, $g(x_2) = f(x_2)$ and $g \subset G$. Since $G \cap B = \emptyset$, we can extend $h \colon [0, x_1] \to \mathbb{R}$ by g to the continuous function $h' \colon [0, x_2] \to \mathbb{R}$ such that $h'(x_2) = f(x_2)$ and $h' \cap B = \emptyset$. So $\langle x_2, f(x_2) \rangle$ belongs to $\mathcal{E}(B)$ rather than $\mathcal{N}(B)$. This is a contradiction. $\quad\square$

Note that analogous facts remain true for functions from $\mathbb{I}$ to $\mathbb{I}$.

Remark 1. *Let $B \subset \mathbb{I} \times \mathbb{R}$ be a blocking set. Suppose $\langle x, y_1 \rangle \in \mathcal{E}(B)$ and $\langle x, y_2 \rangle \in \mathcal{N}(B)$. Then there exists $y \in (y_1, y_2)$ such that $\langle x, y \rangle \in B$.*

$\square$

We will use Theorem 2 in the next part of the proof to analyze uniform limits of sequences of almost continuous functions.

Theorem 2. *Let $f \colon \mathbb{I} \to \mathbb{R}$ be almost continuous and $B \subset \mathbb{I} \times \mathbb{R}$ be a blocking set. For every $a, b \in \mathbb{I}$, if $a < b$, $\langle a, f(a) \rangle \in \mathcal{E}(B)$ and $\langle b, f(b) \rangle \in \mathcal{N}(B)$ then at least one of the following statements holds:*

1. *there exists $x \in [a, b]$ and left or right-open neighbourhood U of $\langle x, f(x) \rangle$ such that $f \cap U \subset B$;*

2. *there exists $x \in [a, b]$ such that for every open neighbourhood V of $\langle x, f(x) \rangle$ there exists $x_1 < x_2$ such that $\langle x_1, f(x_1) \rangle \in V \cap \mathcal{E}(B)$ and $\langle x_2, f(x_2) \rangle \in V \cap \mathcal{N}(B)$.*

(In the second case we will say that f breaks through B at $\langle x, f(x) \rangle$.)

Proof. Suppose, contrary to our claim, that neither (1) nor (2) hold. Denote:

- $B_{EE} = \{\langle x, f(x) \rangle \mid$ there exists an open set $V_{\langle x, f(x) \rangle}$ such that $\langle x, f(x) \rangle \in V_{\langle x, f(x) \rangle}$ & $V_{\langle x, f(x) \rangle} \cap f \subset \mathcal{E}(B) \cup B\}$;

- $B_{NN} = \{\langle x, f(x) \rangle \mid$ there exists an open set $V_{\langle x, f(x) \rangle}$ such that $\langle x, f(x) \rangle \in V_{\langle x, f(x) \rangle}$ & $V_{\langle x, f(x) \rangle} \cap f \subset \mathcal{N}(B) \cup B\}$;

- $B_{NE} = \{\langle x, f(x) \rangle \mid$ there exists an open set $V_{\langle x, f(x) \rangle}$ such that $\langle x, f(x) \rangle \in V_{\langle x, f(x) \rangle}$ & $([0, x] \times \mathbb{R}) \cap V_{\langle x, f(x) \rangle} \cap f \subset \mathcal{N}(B) \cup B$ & $((x, 1] \times \mathbb{R}) \cap V_{\langle x, f(x) \rangle} \cap f \subset \mathcal{E}(B) \cup B\}$.

First, note that sets $B_{EE} \cap ((a,b) \times \mathbb{R})$, $B_{NN} \cap ((a,b) \times \mathbb{R})$ and $B_{NE} \cap ((a,b) \times \mathbb{R})$ are pairwise disjoint. Indeed, suppose there exists $x \in (a,b)$ such that $\langle x, f(x) \rangle \in B_{EE} \cap B_{NE}$. So, there exists an open neighbourhood $V_{\langle x, f(x) \rangle}$ of $\langle x, f(x) \rangle$ such that $([a,x] \times \mathbb{R}) \cap V_{\langle x, f(x) \rangle} \cap f \subset (\mathcal{E}(B) \cup B) \cap (\mathcal{N}(B) \cup B) = B$, contrary to the negation of (1). Analogously we prove that $B_{NN} \cap B_{NE} = \emptyset$ and $B_{EE} \cap B_{NN} = \emptyset$.

Next, note that $f \upharpoonright [a,b] \subset B_{EE} \cup B_{NN} \cup B_{NE}$. Indeed, it is easy to see that $\langle a, f(a) \rangle \in B_{EE}$ and $\langle b, f(b) \rangle \in B_{NN} \cup B_{NE}$. Now, take $x \in (a,b)$. There exists an open neighbourhood U of $\langle x, f(x) \rangle$ such that either $([a,x] \times \mathbb{R}) \cap U \cap f \subset \mathcal{E}(B) \cup B$ or $([a,x] \times \mathbb{R}) \cap U \cap f \subset \mathcal{N}(B) \cup B$. (If it is not the case, since $([a,x] \times \mathbb{R}) \cap U \cap f \not\subset \mathcal{N}(B) \cup B$, we can find $x_1 < x$ such that point $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B) \cap U$. Next, since $((x_1, x) \times \mathbb{R}) \cap U \cap f \not\subset \mathcal{E}(B) \cup B$, we find a point $\langle x_2, f(x_2) \rangle \in ((x_1, x) \times \mathbb{R}) \cap \mathcal{N}(B) \cap U$. We can find such points for arbitrary set U—contrary to the negation of (2).) Using the same arguments we show that there exists an open neighbourhood V of $\langle x, f(x) \rangle$ such that either $((x,b) \times \mathbb{R}) \cap V \cap f \subset \mathcal{E}(B) \cup B$ or $((x,b) \times \mathbb{R}) \cap V \cap f \subset \mathcal{N}(B) \cup B$. Let $G = U \cap V$. Since f is bilaterally dense in itself, the case $([a,x] \times \mathbb{R}) \cap G \cap f \subset \mathcal{E}(B) \cup B$ and $((x,b) \times \mathbb{R}) \cap G \cap f \subset \mathcal{N}(B) \cup B$ contradicts to the negation of (2). In any other case there exists an open neighbourhood $V_{\langle x, f(x) \rangle}$ of $\langle x, f(x) \rangle$ as in the definition of B_{EE} or B_{NN} or B_{NE}. It is equivalent to $\langle x, f(x) \rangle \in B_{EE}$ or $\langle x, f(x) \rangle \in B_{NN}$ or $\langle x, f(x) \rangle \in B_{NE}$.

For $S_{\langle x,y \rangle}$ being an open square with the center $\langle x,y \rangle$ let $3 \cdot S_{\langle x,y \rangle}$ denote the open square with the center $\langle x,y \rangle$ and with the diagonal 3 times that of $S_{\langle x,y \rangle}$. For every $x \in [a,b]$ let $S_{\langle x, f(x) \rangle}$ be an open square with the center $\langle x, f(x) \rangle$ such that:

- $3 \cdot S_{\langle x, f(x) \rangle} \subset V_{\langle x, f(x) \rangle}$;

- $3 \cdot S_{\langle x, f(x) \rangle} \subset (a,b] \times \mathbb{R}$, for $x > a$;

- $3 \cdot S_{\langle x, f(x) \rangle} \subset [a,b) \times \mathbb{R}$, for $x < b$.

Note that for $x > a$ the square $S_{\langle x, f(x) \rangle}$ does not contain points with abscissa a. Respectively, the square $S_{\langle x, f(x) \rangle}$ does not contain points with abscissa b for $x < b$. Note also, that if $S_{\langle a,b \rangle} \cap S_{\langle c,d \rangle} \neq \emptyset$ then either $S_{\langle a,b \rangle} \subset 3 \cdot S_{\langle c,d \rangle}$ or $S_{\langle c,d \rangle} \subset 3 \cdot S_{\langle a,b \rangle}$.

For every $x \in [a,b]$ let $R_{\langle x, f(x) \rangle} = (x_l, x_r) \times (y_l, y_u)$ be an open rectangular neighbourhood of $\langle x,y \rangle$ such that $R_{\langle x, f(x) \rangle} \subset S_{\langle x, f(x) \rangle}$, $f(x_l) \in (y_l, y_u)$ for $x > a$ and $f(x_r) \in (y_l, y_u)$ for $x < b$.

Then for every $x \in [a,b]$ the set $R_{\langle x, f(x) \rangle}$ fulfills the following conditions:

$(\circ_1)$ if $\langle r, f(r) \rangle \in V_{\langle x, f(x) \rangle} \cap \mathcal{E}(B)$ for an $r \le x$ then $\langle x, f(x) \rangle \in B_{EE}$ and for every $t > x_l$ there exists $z < t$ such that $\langle z, f(z) \rangle \in R_{\langle x, f(x) \rangle} \cap \mathcal{E}(B)$;

($\circ_2$) if $\langle r, f(r)\rangle \in V_{\langle x, f(x)\rangle} \cap \mathcal{E}(B)$ for an $r > x$ then $\langle x, f(x)\rangle \in B_{EE} \cup B_{NE}$ and for every $t > x$ there exists $z \in (x, t)$ such that $\langle z, f(z)\rangle \in R_{\langle x, f(x)\rangle} \cap \mathcal{E}(B)$;

($\bullet_1$) if $R_{\langle x, f(x)\rangle} \cap ((r, x) \times \mathbb{R}) \cap f \subset \mathcal{E}(B) \cup B$ for an $r \in (x_l, x)$ then $\langle x, f(x)\rangle \in B_{EE}$ and there exists $z \le r$ such that $\langle z, f(z)\rangle \in R_{\langle x, f(x)\rangle} \cap \mathcal{E}(B)$;

($\bullet_2$) if $R_{\langle x, f(x)\rangle} \cap ((r, x_r) \times \mathbb{R}) \cap f \subset \mathcal{E}(B) \cup B$ for an $r \in (x, x_r)$ then $\langle x, f(x)\rangle \in B_{EE} \cup B_{NE}$ and there exists $z \le r$ such that $\langle z, f(z)\rangle \in R_{\langle x, f(x)\rangle} \cap \mathcal{E}(B)$.

Let $H = \bigcup_{x \in [a, b]} R_{\langle x, f(x)\rangle}$. H is open and $f \subset H$, so there exists a continuous function $g: [a, b] \to \mathbb{R}$ such that $g \subset H$. Let $\mathcal{R}$ be a finite subfamily of $\{R_{\langle x, f(x)\rangle} \mid x \in [a, b]\}$ such that $g \subset \bigcup \mathcal{R}$.

Since only the $R_{\langle a, f(a)\rangle}$ contains points with abscissa a and only the $R_{\langle b, f(b)\rangle}$ contains points with abscissa b, $R_{\langle a, f(a)\rangle} \in \mathcal{R}$ and $R_{\langle b, f(b)\rangle} \in \mathcal{R}$. Moreover, since $\mathcal{R}$ is finite and $3 \cdot S_{\langle x, f(x)\rangle} \subset ([a, b) \times \mathbb{R})$ for every $x < b$,

$$\sup\left\{x \in [a, b] \mid \langle x, y\rangle \in \bigcup(\mathcal{R} \setminus \{R_{\langle b, f(b)\rangle}\})\right\} < b. \tag{$\star$}$$

Let $C = \{x \in [a, b] \mid (\exists R \in \mathcal{R}) \ (\langle x, g(x)\rangle \in R \ \& \ (\exists x_1 \le x) \ \langle x_1, f(x_1)\rangle \in \mathcal{E}(B) \cap R)\}$, and let $s = \sup C$. Since $\langle a, f(a)\rangle \in \mathcal{E}(B)$ and $\langle a, g(a)\rangle \in R_{\langle a, f(a)\rangle}$, there exists $x > a$, $x_1 \in (a, x)$ such that $\langle x, g(x)\rangle \in R_{\langle a, f(a)\rangle}$, $\langle x_1, f(x_1)\rangle \in \mathcal{E}(B) \cap R_{\langle a, f(a)\rangle}$, so $s \ge x > a$. Analogously, the condition ($\star$) implies $s < b$.

Since $\mathcal{R}$ is finite and g is continuous, so there exists $R_{\langle p, f(p)\rangle} \in \mathcal{R}$ and $p_1 \le s$ such that $\langle s, g(s)\rangle \in \overline{R_{\langle p, f(p)\rangle}}$ and $\langle p_1, f(p_1)\rangle \in \mathcal{E}(B) \cap R_{\langle p, f(p)\rangle}$.

Let $R_{\langle q, f(q)\rangle} \in \mathcal{R}$ be an open rectangle such that $\langle s, g(s)\rangle \in R_{\langle q, f(q)\rangle}$. Since $\langle s, g(s)\rangle \in \overline{R_{\langle p, f(p)\rangle}} \cap R_{\langle q, f(q)\rangle}$, $R_{\langle p, f(p)\rangle} \cap R_{\langle q, f(q)\rangle} \ne \emptyset$.

We have two cases:

1. $R_{\langle p, f(p)\rangle} \subset 3 \cdot S_{\langle q, f(q)\rangle} \subset V_{\langle q, f(q)\rangle}$, if diameter of $S_{\langle p, f(p)\rangle}$ is less than diameter of $S_{\langle q, f(q)\rangle}$;

2. $R_{\langle q, f(q)\rangle} \subset 3 \cdot S_{\langle p, f(p)\rangle} \subset V_{\langle p, f(p)\rangle}$, otherwise.

Case 1. Then $\langle p_1, f(p_1)\rangle \in \mathcal{E}(B) \cap V_{\langle q, f(q)\rangle}$. There exists $q_1 \le s$ such that $\langle q_1, f(q_1)\rangle \in R_{\langle q, f(q)\rangle} \cap \mathcal{E}(B)$. Indeed, if $p_1 \le q$ then $\langle p_1, f(p_1)\rangle \in ([a, q] \times \mathbb{R}) \cap V_{\langle q, f(q)\rangle} \cap \mathcal{E}(B)$, so $\langle q, f(q)\rangle \in B_{EE}$ and there exists $q_1 \le s$ such that $\langle q_1, f(q_1)\rangle \in \mathcal{E}(B) \cap R_{\langle q, f(q)\rangle}$ (see the condition ($\circ_1$)). If $p_1 > q$, then $\langle p_1, f(p_1)\rangle \in ((q, b] \times \mathbb{R}) \cap V_{\langle q, f(q)\rangle} \cap \mathcal{E}(B)$ and $s > q$, so $\langle q, f(q)\rangle \in B_{EE} \cup B_{NE}$ and there exists $q_1 \le s$ such that $\langle q_1, f(q_1)\rangle \in \mathcal{E}(B) \cap R_{\langle q, f(q)\rangle}$ (see the condition ($\circ_2$)).

Now, since g is continuous and $s < b$, there exists $s_1 > s$ such that $\langle s_1, g(s_1) \rangle \in R_{\langle q, f(q) \rangle}$, so $s_1 \in C$, a contradiction.

Case 2. There exists $q_1 > s$ such that $\langle q_1, f(q_1) \rangle \in \mathcal{N}(B) \cap R_{\langle q, f(q) \rangle}$. Indeed, suppose by contradiction that no $q_1 > s$ fulfills the claim. Since g is continuous and $s < b$, there exists $s_1 > s$ such that $\langle s_1, g(s_1) \rangle \in R_{\langle q, f(q) \rangle}$ and $s_1 \neq q$. By supposition, $((s_1, b] \times \mathbb{R}) \cap R_{\langle q, f(q) \rangle} \cap \mathcal{N}(B) \cap f = \emptyset$, so $((s_1, b] \times \mathbb{R}) \cap R_{\langle q, f(q) \rangle} \cap f \subset \mathcal{E}(B) \cup B$. If $s_1 < q$ then there exists $z \leq s_1$ such that $\langle z, f(z) \rangle \in \mathcal{E}(B) \cap R_{\langle q, f(q) \rangle}$ (see the condition ($\bullet_1$)). If $s_1 > q$ then there exists $z \leq s_1$ such that $\langle z, f(z) \rangle \in \mathcal{E}(B) \cap R_{\langle q, f(q) \rangle}$ (see the condition ($\bullet_2$)). In any case, $s_1 \in C$. Since this is a contradiction, there exists $q_1 > s$ such that $\langle q_1, f(q_1) \rangle \in \mathcal{N}(B) \cap R_{\langle q, f(q) \rangle}$.

Since $\langle q_1, f(q_1) \rangle \in \mathcal{N}(B) \cap R_{\langle q, f(q) \rangle}$, so $\langle q_1, f(q_1) \rangle \in \mathcal{N}(B) \cap V_{\langle p, f(p) \rangle}$. But $\langle p_1, f(p_1) \rangle \in \mathcal{E}(B) \cap V_{\langle p, f(p) \rangle}$ and $p_1 < q_1$—it is impossible, since $\langle p, f(p) \rangle \in B_{EE} \cup B_{NN} \cup B_{NE}$. $\qquad \square$

3 Some new proofs of old theorems and a proof of the new one

We will show below that some classical theorems can be proved in a shorter way by using the theory of operators $\mathcal{E}(\,\cdot\,)$ and $\mathcal{N}(\,\cdot\,)$.

Theorem 3 ([JB]). *Every* DB_1 *function* $f \colon \mathbb{I} \to \mathbb{R}$ *is almost continuous.*

Proof. Suppose $f \colon \mathbb{I} \to \mathbb{R}$ is DB_1 and there exists blocking set $B \subset \mathbb{I} \times \mathbb{R}$ such that $f \subset \mathcal{E}(B) \cup \mathcal{N}(B)$. Recall that f, as a Darboux function, is bilaterally dense in itself.

Denote:

- $E' = \{x \in \mathbb{I} \mid \langle x, f(x) \rangle \in \mathcal{E}(B)\}$;

- $N' = \{x \in \mathbb{I} \mid \langle x, f(x) \rangle \in \mathcal{N}(B)\}$.

By Theorem 1, we have $E' \cap N' = \emptyset$, $0 \in E'$, $1 \in N'$ and $E' \cup N' = \mathbb{I}$. Let K be the set of all $x \in \mathbb{I}$ such that for every open neighbourhood G of x there exist $x_1, x_2 \in G$ with $x_1 < x_2$ and $x_1 \in E'$, $x_2 \in N'$.

Note that for $x_1 < x_2$, $x_1 \in E'$, $x_2 \in N'$ there exists $x \in [x_1, x_2]$ such that $x \in K$ (e. g. $\sup(E' \cap [x_1, x_2]) \in K$), so $K \neq \emptyset$. It is also easy to see that K is closed. Since f is Baire class 1, there exists $p \in K$ such that $f \upharpoonright K$ is continuous at p. Since $\langle p, f(p) \rangle \notin B$, $\langle p, f(p) \rangle \in A_{NE}(B) \cup A_{NN}(B) \cup A_{EE}(B)$. During the rest of the proof we will suppose $\langle p, f(p) \rangle \in A_{NE}(B)$; the proof in other cases is analogous.

Since $f \restriction K$ is continuous at p, there exists an open neighbourhood G of p such that $[0,p] \cap G \cap K \subset N'$ and $(p,1] \cap G \cap K \subset E'$. We will show that $[0,p] \cap G \subset N'$ and $(p,1] \cap G \subset E'$.

Indeed, suppose there exists $p_1 < p$ such that $p_1 \in G \cap E'$. Then $p_1 \notin K$, so p_1 is contained in a component (a,b) of $\mathbb{I} \setminus K$. Since $b \in K$ and $b \leq p$, $b \in N'$. Since $\langle b, f(b) \rangle \in \mathcal{N}(B)$, $\mathcal{N}(B)$ is left-open and f is bilaterally dense in itself, there exists $b_1 \in (p_1, b)$ such that $b_1 \in N'$. But now there exists $k \in K$ such that $p_1 \leq k \leq b_1$, contrary to $[p_1, b_1] \subset (a,b) \subset \mathbb{I} \setminus K$. Analogous arguments work for the right side of p.

Since $[0,p] \cap G \subset N'$ and $(p,1] \cap G \subset E'$, $p \notin K$. This is a contradiction. $\square$

Theorem 4 ([JB1]). *The Cesáro-Vietoris function $\phi \colon \mathbb{I} \to \mathbb{R}$ defined by:*

$$\phi(x) = \limsup_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n}$$

(where the a_i are given by the unique nonterminating binary expansion of x) is almost continuous.

Proof. Suppose there exists a blocking set $B \subset \mathbb{I} \times \mathbb{I}$, such that $\phi \subset \mathcal{E}(B) \cup \mathcal{N}(B)$.

First, observe that ϕ is Darboux. Let $s = \sup \{ x \mid \langle x, \phi(x) \rangle \in \mathcal{E}(B) \}$. Since $\langle 0, \phi(0) \rangle \in \mathcal{E}(B)$, $\mathcal{E}(B)$ is open and ϕ is bilaterally dense in itself, so $s > 0$ and $\langle s, \phi(s) \rangle \in \mathcal{N}(B)$. By theorem 1 (4) there are a, b and c such that $a < s$, $b < \phi(s) < c$ and $(a,s) \times (b,c) \subset \mathcal{N}(B)$. Let $s' \in (a,s)$ be such that $\langle s', \phi(s') \rangle \in \mathcal{E}(B)$. Take a', b' and c' such that $s' < a' < s$, $b' < \phi(s') < c'$ and $(s', a') \times (b', c') \subset \mathcal{E}(B)$. Then we have the vertical strip $W = [s', a'] \times \mathbb{R}$ such that every vertical line lying in W contain a point of $\mathcal{E}(B)$, a point of $\mathcal{N}(B)$ and therefore a point of B. (See Remark 1.)

The rest of the proof is a modification of Brown's (Vietoris) proof. Let $B_0 = B \cap W$ and $\{W_n\}_n$ be a descending sequence of open neighbourhoods of B_0 such that $B_0 = \bigcap_{n \in \mathbb{N}} W_n$. We will define an ascending sequence $n_1, n_2, \ldots$ of natural numbers and dyadic decimal $0.a_1 a_2 \cdots a_{n_1} a_{n_1+1} \cdots a_{n_2} a_{n_2+1} \cdots$ simultanously.

Take any dyadic rational $\xi_0 = 0.a_1 a_2 \cdots a_i \in (s', a')$ and $\eta_0 > 0$ such that $\langle \xi_0, \eta_0 \rangle \in W_1 \cap W$. Define Q as a square neighbourhood of $\langle \xi_0, \eta_0 \rangle$ which has radius $q < \frac{\eta_0}{2}$ and lies interior to $W_1 \cap W$. Now, take $k > i$ such that if we put $a_{i+1} = a_{i+2} = \cdots = a_k = 0$, then:

1. $\xi_0 < 0.a_1 a_2 \cdots a_k 111 \cdots < \xi_0 + q$; and

2. $M_k < q$, with M_k denoting $\frac{a_1 + a_2 + \cdots a_k}{k}$; and

3. $\frac{1}{k} < q$.

Condition 1 implies that regardless of how $0.a_1a_2\cdots a_k$ is continued it will differ from ξ_0 by less than q. Then define $a_{k+1} = a_{k+2} = \cdots = a_{n_1} = 1$, such that $\langle \xi_1, M_{n_1}\rangle \in Q$, with ξ_1 denoting $0.a_1a_2\cdots a_{n_1}$. (This can be accomplished since $|M_{i+1} - M_i| < \frac{1}{i+1} < q$ for each $i > k$.)

If we have defined $\xi_m = 0.a_1a_2\cdots a_{n_m} \in (s', a')$ such that $\langle \xi_m, M_{n_m}\rangle \in W_m \cap W$, we can repeat the process, starting from the point $\langle \xi_m, \eta_m\rangle$ of $W_{m+1}\cap W$ with $\eta_m > 0$, and define $a_{n_m+1}, a_{n_m+2}, \ldots, a_{n_{m+1}}$ first using consecutive 0's and then using consecutive 1's, so that $\langle \xi_{m+1}, M_{n_{m+1}}\rangle \in W_{m+1}\cap W$.

Define $\xi_\omega = 0.a_1a_2\cdots$. Since consecutive 0's and then consecutive 1's were used in proceeding from a_{n_r} to $a_{n_{r+1}}$ in the above induction, for each i between n_r and n_{r+1} we have $M_i \le \max\{M_{n_r}, M_{n_{r+1}}\}$, so $\phi(\xi_\omega) = \limsup_{i\to\infty} M_i = \limsup_{r\to\infty} M_{n_r}$. Therefore, the point $\langle \xi_\omega, \phi(\xi_\omega)\rangle$ is the limit of some subsequence $\langle \xi_{m_r}, M_{n_{m_r}}\rangle$ of $\langle \xi_m, M_{n_m}\rangle$. Since $\langle \xi_{m_r}, M_{n_{m_r}}\rangle \in W_{m_r}$ and $\bigcap_{r\in\mathbb{N}} W_{m_r} = B_0$, $\langle \xi_\omega, \phi(\xi_\omega)\rangle \in B_0 \subset B$, contrary to $\phi \cap B = \emptyset$. □

The next theorem is connected with a problem: "Given a Darboux function $f\colon \mathbb{I} \to \mathbb{I}$, does there exists almost continuous function $g\colon \mathbb{I} \to \mathbb{I}$ with $g \subset \overline{f}$?" (see [MHM]). This problem has been solved by H. Rosen in more general case of extendable functions (see [HR]). Here we will present an easier and shorter proof for the case of almost continuous function.

Theorem 5. *For every Darboux function $f\colon \mathbb{I} \to \mathbb{I}$ there exists almost continuous function $g\colon \mathbb{I} \to \mathbb{I}$ with $g \subset \overline{f}$.*

Proof. Let:

- $C_-(x_0) = \{y \in \mathbb{I} \mid$ there exists a sequence $\{x_n\}_n \subset [0, x_0], f(x_n) \to y\}$;

- $C_+(x_0) = \{y \in \mathbb{I} \mid$ there exists a sequence $\{x_n\}_n \subset [x_0, 1], f(x_n) \to y\}$.

It is easy to see that $\overline{f} = \bigcup_{x\in\mathbb{I}} (\{x\} \times (C_-(x) \cup C_+(x)))$. It is also known that for every Darboux function f and $x \in \mathbb{I}$, f is bilaterally $\mathfrak{c}$-dense in itself, sets $C_-(x)$ and $C_+(x)$ are closed intervals and by well-known Young theorem, set $\{x \in \mathbb{I} \mid C_-(x) \ne C_+(x)\}$ is countable.

Now we will show that for every blocking set B at least one of the following conditions holds:

1. there exists $x \in \mathbb{I}$ such that $C_-(x) \cap C_+(x) \subset B_x$;

2. the set $\{x \in \mathbb{I} \mid (C_-(x) \cup C_+(x)) \cap B_x \ne \emptyset\}$ has cardinality $\mathfrak{c}$.

Suppose that there exists blocking set B such that neither (1) nor (2) holds. Then for every $x \in \mathbb{I}$ there exists $y \in (C_-(x) \cap C_+(x)) \setminus B_x$ and the set $\{x \in \mathbb{I} \mid \langle x, f(x)\rangle \in B\}$ has cardinality less than $\mathfrak{c}$.

Denote:

- $E' = \{x \in \mathbb{I} \mid \langle x, f(x) \rangle \in \mathcal{E}(B)\}$;

- $N' = \{x \in \mathbb{I} \mid \langle x, f(x) \rangle \in \mathcal{N}(B)\}$;

- $B' = \{x \in \mathbb{I} \mid \langle x, f(x) \rangle \in B\}$.

Sets E', N' and B' are pairwise disjoint, $0 \in E' \cup B'$, $1 \in N' \cup B'$ and $E' \cup N' \cup B' = \mathbb{I}$.

Let K be the set of all $x \in \mathbb{I}$ such that for every open neighbourhood G of x there exists $x_1, x_2 \in G$ with $x_1 < x_2$, $x_1 \in E'$ and $x_2 \in N'$. It is easy to see that K is closed.

Note that $K \subset \{x \in \mathbb{I} \mid (C_-(x) \cup C_+(x)) \cap B_x \neq \emptyset\}$. Indeed, for every open neighbourhood G of $x \in K$, if we take $x_1 < x_2$, $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B)$ and $\langle x_2, f(x_2) \rangle \in \mathcal{N}(B)$, then (by a slight change in the proof of Theorem 1 (5)) there exists $x_3 \in (x_1, x_2)$ and $y_3 \in (f(x_1), f(x_2))$ (we can assume that $f(x_1) < f(x_2)$) such that $\langle x_3, y_3 \rangle \in B$. Having sequences $\{\langle x_1^n, f(x_1^n) \rangle\}_n \subset \mathcal{E}(B)$, $x_1^n \to x$ and $\{\langle x_2^n, f(x_2^n) \rangle\}_n \subset \mathcal{N}(B)$, $x_2^n \to x$, we can build a sequence $\{\langle x_3^n, y_3^n \rangle\}_n \subset B$ such that $|y_3^n - f(x)| \leq \max\{|f(x_1^n) - f(x)|, |f(x_2^n) - f(x)|\}$. Since B is compact and $\{\langle x_3^n, y_3^n \rangle\}_n \subset B$ then there exists a subsequence of $\{\langle x_3^n, y_3^n \rangle\}_n$ which converges to some $\langle x, y \rangle \in B$. Thus $y \in (C_-(x) \cup C_+(x)) \cap B_x$.

Note also that for $x_1 < x_2$, $x_1 \in E'$, $x_2 \in N'$ there exists $x \in [x_1, x_2]$ such that $x \in K$. Indeed, $x = \sup(E' \cap [x_1, x_2]) \in K$. Since f is bilaterally dense in itself, $\langle x, f(x) \rangle \in B \cup \mathcal{N}(B)$. If $x \in B'$, then $(x, x_2] \cap N'$ is dense in x (the cardinality of B' is less then $\mathfrak{c}$). If $x \in N'$ then $[x_1, x) \cap N'$ is dense in x (f is left side dense in itself and $\mathcal{N}(B)$ is left-open). As a corollary we have $K \neq \emptyset$.

Now, we will show that K is dense in itself. Let $x \in K$. There exists $y \in (C_-(x) \cap C_+(x)) \setminus B_x$, so by Theorem 1 (6) $\langle x, y \rangle \in A_{\mathrm{EE}}(B) \cup A_{\mathrm{NN}}(B) \cup A_{\mathrm{NE}}(B)$. The proof for these three cases is analogous, so suppose $\langle x, y \rangle \in A_{\mathrm{NE}}(B)$. Take an open rectangular neighbourhood $(a, b) \times (c, d)$ of $\langle x, y \rangle$ such that $(a, x) \times (c, d) \subset \mathcal{N}(B)$ and $(x, b) \times (c, d) \subset \mathcal{E}(B)$. Since $x \in K$, there exists $x_1 \in (a, b) \cap E'$ and $x_2 \in (a, b) \cap N'$, $x_1 < x_2$. Either $x_1 < x$ or $x_2 > x$.

If $x_1 < x$ then $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B)$. Since $y \in C_-(x)$ and $\langle x, y \rangle \in A_{\mathrm{NE}}(B)$, there exists $x' \in (x_1, x)$ such that $\langle x', f(x') \rangle \in \mathcal{N}(B)$. Now we have $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B)$, $\langle x', f(x') \rangle \in \mathcal{N}(B)$ and $x_1 < x'$, so there exists $x'' \in [x_1, x'] \cap K$.

Analogously, if $x_2 > x$ then we can find $x' \in (x, x_2)$ such that $\langle x', f(x') \rangle \in \mathcal{E}(B)$, so there exists $x'' \in [x', x_2] \cap K$.

So, $K \subset \{x \in \mathbb{I} \mid (C_-(x) \cup C_+(x)) \cap B_x \neq \emptyset\}$. But K is closed and dense in itself, so K has cardinality $\mathfrak{c}$. This is a contradiction with the negation of condition (2).

Now we can construct by transfinite induction a function g such that $g \subset \overline{f}$ and $g \cap B \neq \emptyset$ for every blocking set B. Let $A = \{x \in \mathbb{I} \mid C_-(x) \neq C_+(x)\}$.

(By Young theorem, A is countable.) Let $\{B_\alpha\}_{\alpha<\mathfrak{c}}$ be an ordered family of all blocking sets B such that the set $\{x \in \mathbb{I} \mid (C_-(x) \cup C_+(x)) \cap B_x \neq \emptyset\}$ has cardinality $\mathfrak{c}$. For every $\alpha < \mathfrak{c}$ choose $\langle x_\alpha, y_\alpha \rangle \in B_\alpha$ such that $x_\alpha \in \mathbb{I} \setminus (A \cup \{x_\beta \mid \beta < \alpha\})$ and $y_\alpha \in C_-(x_\alpha) \cap C_+(x_\alpha)$. Define g by

$$g(x) = \begin{cases} y_\alpha & \text{if } x = x_\alpha \text{ for an } \alpha < \mathfrak{c}; \\ y \in C_-(x) \cap C_+(x) & \text{otherwise.} \end{cases}$$

It is easy to see that $g \subset \bigcup_{x \in \mathbb{I}} (\{x\} \times (C_-(x) \cap C_+(x))) \subset \overline{f}$ and g is almost continuous. $\qquad\square$

It is known, that if $f \colon \mathbb{I} \to \mathbb{R}$ is almost continuous, then $\max(f, g)$ is almost continuous for every Darboux upper semicontinuous function $g \colon \mathbb{I} \to \mathbb{R}$ ([AM]). We don't know if the converse statement is true, but we can prove the following fact.

Theorem 6. *Suppose $f \colon \mathbb{I} \to \mathbb{R}$. If $\max(f, c) \in \mathrm{AC}$ and $\min(f, c) \in \mathrm{AC}$ for every real number c, then $f \in \mathrm{AC}$.*

Proof. Suppose, $\max(f, c) \in \mathrm{AC}$ and $\min(f, c) \in \mathrm{AC}$ for every $c \in \mathbb{R}$ and there exists a blocking set B with $B \cap f = \emptyset$. Let $s = \sup \{x \in \mathbb{I} \mid \langle x, f(x) \rangle \in \mathcal{E}(B)\}$. Then either $\langle s, f(s) \rangle \in \mathcal{E}(B)$ or $\langle s, f(s) \rangle \in \mathcal{N}(B)$.

If $\langle s, f(s) \rangle \in \mathcal{E}(B)$ then $s < 1$ and there exists a rectangular right-open neighbourhood $[s, x_1) \times (y_1, y_2) \subset \mathcal{E}(B)$ of $\langle s, f(s) \rangle$. Fix $t \in (s, x_1)$ such that $\langle t, f(t) \rangle \in \mathcal{N}(B)$. Suppose $f(t) < f(s)$ ($f(t) > f(s)$) and consider $g = \min(f, f(s))$ ($g = \max(f, f(s))$). Since $g \in \mathrm{AC}$ and $\langle s, g(s) \rangle = \langle s, f(s) \rangle \in \mathcal{E}(B)$ and $\langle t, g(t) \rangle = \langle t, f(t) \rangle \in \mathcal{N}(B)$, by Theorem 1 (7) there exists $x \in (s, t)$ such that $\langle x, g(x) \rangle \in B$. This is the contradiction, because $g(x) \neq f(x)$ (since $f \cap B = \emptyset$) and $g(x) \neq f(s)$ (since $\langle x, f(s) \rangle \in [s, x_1) \times (y_1, y_2) \subset \mathcal{E}(B)$).

The case $\langle s, f(s) \rangle \in \mathcal{N}(B)$ is analogous if we take into consideration left-open rectangular neighbourhood of $\langle s, f(s) \rangle$ contained in $\mathcal{N}(B)$. $\qquad\square$

Note that in the above proof we use only the following property of the family C of constant functions: for every $\langle x, y \rangle \in \mathbb{I} \times \mathbb{R}$ there exists right-open (left-open) set $G = [x, x_2) \times (y_1, y_2)$ ($G = (x_1, x] \times (y_1, y_2)$) and a function $c \in C$ such that $\langle x, y \rangle \in G$ and $c \restriction [x, x_2) \subset G$ ($c \restriction (x_1, x] \subset G$). So, we can formulate the general corollary.

Corollary 1. *Suppose C is a subfamily of continuous real functions such that $\bigcup C$ is dense in the plane. If $f \colon \mathbb{I} \to \mathbb{R}$ is such that $\max(f, g) \in \mathrm{AC}$ and $\min(f, g) \in \mathrm{AC}$ for every $g \in C$, then f is almost continuous.*

$\qquad\square$

Note that, in particular, C can be countable.

4   Uniform limits of almost continuous functions

The problem of characterization of uniform limits of AC functions has been posed by K. Kellum in [KK]. This problem remains still open ([GN1]) and it seems to be the most interesting unsolved problem concerning almost continuous functions. We will apply Theorem 2 to obtain some partial results concerning this problem.

First, recall two notions defined by Bruckner, Ceder and Weiss.

Definition 3. *Let $f \colon \mathbb{I} \to \mathbb{R}$.*

- *f is in class $\mathcal{U}_0$ if for every interval $J \subset \mathbb{I}$ the set $f(J)$ is dense in the interval $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$;*

- *f is in class $\mathcal{U}$ if for every interval $J \subset \mathbb{I}$ and every set $A \subset \mathbb{I}$ such that cardinality of A is less than $\mathfrak{c}$ the set $f(J \setminus A)$ is dense in the interval $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$.*

Note that $\overline{\mathrm{D}} = \mathcal{U}$ ([BCW]).

Of course, $\overline{\mathrm{AC}} \subset \mathcal{U}$. An example showing that $\overline{\mathrm{AC}} \neq \mathcal{U}$ was found by Kellum ([KK], see also [JJ]). The next definition is connected with necessary and sufficient conditions for a function $f \colon \mathbb{I} \to \mathbb{R}$ to be in $\overline{\mathrm{AC}}$.

Definition 4. *Let B be a blocking set and $\varepsilon > 0$.*

- *$\alpha(B, \varepsilon)$ is the class of all $f \colon \mathbb{I} \to \mathbb{R}$ for which at least one of the following conditions holds:*

  1. *the set $\{x \in \mathbb{I} \mid |f(x) - B_x| < \varepsilon\}$ has cardinality $\mathfrak{c}$;*
  2. *there exists $x \in \mathbb{I}$ such that $[f(x) - \varepsilon, f(x) + \varepsilon] \subset B_x$.*

- *$\beta(B, \varepsilon)$ is the class of all $f \colon \mathbb{I} \to \mathbb{R}$ for which at least one of the following conditions holds:*

  1. *the set $\{x \in \mathbb{I} \mid |f(x) - B_x| < \varepsilon\}$ has cardinality $\mathfrak{c}$;*
  2. *there exists $x \in \mathbb{I}$ such that $[f(x) - \varepsilon, f(x) + \varepsilon] \cap B_x$ has non-empty interior.*

We say that f is in class α (respectively f is in β) iff $f \in \alpha(B, \varepsilon)$ (respectively $f \in \beta(B, \varepsilon)$) for every blocking set B and $\varepsilon > 0$.

T. Natkaniec proved in [TN] that $\alpha \subset \overline{\mathrm{AC}}$ and, under CH, $\overline{\mathrm{AC}} \subset \beta$ (we will show later that $\overline{\alpha} = \overline{\mathrm{AC}} \subset \beta$ in ZFC)[2]. He also proved the following theorem.

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[2]The definition of the class α given by Natkaniec is slightly weaker than used in this paper, since for a function f to belong to α he requires $f \in \alpha(B, \varepsilon)$ for sufficiently small $\varepsilon > 0$ and every blocking set B. Author does not know if both definitions are equivalent.

Theorem 7. *Let $f\colon \mathbb{I} \to \mathbb{R}$ and $\varepsilon > 0$. Suppose that $\{x \in \mathbb{I} \mid |f(x) - q| \leq \varepsilon\}$ is $\mathfrak{c}$-dense in $\mathbb{I}$ for every rational q. Then f fulfills the first condition from the definition of the class $\alpha(B, \varepsilon)$.*

In the particular, if $f \in \mathcal{U} = \overline{\mathrm{D}}$ and the graph of f is dense in the plane, then $f \in \overline{\mathrm{AC}}$.

The first from the following definitions belongs to Kellum[3]:

Definition 5. *Let $f\colon \mathbb{I} \to \mathbb{R}$.*

- *$f \in \mathrm{AAC}_0$ (f is away-almost continuous) iff $\{x \mid |f(x) - B_x| < \varepsilon\}$ is non-empty for every blocking set B and $\varepsilon > 0$;*

- *$f \in \mathrm{AAC}$ iff $f \cap B \neq \emptyset$ or $\{x \mid |f(x) - B_x| < \varepsilon\}$ has cardinality $\mathfrak{c}$ for every blocking set B and $\varepsilon > 0$.*

Obviously $\mathrm{AAC} \subset \mathrm{AAC}_0$. It is also easy to see that $\overline{\mathrm{AC}} \subset \mathrm{AAC}_0$.

During Miniconference in Real Analysis in Auburn in 1999 Kellum formulated conjecture that f is in $\overline{\mathrm{AC}}$ iff $f \in \mathcal{U} \cap \mathrm{AAC}_0$.

Note the alternate definition of AAC.

Remark 2. *A function f is in AAC iff for every $\varepsilon > 0$, every set $J \subset \mathbb{I}$ of cardinality less than $\mathfrak{c}$ and every open neighbourhood G of the set $f \cup \bigcup_{x \in \mathbb{I} \setminus J} (\{x\} \times [f(x) - \varepsilon, f(x) + \varepsilon])$ there exists a continuous function $g\colon \mathbb{I} \to \mathbb{R}$ such that $g \subset G$.*

$\square$

Theorem 8.

1. $\mathrm{AC} \subset \alpha$;

2. $\overline{\alpha} \subset \beta \cap \mathrm{AAC}$;

3. $\mathrm{AAC} \subset \mathcal{U} \cap \mathrm{AAC}_0$.

Lemma 1. *Suppose $f\colon \mathbb{I} \to \mathbb{R}$ is bilaterally $\mathfrak{c}$-dense in itself, $c \in \mathbb{I}$ and either*

1. *$\langle c, f(c) \rangle \in \mathcal{E}(B)$ and $|f(c) - \mathcal{N}(B)_c| < \varepsilon$, or*

2. *$\langle c, f(c) \rangle \in \mathcal{N}(B)$ and $|f(c) - \mathcal{E}(B)_c| < \varepsilon$.*

Then the set $\{x \in \mathbb{I} \mid |f(x) - B_x| < \varepsilon\}$ has cardinality $\mathfrak{c}$.

---

[3]Originally the class AAC_0 was called AAC. We decide to change its name to AAC_0 to obtain duality with classes $\mathcal{U}$ and $\mathcal{U}_0$.

Proof. We will prove this lemma only for the case (1).

Take $y \in \mathbb{R}$ such that $\langle c, y \rangle \in \mathcal{N}(B)$ and $|y - f(c)| < \varepsilon$. By Theorem 1 (3) and (4), there exists $\tau > 0$ such that $(c - \tau, c + \tau) \times (f(c) - \frac{\tau}{2}, f(c) + \frac{\tau}{2}) \subset \mathcal{E}(B)$ and $(c - \tau, c] \times (y - \tau, y + \tau) \subset \mathcal{N}(B)$. Since f is bilaterally $\mathfrak{c}$-dense in itself, the set $A = (c - \tau, c] \times (f(c) - \frac{\tau}{2}, f(c) + \frac{\tau}{2}) \cap f$ has cardinality $\mathfrak{c}$.

For every $\langle a, f(a) \rangle \in A$, $\left| f(a) - (y - \frac{\tau}{2}, y + \frac{\tau}{2}) \right| < \varepsilon$, so since $\langle a, f(a) \rangle \in \mathcal{E}(B)$ and $\{a\} \times (y - \frac{\tau}{2}, y + \frac{\tau}{2}) \subset \mathcal{N}(B)$, $|f(a) - B_a| < \varepsilon$ (see Remark 1). $\square$

Proof of theorem 8. (1) Suppose $f \in \mathrm{AC}$ and there exists blocking set $B \subset \mathbb{I} \times \mathbb{R}$ and $\varepsilon > 0$ such that $f \notin \alpha(B, \varepsilon)$.

First note, that there exist $a, b \in \mathbb{I}$ such that $a < b$, $\langle a, f(a) \rangle \in \mathcal{E}(B)$ and $\langle b, f(b) \rangle \in \mathcal{N}(B)$. To choose $a \in [0, 1)$ with $\langle a, f(a) \rangle \in \mathcal{E}(B)$ assume $\langle 0, f(0) \rangle \notin \mathcal{E}(B)$. Since $f \notin \alpha(B, \varepsilon)$ and $\{0\} \times \mathbb{R} \subset B \cup \mathcal{E}(B)$ (see Theorem 1 (1)), there exists $y_0 \in \mathbb{R}$ such that $\langle 0, y_0 \rangle \in \mathcal{E}(B)$ and $|f(0) - y_0| < \varepsilon$. Take $\tau > 0$ such that $[0, \tau) \times (y_0 - \tau, y_0 + \tau) \subset \mathcal{E}(B)$ (by Th. 1 (3), $\mathcal{E}(B)$ is right-open). Since f is bilaterally $\mathfrak{c}$-dense in itself and cardinality of $f \cap B$ is less than $\mathfrak{c}$, the set $A = [0, \tau) \times (f(0) - \frac{\tau}{2}, f(0) + \frac{\tau}{2}) \cap f \cap (\mathcal{E}(B) \cup \mathcal{N}(B))$ is not empty. Take $\langle a, f(a) \rangle \in A$. We will show that $\langle a, f(a) \rangle \in \mathcal{E}(B)$. Indeed, if $\langle a, f(a) \rangle \in \mathcal{N}(B)$ then, since $\left| f(a) - (y_0 - \frac{\tau}{2}, y_0 + \frac{\tau}{2}) \right| < \varepsilon$ and $\{a\} \times (y_0 - \frac{\tau}{2}, y_0 + \frac{\tau}{2}) \subset \mathcal{E}(B)$, lemma 1 implies $f \in \alpha(B, \varepsilon)$. So $\langle a, f(a) \rangle \in \mathcal{E}(B)$. Analogously we can find $b \in (a, 1]$ such that $\langle b, f(b) \rangle \in \mathcal{N}(B)$.

By Theorem 2, there exists $c \in [a, b]$ such that f breaks through B at $\langle c, f(c) \rangle$.

Since $f \notin \alpha(B, \varepsilon)$, there exists $y_c \in \mathbb{R}$ such that $\langle c, y_c \rangle \notin B$ and $|f(c) - y_c| < \varepsilon$. Theorem 1 (6) shows that $\langle c, y_c \rangle \in \mathrm{A_{EE}}(B) \cup \mathrm{A_{NN}}(B) \cup \mathrm{A_{NE}}(B)$. All cases are analogous, so assume $\langle c, y_c \rangle \in \mathrm{A_{NE}}(B)$. Fix $\tau > 0$ such that $(c - \tau, c] \times (y_c - \tau, y_c + \tau) \subset \mathcal{N}(B)$ and $(c, c + \tau) \times (y_c - \tau, y_c + \tau) \subset \mathcal{E}(B)$. Since f breaks through B at $\langle c, f(c) \rangle$, there are $x_1 < x_2$ with $\langle x_1, f(x_1) \rangle, \langle x_2, f(x_2) \rangle \in (c - \tau, c + \tau) \times (f(c) - \frac{\tau}{2}, f(c) + \frac{\tau}{2})$ and $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B)$, $\langle x_2, f(x_2) \rangle \in \mathcal{N}(B)$. Now, we have $x_1 \leq c$ or $x_2 > c$.

In the first case $\langle x_1, f(x_1) \rangle \in \mathcal{E}(B)$, $\{x_1\} \times (y_c - \frac{\tau}{2}, y_c + \frac{\tau}{2}) \subset \mathcal{N}(B)$ and $\left| f(x_1) - (y_c - \frac{\tau}{2}, y_c + \frac{\tau}{2}) \right| < \varepsilon$, so lemma 1 implies $f \in \alpha(B, \varepsilon)$.

In the second case $\langle x_2, f(x_2) \rangle \in \mathcal{N}(B)$, $\{x_2\} \times (y_c - \frac{\tau}{2}, y_c + \frac{\tau}{2}) \subset \mathcal{E}(B)$ and $\left| f(x_2) - (y_c - \frac{\tau}{2}, y_c + \frac{\tau}{2}) \right| < \varepsilon$, and again lemma 1 gives $f \in \alpha(B, \varepsilon)$. A contradiction.

(2) Suppose $\{f_n\}_n \subset \alpha$ uniformly converges to f. Let $\varepsilon > 0$ and $B \subset \mathbb{I} \times \mathbb{R}$ be a blocking set.

Fix $n \in \mathbb{N}$ such that $|f_n - f| < \frac{\varepsilon}{2}$. Since $f_n \in \alpha(B, \frac{\varepsilon}{2})$, we have two cases:

- The set $C = \left\{ x \mid |f_n(x) - B_x| < \frac{\varepsilon}{2} \right\}$ has cardinality $\mathfrak{c}$. Then for every $x \in C$ we have $|f(x) - B_x| \leq |f(x) - f_n(x)| + |f_n(x) - B_x| < \varepsilon$.

- There exists $c \in \mathbb{I}$ such that $[f_n(c) - \frac{\varepsilon}{2}, f_n(c) + \frac{\varepsilon}{2}] \subset B_c$. Because $f(c) \in (f_n(c) - \frac{\varepsilon}{2}, f_n(c) + \frac{\varepsilon}{2})$, so $\langle c, f(c) \rangle \in B$ and $(f_n(c) - \frac{\varepsilon}{2}, f_n(c) + \frac{\varepsilon}{2}) \subset B_c \cap (f(c) - \varepsilon, f(c) + \varepsilon)$.

In both cases $f \in \beta \cap \mathrm{AAC}$.

(3) We will only show that $\mathrm{AAC} \subset \mathcal{U}$.

Suppose $f \in \mathrm{AAC}$ and $f \notin \mathcal{U}$. Since every function in the class $\mathcal{U}$ is $\mathfrak{c}$-dense in itself ([BCW]), then there exist an interval $J = [a,b]$, a set A of cardinality less than $\mathfrak{c}$, $y \in [f(a), f(b)]$ (without lost of generality we can assume that $f(a) < f(b)$) and $\varepsilon > 0$ such that $a, b \notin A$, $(y - 3\varepsilon, y + 3\varepsilon) \subset (f(a), f(b))$ and $f(J \setminus A) \cap (y - 3\varepsilon, y + 3\varepsilon) = \emptyset$. Since cardinality of A is less than $\mathfrak{c}$, we can also find $y' \in (y - \varepsilon, y + \varepsilon)$ such that $y' \notin f(A)$. Now, for every $\langle x, f(x) \rangle$ define open set $S_{\langle x, f(x) \rangle}$:

- $S_{\langle a, f(a) \rangle} = [0, \frac{a+b}{2}) \times (f(a) - 2\varepsilon, f(a) + 2\varepsilon)$;

- $S_{\langle b, f(b) \rangle} = (\frac{a+b}{2}, 1] \times (f(b) - 2\varepsilon, f(b) + 2\varepsilon)$;

- $S_{\langle x, f(x) \rangle} = [0, a) \times (f(x) - 2\varepsilon, f(x) + 2\varepsilon)$ for $x \in [0, a)$;

- $S_{\langle x, f(x) \rangle} = (b, 1] \times (f(x) - 2\varepsilon, f(x) + 2\varepsilon)$ for $x \in (b, 1]$;

- $S_{\langle x, f(x) \rangle} = (a, b) \times (f(x) - 2\varepsilon, f(x) + 2\varepsilon)$ for $x \in (a, b) \setminus A$;

- $S_{\langle x, f(x) \rangle} = (a, b) \times U_x$, where U_x is an open interval containing $f(x)$ and not containing y' for $x \in (a, b) \cap A$.

Since $V = \bigcup_{x \in \mathbb{I}} S_{\langle x, f(x) \rangle}$ is an open neighbourhood of f fullfilling all assumptions of Remark 2, there exists continuous function $g \colon \mathbb{I} \to \mathbb{R}$, $g \subset V$. We have built V such that $g(a) < y'$, $g(b) > y'$ and $((a,b) \times \{y'\}) \cap V = \emptyset$. So, g cannot take value y' between a and b, contrary to Darboux property of g. $\square$

Corollary 2. $\overline{\mathrm{AC}} \subset \beta \cap \mathrm{AAC}$.[4]

$\square$

Corollary 3. $\overline{\mathrm{AC}} = \overline{\alpha}$.

Proof. This is a consequence of inclusions $\alpha \subset \overline{\mathrm{AC}}$ and $\mathrm{AC} \subset \alpha$. $\square$

Finally, we would like to state a problem.

Problem 1. *Does the equality* $\mathrm{AAC} = \mathcal{U} \cap \mathrm{AAC}_0$ *hold?*

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[4]Using technique from the proof of Theorem 2 it is possible to show that $\mathrm{AAC} \subset \beta$, but such proof is much longer than the one presented above.

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