ON FUNCTIONS WHOSE GRAPH IS A HAMEL BASIS

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Abstract. We say that a function $h\colon \mathbb{R} \to \mathbb{R}$ is a *Hamel function* ($h \in \mathrm{HF}$) if h, considered as a subset of $\mathbb{R}^2$, is a Hamel basis for $\mathbb{R}^2$. We prove that *every function from* $\mathbb{R}$ *into* $\mathbb{R}$ *can be represented as a pointwise sum of two Hamel functions*. The latter is equivalent to the statement: for all $f_1, f_2 \in \mathbb{R}^{\mathbb{R}}$ there is a $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + f_1, g + f_2 \in \mathrm{HF}$. We show that this fails for infinitely many functions.

1. Introduction

The terminology is standard and follows [2]. The symbols $\mathbb{R}$ and $\mathbb{Q}$ stand for the sets of all real and all rational numbers, respectively. A basis of $\mathbb{R}^n$ as a linear space over $\mathbb{Q}$ is called *Hamel basis*. For $Y \subset \mathbb{R}^n$, the symbol $\mathrm{Lin}_{\mathbb{Q}}(Y)$ stands for the smallest linear subspace of $\mathbb{R}^n$ over $\mathbb{Q}$ that contains Y. The zero element of $\mathbb{R}^n$ is denoted by 0. The cardinality of a set X we denote by $|X|$. In particular, $\mathfrak{c}$ stands for $|\mathbb{R}|$. Given a cardinal κ, we let $\mathrm{cf}(\kappa)$ denote the cofinality of κ. We say that a cardinal κ is regular if $\mathrm{cf}(\kappa) = \kappa$. For any set X, the symbol $[X]^{<\kappa}$ denotes the set $\{Z \subseteq X\colon |Z| < \kappa\}$. For $A, B \subseteq \mathbb{R}^n$, $A + B$ stands for $\{a + b\colon a \in A, b \in B\}$.

We consider only real-valued functions. No distinction is made between a function and its graph. For any two partial real functions f, g we write $f + g$, $f - g$ for the sum and difference functions defined on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. The class of all functions from a set X into a set Y is denoted by Y^X. We write $f|A$ for the restriction of $f \in Y^X$ to the set $A \subseteq X$. For $B \subseteq \mathbb{R}^n$ its characteristic function is denoted by χ_B. For any function $g \in \mathbb{R}^X$ and any family of functions $F \subseteq \mathbb{R}^X$ we define $g + F = \{g + f\colon f \in F\}$. For any planar set P, we denote its x-projection by $\mathrm{dom}(P)$.

The cardinal function $\mathrm{A}(\mathrm{F})$, for $\mathrm{F} \subsetneq \mathbb{R}^X$, is defined as the smallest cardinality of a family $G \subseteq \mathbb{R}^X$ for which there is no $g \in \mathbb{R}^X$ such that $g + G \subseteq \mathrm{F}$. It was investigated for many different classes of real functions, see e.g. [4, 5, 10]. Recall here that $\mathrm{A}(\mathrm{F}) \geq 3$ is equivalent to $\mathrm{F} - \mathrm{F} = \mathbb{R}^X$ (see [12, Proposition 1].)

One of the very important concepts in Real Analysis is *additivity*. It dates back to the early 19th century when the following functional equation was considered for the first time

$$f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

An obvious solution to this equation is a linear function, that is, a function defined by $f(x) = ax$ for all $x \in \mathbb{R}$, where a is some constant. The fact that the linear

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functions are the only continuous solutions, was first proved by A. L. Cauchy [1]. Because of this, the above equation is known as Cauchy's Functional Equation. The problem of the existence of a discontinuous solutions of the Cauchy equation was solved by G. Hamel in 1905 [6] who constructed a discontinuous function which satisfies the desired equation. To construct an example of such a function observe first that $f \in \mathbb{R}^{\mathbb{R}}$ satisfies the Cauchy equation if and only if it is linear over $\mathbb{Q}$, i.e., for all $p, q \in \mathbb{Q}$ and $x, y \in \mathbb{R}$ we have $f(px + qy) = pf(x) + qf(y)$. So to define an additive function it is enough to define it on a Hamel basis. Thus, if $H \subseteq \mathbb{R}$ is a Hamel basis and f identically equals 1 on H then clearly f is not continuous. The family of all solutions of Cauchy's Functional Equation is called the family of *additive functions* and we denote it by AD. For the class of additive functions defined on $\mathbb{R}^n$ we use the symbol $\mathrm{AD}(\mathbb{R}^n)$.

It is obvious that not all functions are additive. But one could wonder how "badly" the additive condition can be violated. In particular, does there exist a function f for which the condition $f(x + y) = f(x) + f(y)$ fails for all x and y? It turns out that the answer is positive. We give two examples of families of such functions. In Section 2 we define and discuss a class of functions whose graph is a linearly independent set over $\mathbb{Q}$. Then, in Section 3, we investigate a proper subfamily of this class: functions whose graph is a Hamel basis. In this section we state and prove the main result of this paper (Theorem 3.4) which says that every real function is the pointwise sum of two Hamel functions.

2. Functions with linearly independent graphs

Definition 2.1. We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *linearly independent over* $\mathbb{Q}$ (shortly: *linearly independent*) if f is linearly independent subset of the space $\langle \mathbb{R}^{n+1}; \mathbb{Q}; +; \cdot \rangle$.

The symbol $\mathrm{LIF}(\mathbb{R}^n)$ stands for the family of all linearly independent functions. In the case when $n = 1$, we simply write LIF. An easy example shows that the family $\mathrm{LIF}(\mathbb{R}^n)$ is non-empty for all $n \geq 1$.

Example 2.2. *Every injection from $\mathbb{R}^n$ into a linearly independent set $H \subseteq \mathbb{R}$ is linearly independent over $\mathbb{Q}$.*

Proof. Let $f\colon \mathbb{R}^n \to H$ be an injection. Assume that for some $p_1, \ldots, p_n \in \mathbb{Q}$ and pairwise different $x_1, \ldots, x_n \in \mathbb{R}^n$ we have $\sum_1^n p_i \langle x_i, f(x_i) \rangle = 0$. Since $f(x_1), \ldots, f(x_n) \in H$ are all different and H is linearly independent over $\mathbb{Q}$, we conclude that $p_1 = \cdots = p_n = 0$. ■

As mentioned in the introductory part of this paper, the linearly independent functions lack the additive property. Thus, $\mathrm{AD}(\mathbb{R}^n) \cap \mathrm{LIF}(\mathbb{R}^n) = \emptyset$.

Below we give some basic properties of the class $\mathrm{LIF}(\mathbb{R}^n)$. Note that Fact 2.3 (i) has its counterpart in the case of continuous and Sierpiński-Zygmund functions (for the definition see [12].)

Fact 2.3.

- **(i):** $\mathrm{LIF}(\mathbb{R}^n) + \mathrm{AD}(\mathbb{R}^n) = \mathrm{LIF}(\mathbb{R}^n)$.
- **(ii):** If $f \in \mathrm{LIF}(\mathbb{R}^n)$ then $|f[\mathbb{R}^n]| = \mathfrak{c}$.
- **(iii):** If $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous on a non-empty open set then $f \notin \mathrm{LIF}(\mathbb{R}^n)$.
- **(iv):** There exists an $f \in \mathrm{LIF}(\mathbb{R}^n)$ which is the union of countably many partial continuous functions.

(v): $\mathrm{A}(\mathrm{LIF}(\mathbb{R}^n)) = \mathfrak{c}$.

PROOF. **(i)** Let $f \in \mathrm{LIF}(\mathbb{R}^n)$ and $g \in \mathrm{AD}(\mathbb{R}^n)$. Fix $x_1, \ldots, x_k \in \mathbb{R}^n$ and $q_1, \ldots q_k \in \mathbb{Q}$. Now suppose that $\sum_1^k q_i \langle x_i, f(x_i) + g(x_i)\rangle = 0$. Thus, in particular, $\sum_1^k q_i x_i = 0$. Since g is additive we have $\sum_1^k q_i g(x_i) = 0$. Consequently, $\sum_1^k q_i \langle x_i, f(x_i)\rangle = 0$. The linear independence of f implies that $q_1 = \cdots = q_k = 0$. So $f + g \in \mathrm{LIF}(\mathbb{R}^n)$.

(ii) Notice that it suffices to prove part (ii) for $n = 1$. Assume, by the way of contradiction, that $f \in \mathrm{LIF}$ and $|f[\mathbb{R}]| = \kappa < \mathfrak{c}$. We claim that there exist $x_1, x_2 \in \mathbb{R}$ with the following properties:

$$x_1 \neq x_2,\ f(x_1) = f(x_2),\ \text{and } f(-x_1) = f(-x_2).$$

To see the claim choose $y_0 \in \mathbb{R}$ such that $|f^{-1}(y_0) \cap (0, \infty)| \geq \kappa^+$. Such an element exists because $(0, \infty) \subseteq \bigcup_{y \in \mathbb{R}} f^{-1}(y)$ and $|f[\mathbb{R}]| = \kappa < \mathfrak{c}$. Since y_0 satisfies the condition $|f[-f^{-1}(y_0)]| \leq \kappa < \kappa^+ \leq |-f^{-1}(y_0)|$, there exist different $x_1, x_2 \in f^{-1}(y_0) \cap (0, \infty)$ satisfying the equality $f(-x_1) = f(-x_2)$. Note that x_1 and x_2 are the required points. Next observe that

$$\langle x_1, f(x_1)\rangle + \langle -x_1, f(-x_1)\rangle = \langle x_2, f(x_2)\rangle + \langle -x_2, f(-x_2)\rangle.$$

This leads to a contradiction with $f \in \mathrm{LIF}$.

(iii) Like in part (ii), it is enough to prove the case $n = 1$. Let $(a-h, a+h) \subseteq \mathbb{R}$ be a non-empty open interval such that $f|(a-h, a+h)$ is continuous. Consider a function $g\colon [0, h) \to \mathbb{R}$ defined by $g(x) = f(a-x) + f(a+x)$. Obviously, g is also continuous. If $g(x) = g(0) = 2f(a)$ for all $x \in [0, h)$ then f is not linearly independent. Hence we may suppose that there exist two different $x_1, x_2 \in (0, h)$ such that $g(x_1) = 2f(a) + p_1$ and $g(x_2) = 2f(a) + p_2$ for some non-zero rationals p_1, p_2. Then we have

$$p_2\langle 2a, g(x_1)\rangle - p_1\langle 2a, g(x_2)\rangle \in \mathrm{Lin}_{\mathbb{Q}}(\langle 2a, 2f(a)\rangle) = \mathrm{Lin}_{\mathbb{Q}}(\langle a, f(a)\rangle). \tag{2.1}$$

Now, recall the definition of g and note $\langle a - x_i, f(a - x_i)\rangle + \langle a + x_i, f(a + x_i)\rangle = \langle 2a, g(x_i)\rangle$ for $i = 1, 2$. Based on (2.1), we see that f is not linearly independent.

(iv) Let us first recall that $\mathbb{R}^n$ can be decomposed into $(n+1)$ 0-dimensional spaces $E_0, \ldots, E_n$. For every perfect set $Q \subseteq \mathbb{R}$ and 0-dimensional space E there exists an embedding $h_Q^E\colon E \to Q$. (See e.g., [7].) It is also known that there exists a perfect set $P \subseteq \mathbb{R}$ which is linearly independent over $\mathbb{Q}$. (See e.g., [8].) Now, if $P = P_0 \cup P_1 \cup \cdots \cup P_n$ is a partition of P into $(n+1)$ perfect sets then, by Example 2.2, $h_{P_i}^{E_i}\colon E_i \to P_i$ $(i = 0, \ldots, n)$ is a linearly independent subset of $\mathbb{R}^{n+1}$. It is easy to see that $h = \bigcup_0^n h_{P_i}^{E_i}\colon \mathbb{R}^n \to P$ is one-to-one. So, again by Example 2.2, h is linearly independent. Obviously, h is the union of countably many partial continuous functions.

(v) We start with showing that $\mathrm{A}(\mathrm{LIF}(\mathbb{R}^n)) \geq \mathfrak{c}$. Let $\mathbb{R}^n = \{x_\xi \colon \xi < \mathfrak{c}\}$. Fix an $F \subseteq \mathbb{R}^{\mathbb{R}^n}$ of cardinality less than continuum. We will define, by induction, a function $h\colon \mathbb{R}^n \to \mathbb{R}$ such that for every $f \in F$, $h + f$ is one-to-one and $(h+f)[\mathbb{R}^n]$ is linearly independent. Then, by Example 2.2, $h + F \subseteq \mathrm{LIF}(\mathbb{R}^n)$.

Let $\alpha < \mathfrak{c}$. Assume that h is defined on $\{x_\xi \colon \xi < \alpha\}$, for all $f \in F$ the function $h + f$ is one-to-one, and $(h+f)[\{x_\xi \colon \xi < \alpha\}]$ is linearly independent. We will define $h(x_\alpha)$. Choose

$$h(x_\alpha) \in \mathbb{R} \setminus \mathrm{Lin}_{\mathbb{Q}}\left(\bigcup_{f \in F} ((h+f)[\{x_\xi \colon \xi < \alpha\}] \cup \{f(x_\alpha)\})\right).$$

This choice is possible since

$$\left|\bigcup_{f\in F}\left((h+f)[\{x_\xi\colon \xi<\alpha\}]\cup\{f(x_\alpha)\}\right)\right|\le(\alpha+1)|F|<\mathfrak{c}.$$

It is easy to see that all the required properties of h are preserved. This ends the proof of $\mathrm{A}(\mathrm{LIF}(\mathbb{R}^n))\ge\mathfrak{c}$.

To see the opposite inequality consider F consisting of all constant functions. Then for any function $h\colon\mathbb{R}^n\to\mathbb{R}$ there is an $f\in F$ such that $h(0)+f(0)=0$. Therefore $h+f\notin\mathrm{LIF}(\mathbb{R}^n)$. ■

3. Hamel functions

In this section we confine ourselves to a proper subclass of linearly independent functions. More precisely, we consider the class of *Hamel functions*. We say that a function $f\colon\mathbb{R}^n\to\mathbb{R}$ is a Hamel function ($f\in\mathrm{HF}(\mathbb{R}^n)$ or $f\in\mathrm{HF}$ for $n=1$) if f, considered as a subset of $\mathbb{R}^{n+1}$, is a Hamel basis for $\mathbb{R}^{n+1}$. Clearly, $\mathrm{HF}(\mathbb{R}^n)\subseteq\mathrm{LIF}(\mathbb{R}^n)$. A little more challenging argument, comparing with the case of linearly independent functions, proves the existence of a Hamel function. We do not present it here since this observation is a corollary of Theorem 3.4.

Fact 2.3 states some basic properties of the class $\mathrm{LIF}(\mathbb{R}^n)$. It is interesting whether the same statements are true for $\mathrm{HF}(\mathbb{R}^n)$. Since $\mathrm{HF}(\mathbb{R}^n)\subseteq\mathrm{LIF}(\mathbb{R}^n)$, the properties (ii) and (iii) hold trivially. A short additional argument shows that (i) is also true. So we can state

Fact 3.1.

- **(i):** $\mathrm{HF}(\mathbb{R}^n)+\mathrm{AD}(\mathbb{R}^n)=\mathrm{HF}(\mathbb{R}^n)$.
- **(ii):** If $f\in\mathrm{HF}(\mathbb{R}^n)$ then $|f[\mathbb{R}^n]|=\mathfrak{c}$.
- **(iii):** If $f\colon\mathbb{R}^n\to\mathbb{R}$ is continuous on a non-empty open set then $f\notin\mathrm{HF}(\mathbb{R}^n)$.

However, it remains an open problem whether Fact 2.3 (iv) still holds when $\mathrm{LIF}(\mathbb{R}^n)$ is replaced by $\mathrm{HF}(\mathbb{R}^n)$.

Problem 3.2. Does there exist an $h\in\mathrm{HF}(\mathbb{R}^n)$ which is the union of countably many partial continuous functions?

But it turns out that the statement of the last part of Fact 2.3 is false for the class $\mathrm{HF}(\mathbb{R}^n)$.

Fact 3.3. $\mathrm{A}(\mathrm{HF}(\mathbb{R}^n))\le\omega$ for every $n\ge1$.

Proof. For each $q\in\mathbb{Q}$ and each open ball B with rational center and radius (rational ball), let us define a function $f_q^B\colon\mathbb{R}^n\to\mathbb{R}$ by $f_q^B=q\chi_B$. We claim that for every function $f\colon\mathbb{R}^n\to\mathbb{R}$ there exist a $q\in\mathbb{Q}$ and a rational ball B such that $f+f_q^B\notin\mathrm{HF}(\mathbb{R}^n)$. To see this, first note that we may assume that $f=f+f_0^B\in\mathrm{HF}(\mathbb{R}^n)$. Thus, $\langle0,1\rangle\in\mathrm{Lin}_{\mathbb{Q}}(f)$. Consequently, there exist $x_1,\ldots,x_k\in\mathbb{R}^n$ and $p_1,\ldots,p_k\in\mathbb{Q}$ satisfying $\sum_{i=1}^k p_i\langle x_i,f(x_i)\rangle=\langle0,1\rangle$.

Without loss of generality we may assume that $p_1\ne0$. Now let $q=\frac{-1}{p_1}$ and B be a rational ball containing x_1 but not $x_2,\ldots,x_k$. It follows easily that $f+f_q^B$ is not linearly independent over $\mathbb{Q}$. Indeed,

$$\sum_{i=1}^k p_i\langle x_i,f(x_i)+f_q^B(x_i)\rangle=\sum_{i=1}^k p_i\langle x_i,f(x_i)\rangle+\sum_{i=1}^k p_i\langle0,f_q^B(x_i)\rangle=\langle0,1\rangle+p_1\langle0,q\rangle=0.$$

■

Notice here that $\mathrm{A}(\mathrm{LIF}) = \mathfrak{c}$ (Fact 2.3 (v)) implies, in particular, that every function from $\mathbb{R}^{\mathbb{R}}$ can be written as the algebraic sum of two linearly independent functions. In other words $\mathrm{LIF} + \mathrm{LIF} = \mathbb{R}^{\mathbb{R}}$. Since we only found the upper bound for $\mathrm{A}(\mathrm{HF})$, it would be very interesting to determine whether $\mathrm{HF} + \mathrm{HF} = \mathbb{R}^{\mathbb{R}}$. The answer to the latter is given in the main result of this paper - Theorem 3.4.

Theorem 3.4.. *Every real function $f \colon \mathbb{R}^n \to \mathbb{R}$ can be represented as a sum of two Hamel functions. In other words,* $\mathbb{R}^{\mathbb{R}^n} = \mathrm{HF}(\mathbb{R}^n) + \mathrm{HF}(\mathbb{R}^n)$.

Theorem 3.4 and Fact 3.3 give us the bounds for $\mathrm{A}(\mathrm{HF}(\mathbb{R}^n))$. Namely, $3 \leq \mathrm{A}(\mathrm{HF}(\mathbb{R}^n)) \leq \omega$. It is not known whether the techniques used in the proofs of these two results could also be used to determine $\mathrm{A}(\mathrm{HF}(\mathbb{R}^n))$ exactly. We state the next open problem.

Problem 3.5. $\mathrm{A}(\mathrm{HF}(\mathbb{R}^n)) = \omega$?

Before proving the theorem we introduce some definitions and show auxiliary results. For $f \colon \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, and $1 \leq k < \omega$ let

$$\mathrm{LC}(f,k,x) = \left\{ \sum_1^k p_i f(x_i) \colon p_j \in \mathbb{Q},\ x_j \in \mathbb{R}^n\ (j = 1, \dots, k),\ \sum_1^k p_i x_i \ = \ x \right\}.$$

When $x = 0$ we write $\mathrm{LC}(f,k)$. We also use $\mathrm{LC}(f)$ to denote $\bigcup_{1 \leq k < \omega} \mathrm{LC}(f,k)$. Observe that $\mathrm{LC}(f)$ is a linear subspace of $\mathbb{R}$ over $\mathbb{Q}$, i.e., $\mathrm{LC}(f) = \mathrm{Lin}_{\mathbb{Q}}(\mathrm{LC}(f))$. This is so because $\mathrm{LC}(f)$ is linearly isomorphic to $\mathrm{Lin}_{\mathbb{Q}}(f) \cap (\{0\} \times \mathbb{R})$.

The sets $\mathrm{LC}(f)$ will play an important role in the proof of Theorem 3.4. Hence, we will investigate properties of these sets.

Property 3.6. $\mathrm{LC}(f,k) \subseteq \mathrm{LC}(f,3) + \mathrm{LC}(f,k-1)$ for every $f \in \mathbb{R}^{\mathbb{R}^n}$ and $3 \leq k < \omega$.

Proof. Let $y \in \mathrm{LC}(f,k)$. So $y = \sum_1^k p_i f(x_i)$ for some $x_1, \dots, x_k \in \mathbb{R}^n$ and $p_1, \dots, p_k \in \mathbb{Q}$ satisfying $\sum_1^k p_i x_i = 0$. Define $x' = p_1 x_1 + p_2 x_2$, $q = 1$, and $r = -1$. Observe that

$$p_1 x_1 + p_2 x_2 + r x' = q x' + p_3 x_3 + \cdots + p_n x_k = 0.$$

Hence, $p_1 f(x_1) + p_2 f(x_2) + r f(x') \in \mathrm{LC}(f,3)$ and $q f(x') + p_3 f(x_3) + \cdots + p_n f(x_k) \in \mathrm{LC}(f, k-1)$. Since $y = p_1 f(x_1) + p_2 f(x_2) + r f(x') + q f(x') + p_3 f(x_3) + \cdots + p_n f(x_k)$ we conclude that $y \in \mathrm{LC}(f,3) + \mathrm{LC}(f,k-1)$. ∎

Notice that Property 3.6 implies that

$$\text{if } |\mathrm{LC}(f)| = \mathfrak{c} \text{ then } m_0 = \min\{k \geq 1 \colon |\mathrm{LC}(f,k)| = \mathfrak{c}\} \leq 3. \tag{3.1}$$

Next we show another property which is important for the proof of Theorem 3.4. Note that if $\mathfrak{c}$ is regular (i.e., $\mathrm{cf}(\mathfrak{c}) = \mathfrak{c}$), then the set Z from part (a) can be taken as a singleton.

Property 3.7. Assume that $|\mathrm{LC}(f)| = \mathfrak{c}$. Then at least one of the following two cases hold.

- **(a):** There exists a set $Z \in [\mathbb{R}^n]^{<\mathfrak{c}}$ such that $\left|\bigcup_{z \in Z} \mathrm{LC}(f,2,z)\right| = \mathfrak{c}$.
- **(b):** For all $X \in [\mathbb{R}^n]^{<\mathfrak{c}}, Y \in [\mathbb{R}]^{<\mathfrak{c}}$ there exist $q_1, q_2, q_3 \in \mathbb{Q} \setminus \{0\}$ and pairwise linearly independent $x_1, x_2, x_3 \in \mathbb{R}^n$ such that $\sum_1^3 q_i f(x_i) \notin Y$, $\sum_1^3 q_i x_i = 0$, and $\mathrm{Lin}_{\mathbb{Q}}(x_1, x_2, x_3) \cap \mathrm{Lin}_{\mathbb{Q}}(X) = \{0\}$.

PROOF. Notice first that if $|\mathrm{LC}(f,2)| = \mathfrak{c}$ then case (a) holds with $Z = \{0\}$. Hence, using (3.1), we may assume that

$$|\mathrm{LC}(f,2)| < \mathfrak{c} \text{ and } |\mathrm{LC}(f,3)| = \mathfrak{c}. \tag{3.2}$$

Based on the above assumption and the definition of the set $\mathrm{LC}(f,3)$, we conclude that there exist continuum many triples $\langle x_1,x_2,x_3\rangle \in (\mathbb{R}^n)^3$ and $\langle p_1,p_2,p_3\rangle \in (\mathbb{Q}\setminus\{0\})^3$ such that $\sum_1^3 p_i x_i = 0$ and $\sum_1^3 p_i f(x_i)$ are all different. Thus, an easy cardinal argument implies the existence of a sequence $\langle\langle x_1^\xi, x_2^\xi, x_3^\xi\rangle \in (\mathbb{R}^n)^3 \colon \xi < \mathfrak{c}\rangle$ and some non-zero rationals q_1,q_2,q_3 with the property that $q_1x_1^\xi + q_2x_2^\xi + q_3x_3^\xi = 0$ for every $\xi < \mathfrak{c}$, and all $q_1 f(x_1^\xi) + q_2 f(x_2^\xi) + q_3 f(x_3^\xi)$ are different.

Notice that, if $\dim(\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi)) = 1$ for some ξ then $\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) = \mathrm{Lin}_{\mathbb{Q}}(x_i^\xi)$ for some $i \in \{1,2,3\}$. Say i=1. So there is an $s \in \mathbb{Q}$ such that $sq_1x_1^\xi + q_2x_2^\xi = 0$. Combining this with the equality $q_1x_1^\xi + q_2x_2^\xi + q_3x_3^\xi = 0$ we obtain that $sq_1x_1^\xi + q_2x_2^\xi = (1-s)q_1x_1^\xi + q_3x_3^\xi = 0$. Consequently,

$$\begin{aligned}
[sq_1 f(x_1^\xi) + q_2 f(x_2^\xi)], [(1-s)q_1 f(x_1^\xi) + q_3 f(x_3^\xi)] &\in \mathrm{LC}(f,2) \\
&\text{and} \\
q_1 f(x_1^\xi) + q_2 f(x_2^\xi) + q_3 f(x_3^\xi) &= \\
sq_1 f(x_1^\xi) + q_2 f(x_2^\xi) + (1-s)q_1 f(x_1^\xi) + q_3 f(x_3^\xi) &\in \mathrm{LC}(f,2) + \mathrm{LC}(f,2).
\end{aligned}$$

So, if $\dim(\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi)) = 1$ for continuum many ξ then $|\mathrm{LC}(f,2)| = \mathfrak{c}$. This contradicts (3.2). Thus, we may assume that $\dim(\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi)) = 2$ for all $\xi < \mathfrak{c}$.

Now choose $X \in [\mathbb{R}^n]^{<\mathfrak{c}}$ and $Y \in [\mathbb{R}]^{<\mathfrak{c}}$. Notice that

($\bullet$) if $\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(X) \neq \{0\}$ and $Z = \mathrm{Lin}_{\mathbb{Q}}(X)$ then

$$q_1 f(x_1^\xi) + q_2 f(x_2^\xi) + q_3 f(x_3^\xi) \in \bigcup_{z\in Z} \mathrm{LC}(f,2,z) + \bigcup_{z\in Z} \mathrm{LC}(f,2,z).$$

Indeed, if $\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi)\ \cap \mathrm{Lin}_{\mathbb{Q}}(X) \neq \{0\}$ then there exist $a,b,c \in \mathbb{Q}$ such that $ax_1^\xi + bx_2^\xi + cx_3^\xi \in \mathrm{Lin}_{\mathbb{Q}}(X)\setminus\{0\}$. At least one of the numbers a,b,c is not equal to zero because $ax_1^\xi + bx_2^\xi + cx_3^\xi \neq 0$. Without loss of generality we may suppose that $c \neq 0$ and consequently $c = q_3$ (multiply the above equation by $\frac{q_3}{c}$.) Then, by subtracting $ax_1^\xi + bx_2^\xi + q_3x_3^\xi$ from $q_1x_1^\xi + q_2x_2^\xi + q_3x_3^\xi = 0$, we obtain that $(q_1-a)x_1^\xi + (q_2-b)x_2^\xi \in \mathrm{Lin}_{\mathbb{Q}}(X)\setminus\{0\}$. So at least one of $(q_1-a),(q_2-b)$ is not 0. We may assume that $(q_2-b) \neq 0$. (If $(q_1-b) \neq 0$ then the following argument works analogously.) Now multiply $(q_1-a)x_1^\xi + (q_2-b)x_2^\xi$ by $\frac{q_2}{q_2-b}$. We get that $rq_1x_1^\xi + q_2x_2^\xi \in \mathrm{Lin}_{\mathbb{Q}}(X)$ and consequently $(1-r)q_1x_1^\xi + q_3x_3^\xi = [q_1x_1^\xi + q_2x_2^\xi + q_3x_3^\xi] - [rq_1x_1^\xi + q_2x_2^\xi] = -[rq_1x_1^\xi + q_2x_2^\xi] \in \mathrm{Lin}_{\mathbb{Q}}(X)$, for some $r \in \mathbb{Q}$. Hence

$$[rq_1 f(x_1^\xi) + q_2 f(x_2^\xi)], [(1-r)q_1 f(x_1^\xi) + q_3 f(x_3^\xi)] \in \bigcup_{z\in Z} \mathrm{LC}(f,2,z).$$

Now the claim ($\bullet$) follows from

$$\begin{aligned}
q_1 f(x_1^\xi) + q_2 f(x_2^\xi) + q_3 f(x_3^\xi) &= rq_1 f(x_1^\xi) + q_2 f(x_2^\xi) + (1-r)q_1 f(x_1^\xi) + q_3 f(x_3^\xi) \\
&\in \bigcup_{z\in Z} \mathrm{LC}(f,2,z) + \bigcup_{z\in Z} \mathrm{LC}(f,2,z).
\end{aligned}$$

From (•) we see that if $\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(X) \neq \{0\}$ holds for $\mathfrak{c}$-many ξ then the set Z satisfies the condition $|\bigcup_{z\in Z} \mathrm{LC}(f,2,z)| = \mathfrak{c}$. Obviously $Z \in [\mathbb{R}^n]^{<\mathfrak{c}}$. Thus, case (a) holds.

Summarizing the above discussion, we just need to consider a situation when $\dim(\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi)) = 2$ and $\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(X) = \{0\}$ for all ξ. Recall that $q_1 x_1^\xi + q_2 x_2^\xi + q_3 x_3^\xi = 0$, where $q_1, q_2, q_3 \in \mathbb{Q} \setminus \{0\}$. If two of $x_1^\xi, x_2^\xi, x_3^\xi$ were dependent over $\mathbb{Q}$ then we would have $\dim(\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi)) \leq 1$. Thus, $x_1^\xi, x_2^\xi, x_3^\xi$ are pairwise independent. Now it is easy to see that case (b) holds. ∎

Lemma 3.8. *Let $X \in [\mathbb{R}^n]^{<\mathfrak{c}}$, $x \notin X$, and $y \in \mathbb{R}$. Suppose also that $h, g \colon X \to \mathbb{R}$ are functions linearly independent over $\mathbb{Q}$. Then there exist extensions h', g' of h and g onto $X \cup \{x\}$ such that h' and g' are linearly independent over $\mathbb{Q}$ and $h'(x) + g'(x) = y$.*

Proof. Choose $h'(x) \in \mathbb{R} \setminus \mathrm{Lin}_{\mathbb{Q}}(h[X] \cup g[X] \cup \{y\})$. This choice is possible since $|\mathrm{Lin}_{\mathbb{Q}}(h[X] \cup g[X] \cup \{y\})| < \mathfrak{c}$. Then define $g'(x) = y - h'(x)$. It is easy to see that $h' = h \cup \{\langle x, h'(x)\rangle\}$ and $g' = g \cup \{\langle x, g'(x)\rangle\}$ are the desired extensions. ∎

Proof of Theorem 3.4. Let us start with fixing a function $f \colon \mathbb{R}^n \to \mathbb{R}$ and enumerations $\{x_\xi \colon \xi < \mathfrak{c}\}$, $\{v_\xi \colon \xi < \mathfrak{c}\}$ of $\mathbb{R}^n$ and $\{0\} \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$, respectively. We will construct functions $h, g \colon \mathbb{R}^n \to \mathbb{R}$ which are linearly independent over $\mathbb{Q}$ and satisfy the property that $h + g = f$ and $\{0\} \times \mathbb{R} \subseteq \mathrm{Lin}_{\mathbb{Q}}(h) \cap \mathrm{Lin}_{\mathbb{Q}}(g)$.

First, let us argue that this is enough to prove the theorem. What we have to show is that $\mathrm{Lin}_{\mathbb{Q}}(h) = \mathrm{Lin}_{\mathbb{Q}}(g) = \mathbb{R}^{n+1}$. To see $\mathrm{Lin}_{\mathbb{Q}}(h) = \mathbb{R}^{n+1}$ note that

$$\forall x \in \mathbb{R}^n\ \forall z \in \mathbb{R}\ \langle x, z\rangle = \langle x, h(x)\rangle + \langle 0, z - h(x)\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h) + \mathrm{Lin}_{\mathbb{Q}}(h) = \mathrm{Lin}_{\mathbb{Q}}(h).$$

By the same argument $\mathrm{Lin}_{\mathbb{Q}}(g) = \mathbb{R}^{n+1}$.

To construct the desired functions h and g, we consider three cases. In the first case we assume that $|\mathrm{LC}(f)| < \mathfrak{c}$. If the latter fails, that is $|\mathrm{LC}(f)| = \mathfrak{c}$, then either part (a) (Case 2) or part (b) (Case 3) of Property 3.7 holds.

Case 1: $|\mathrm{LC}(f)| < \mathfrak{c}$**.**

Let $\kappa < \mathfrak{c}$ denote the cardinality of the basis of $\mathrm{LC}(f)$ over $\mathbb{Q}$. There exist $c \in \mathrm{LC}(f)$ and a linearly independent set $A \subseteq \mathbb{R}^n$ such that $|A| = \kappa$ and $f(-a) + f(a) \equiv c = const$ for all $a \in A$. Such a set can be found since $|\mathrm{LC}(f)| < \mathfrak{c}$ and $f(x) + f(-x) \in \mathrm{LC}(f)$ for every $x \in \mathbb{R}^n$. Put $B = (-A) \cup A$.

First, we will construct functions $h, g \colon B \to \mathbb{R}$ linearly independent over $\mathbb{Q}$ for which $h + g \subseteq f$ and

$$\{0\} \times \mathrm{LC}(f) \subseteq \mathrm{Lin}_{\mathbb{Q}}(h) \cap (\{0\} \times \mathbb{R}) = \mathrm{Lin}_{\mathbb{Q}}(g) \cap (\{0\} \times \mathbb{R}). \tag{3.3}$$

To accomplish this let us fix enumerations $\{a_\xi \colon \xi < \kappa\}$ of A and $\{m_\xi \colon \xi < \kappa\}$ of a linear basis of $\mathrm{LC}(f)$ over $\mathbb{Q}$. We may assume that $m_0 = c$ if $c \neq 0$. The construction of h and g is by induction. At every step $\alpha < \kappa$ we will define h and g on $\{-a_\alpha, a_\alpha\}$, assuring that

(a): $h|A_\alpha, g|A_\alpha$ are linearly independent and $(h+g)|A_\alpha \subseteq f$,

(b): $\langle 0, m_\alpha\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h|A_\alpha) \cap (\{0\} \times \mathbb{R}) = \mathrm{Lin}_{\mathbb{Q}}(g|A_\alpha) \cap (\{0\} \times \mathbb{R})$,

where $A_\alpha = \{ia_\xi \colon \xi \leq \alpha, i = -1, 1\}$.

For $\alpha = 0$ and $x = \pm a_0$ put $h(x) = \frac{1}{4} m_0$ and $g(x) = f(x) - h(x)$. Observe that $g(-a_0) + g(a_0) = [f(-a_0) + f(a_0)] - [h(-a_0) + h(a_0)] \in \{-\frac{1}{2} m_0, \frac{1}{2} m_0\}$. This holds because $f(-a_0) + f(a_0) = c$ and $m_0 = c$ if $c \neq 0$. Thus $\langle 0, c\rangle, \langle 0, m_0\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h|A_0) \cap \mathrm{Lin}_{\mathbb{Q}}(g|A_0)$. It is easily seen that $h|A_0$ and $g|A_0$ satisfy (a) and (b).

Now suppose that h and g are defined on $A_{<\alpha} = \bigcup_{\xi<\alpha} A_\xi$, $\alpha < \kappa$, and they satisfy the conditions (a) and (b) for all $\xi < \alpha$. We will extend h and g onto A_α preserving the desired properties.

We may assume that $\langle 0, m_\alpha\rangle \notin \mathrm{Lin}_{\mathbb{Q}}(h|A_{<\alpha}) \cup \mathrm{Lin}_{\mathbb{Q}}(g|A_{<\alpha})$. (Otherwise we could extend h and g using Lemma 3.8 preserving the condition (a).) Put $h(x) = \frac{1}{2} m_\alpha$ and $g(x) = f(x) - h(x)$ for $x \in \{-a_\alpha, a_\alpha\}$. We claim that (a) and (b) are satisfied. Obviously, $(h+g)|A_\alpha \subseteq f$. To see the linear independence of $h|A_\alpha$ and $g|A_\alpha$ first note that, based on the inductive assumption, $h|A_{<\alpha}$ and $g|A_{<\alpha}$ are linearly independent. Next suppose that

$$p\langle -a_\alpha, h(-a_\alpha)\rangle + q\langle a_\alpha, h(a_\alpha)\rangle = v \text{ for some } p, q \in \mathbb{Q} \text{ and } v \in \mathrm{Lin}_{\mathbb{Q}}(h|A_{<\alpha}).$$

Since $a_\alpha \notin \mathrm{Lin}_{\mathbb{Q}}(A_{<\alpha})$ we conclude that $p = q$. Therefore we have

$$p\langle -a_\alpha, h(-a_\alpha)\rangle + q\langle a_\alpha, h(a_\alpha)\rangle = p\langle 0, h(-a_\alpha) + h(a_\alpha)\rangle = p\langle 0, m_\alpha\rangle = v.$$

But we assumed that $\langle 0, m_\alpha\rangle \notin \mathrm{Lin}_{\mathbb{Q}}(h|A_{<\alpha}) \cup \mathrm{Lin}_{\mathbb{Q}}(g|A_{<\alpha})$, so $p = 0$ and $v = 0$. This shows linear independence of $h|A_\alpha$. Very similar argument works for $g|A_\alpha$: just notice that $g(-a_\alpha) + g(a_\alpha) = [f(-a_\alpha) + f(a_\alpha)] - [h(-a_\alpha) + h(a_\alpha)] = c - m_\alpha$ and recall that $\langle 0, c\rangle \in \mathrm{Lin}_{\mathbb{Q}}(g|A_0) \subseteq \mathrm{Lin}_{\mathbb{Q}}(g|A_{<\alpha})$.

Now we show that (b) is also satisfied. From what has already been proved, we conclude that $\langle 0, m_\alpha\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h|A_\alpha) \cup \mathrm{Lin}_{\mathbb{Q}}(g|A_\alpha)$.

Thus, what remains to prove is the equality part of (b). (The following argument is also needed in the case when Lemma 3.8 was used to define h and g on $\{-a_\alpha, a_\alpha\}$.) It follows from the fact that $\langle 0, y\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h|A_\alpha)$ provided there exist $p_i \in \mathbb{Q}$ and $a_i \in A_\alpha$, $i \leq n$ such that

$$\begin{aligned}
\langle 0, y\rangle &= \sum_1^m p_i[\langle -a_i, h(-a_i)\rangle + \langle a_i, h(a_i)\rangle] \\
&= \sum_1^m p_i\langle 0, h(-a_i) + h(a_i)\rangle \\
&= \sum_1^m p_i\langle 0, f(-a_i) + f(a_i)\rangle - p_i\langle 0, g(-a_i) + g(a_i)\rangle \\
&= \sum_1^m p_i\langle 0, c\rangle - \sum_1^m p_i[\langle -a_i, g(-a_i)\rangle + \langle a_i, g(a_i)\rangle] \\
&\in \mathrm{Lin}_{\mathbb{Q}}(g|A_\alpha).
\end{aligned}$$

This completes the inductive definition of h and g. Note that (3.3) implies that any extensions h', g' of h and g, with $h' + g' \subseteq f$, satisfy also

$$\{0\} \times \mathrm{LC}(f) \subseteq \mathrm{Lin}_{\mathbb{Q}}(h') \cap (\{0\} \times \mathbb{R}) = \mathrm{Lin}_{\mathbb{Q}}(g') \cap (\{0\} \times \mathbb{R}). \tag{3.4}$$

To see this choose $\langle 0, y\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h') \cap (\{0\} \times \mathbb{R})$. So for some $p_i \in \mathbb{Q}$ and $x_i \in \mathbb{R}^n$ we have $\langle 0, y\rangle = \sum_1^m p_i\langle x_i, h'(x_i)\rangle = \sum_1^m p_i\langle x_i, f(x_i)\rangle - \sum_1^m p_i\langle x_i, g'(x_i)\rangle \in \mathrm{Lin}_{\mathbb{Q}}(g') \cap (\{0\} \times \mathbb{R})$. The latter holds because $\sum_1^m p_i x_i = 0$ and consequently $\sum_1^m p_i f(x_i) \in \mathrm{LC}(f)$. This ends the proof of (3.4).

Next we extend h and g onto $\mathbb{R}^n = \{x_\xi : \xi < \mathfrak{c}\}$, preserving the linear independence and the property that at a step ξ of the inductive definition we assure that

$x_\xi \in \mathrm{dom}(h_\xi) = \mathrm{dom}(g_\xi)$ and $v_\xi \in \mathrm{Lin}_{\mathbb{Q}}(h_\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(g_\xi)$, where h_ξ and g_ξ denote the extensions obtained in the step ξ.

Let $\beta < \mathfrak{c}$. Assume that $v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(\bigcup_{\xi<\beta} h_\xi) \cup \mathrm{Lin}_{\mathbb{Q}}(\bigcup_{\xi<\beta} g_\xi)$. Choose an $a \in \mathbb{R} \setminus \mathrm{Lin}_{\mathbb{Q}}(\mathrm{dom}(\bigcup_{\xi<\beta} h_\xi))$ and define $h(x)$ by $\langle 0, h(x)\rangle = \frac{1}{2} v_\beta$ for $x \in \{-a, a\}$. Put also $g(x) = f(x) - h(x)$. Since $f(-a) + f(a) \in \mathrm{LC}(f)$, (3.3) implies that $v_\beta \in \mathrm{Lin}_{\mathbb{Q}}(h) \cap \mathrm{Lin}_{\mathbb{Q}}(g)$. What remains to show is that h and g are still linearly independent. But this follows from (3.4) and almost the same argument which is used to show linear independence of $h|A_\alpha$ and $g|A_\alpha$ in the previous part of the proof. (Replace a_α, $h|A_{<\alpha}$, and $g|A_{<\alpha}$ by a, $\bigcup_{\xi<\beta} h_\xi$, and $\bigcup_{\xi<\beta} g_\xi$, respectively.)

To finish the step β of the inductive definition we need to make sure that h and g are defined at x_β. If $x_\beta \notin \mathrm{dom}(h) = \mathrm{dom}(g)$ then we can use Lemma 3.8 to define these functions at x_β, preserving all the required properties. This ends the construction in Case 1.

Case 2: Property 3.7 (a) holds.

Let $Z \in [\mathbb{R}^n]^{<\mathfrak{c}}$ be a set satisfying $\left|\bigcup_{z\in Z} \mathrm{LC}(f, 2, z)\right| = \mathfrak{c}$. We start with defining functions $h, g\colon Z \to \mathbb{R}$ which are linearly independent over $\mathbb{Q}$ and whose sum is contained in f (i.e., $h + g \subseteq f$.) It can be easily done by using Lemma 3.8.

We will extend h and g onto $\mathbb{R}^n$ by induction. Let $\beta < \mathfrak{c}$. Assume that h and g are linearly independent, $h + g \subseteq f$, $\{x_\xi \colon \xi < \beta\} \subseteq D_\beta = \mathrm{dom}(h) = \mathrm{dom}(g)$, $\{v_\xi \colon \xi < \beta\} \subseteq \mathrm{Lin}_{\mathbb{Q}}(h) \cap \mathrm{Lin}_{\mathbb{Q}}(g)$, and $v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(h)$. The property of the set Z implies the existence of a $z \in Z$ satisfying $|\mathrm{LC}(f, 2, z)| > \max(|h|, \omega) = \max(|g|, \omega)$. Thus, an easy cardinal argument shows that we can find $z_1, z_2 \in \mathbb{R}^n \setminus \mathrm{Lin}_{\mathbb{Q}}(D_\beta)$ and $p_1, p_2 \in \mathbb{Q} \setminus \{0\}$ which satisfy

$$p_1 z_1 + p_2 z_2 = z \text{ and } \langle z, p_1 f(z_1) + p_2 f(z_2)\rangle \notin \mathrm{Lin}_{\mathbb{Q}}(g \cup \{\langle 0, h(z)\rangle, v_\beta\}). \tag{3.5}$$

Define the values of h at z_1 and z_2 so that

$$p_1\langle z_1, h(z_1)\rangle + p_2\langle z_2, h(z_2)\rangle = \langle z, p_1 h(z_1) + p_2 h(z_2)\rangle = v_\beta + \langle z, h(z)\rangle.$$

Observe that $v_\beta = [v_\beta + \langle z, h(z)\rangle] - \langle z, h(z)\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h)$.

Now we argue that h and g are linearly independent. To see linear independence of h suppose that for some $q, r \in \mathbb{Q}$ (not both equal 0) we have

$$q\langle z_1, h(z_1)\rangle + r\langle z_2, h(z_2)\rangle = \langle q z_1 + r z_2, q h(z_1) + r h(z_2)\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h|D_\beta).$$

Since $z_1, z_2 \notin \mathrm{Lin}_{\mathbb{Q}}(D_\beta)$ and $p_1 z_1 + p_2 z_2 = z \in Z \subseteq \mathrm{Lin}_{\mathbb{Q}}(D_\beta)$ we conclude that $\langle q, r\rangle$ and $\langle p_1, p_2\rangle$ are linearly dependent. So we may assume that $\langle q, r\rangle = \langle p_1, p_2\rangle$. Consequently, $v_\beta + \langle z, h(z)\rangle = \langle z, p_1 h(z_1) + p_2 h(z_2)\rangle \in \mathrm{Lin}_{\mathbb{Q}}(h|D_\beta)$. This contradicts the assumption $v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(h|D_\beta)$. Hence, h is linearly independent.

Based on the above argument, we see that linear independence of g will follow from $\langle z, p_1 g(z_1) + p_2 g(z_2)\rangle \notin \mathrm{Lin}_{\mathbb{Q}}(g|D_\beta)$. But this holds since (3.5) implies

$$\begin{aligned} \langle z, p_1 g(z_1) + p_2 g(z_2)\rangle &= \\ \langle z, p_1 f(z_1) + p_2 f(z_2) - [p_1 h(z_1) + p_2 h(z_2)]\rangle &= \\ \langle z, p_1 f(z_1) + p_2 f(z_2)\rangle - \langle 0, h(z)\rangle - v_\beta &\notin \mathrm{Lin}_{\mathbb{Q}}(g|D_\beta). \end{aligned}$$

To assure that $v_\beta \in \mathrm{Lin}_{\mathbb{Q}}(g)$ we repeat the same procedure as above for the function g. Finally, if $x_\beta \notin \mathrm{dom}(h) = \mathrm{dom}(g)$ then we use Lemma 3.8 to define the functions at x_β. This ends the construction in Case 2.

Case 3: Property 3.7 (b) holds.

The inductive construction of functions h and g is somewhat similar to the one from the previous case. So assume that $\beta < \mathfrak{c}$ and the construction has been carried out for all $\xi < \beta$. If $v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(h)$ then let $X = \mathrm{dom}(h) = \mathrm{dom}(g)$ and $Y \in [\mathbb{R}]^{<\mathfrak{c}}$ be such a set that $\mathrm{Lin}_{\mathbb{Q}}(g \cup \{v_\beta\}) \subseteq \mathbb{R}^n \times Y$. By Property 3.7 (b), there exist $p_1, p_2, p_3 \in \mathbb{Q} \setminus \{0\}$ and pairwise independent $x_1, x_2, x_3 \in \mathbb{R}^n$ such that $\sum_1^3 p_i x_i = 0$, $\mathrm{Lin}_{\mathbb{Q}}(x_1, x_2, x_3) \cap \mathrm{Lin}_{\mathbb{Q}}(X) = \{0\}$, and $\sum_1^3 p_i f(x_i) \notin Y$.

We extend h and g onto $\{x_1, x_2, x_3\}$. Choose $h(x_1), h(x_2), h(x_3) \in \mathbb{R}$ in such a way that

$$\sum_1^3 p_i \langle x_i, h(x_i)\rangle = \left\langle 0, \sum_1^3 p_i h(x_i) \right\rangle = v_\beta.$$

Then put $g(x_i) = f(x_i) - h(x_i)$ for $i \leq 3$. Obviously $v_\beta \in \mathrm{Lin}_{\mathbb{Q}}(h)$ and $h + g \subseteq f$. We claim that the linear independence of h and g is also preserved.

To show this claim note first that, if $\sum_1^3 p'_i x_i \in \mathrm{Lin}_{\mathbb{Q}}(X)$ for some $p'_1, p'_2, p'_3 \in \mathbb{Q}$ then $\sum_1^3 p'_i x_i = 0$. Pairwise independence of x_1, x_2, x_3 implies that $\sum_1^3 p'_i x_i = 0$ holds only for triples $\langle p'_1, p'_2, p'_3\rangle \in \mathrm{Lin}_{\mathbb{Q}}(\langle p_1, p_2, p_3\rangle)$. Thus, our claim holds if $\sum_1^3 p_i \langle x_i, h(x_i)\rangle \notin \mathrm{Lin}_{\mathbb{Q}}(h|X)$ and $\sum_1^3 p_i \langle x_i, g(x_i)\rangle \notin \mathrm{Lin}_{\mathbb{Q}}(g|X)$. But these two conditions follow from

- $\sum_1^3 p_i \langle x_i, h(x_i)\rangle = v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(h|X)$ and
- $\sum_1^3 p_i \langle x_i, g(x_i)\rangle = \sum_1^3 p_i \langle x_i, f(x_i) - h(x_i)\rangle = \langle 0, \sum_1^3 p_i f(x_i)\rangle - v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(g|X)$
  ("$\notin$" part holds because $\mathrm{Lin}_{\mathbb{Q}}((g|X) \cup \{v_\beta\}) \subseteq \mathbb{R}^n \times Y$ and $\sum_1^3 p_i f(x_i) \notin Y$.)

To assure that $v_\beta \in \mathrm{Lin}_{\mathbb{Q}}(g)$ we repeat the same steps as above for the function g and then, if necessary, define h and g at x_β using Lemma 3.8. This ends the construction in Case 3. ∎

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