Ideals of compact sets associated with Borel functions

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Abstract

We investigate the connection between the Borel class of a function f and the Borel complexity of the set $\mathcal{C}(f) = \{C \in \mathrm{J}(X) \colon f|_C \text{ is continuous}\}$ where $\mathrm{J}(X)$ denotes the compact subsets of X with the Hausdorff metric. For example, we show that for a function $f \colon X \to Y$ between Polish spaces; if $\mathcal{C}(f)$ is $F_{\sigma\delta}$ in $\mathrm{J}(X)$, then f is Borel class one.

1 Introduction

Given a Polish space X let $\mathrm{J}(X)$ denote the collection of nonempty compact subsets of X with the Hausdorff metric. We investigate the connection between the Borel class a function f and the Borel complexity of the set $\mathcal{C}(f) = \{C \in \mathrm{J}(X) \colon f|_C \text{ is continuous}\}$. Generally, the set $\mathcal{C}(f)$ is an ideal. One can see the subject of this paper from at least two directions. First, one can see the complexity of $\mathcal{C}(f)$ as a measure of how discontinuous f is, since for f continuous $\mathcal{C}(f) = \mathrm{J}(X)$ which is a very simple set. Secondly, descriptive set theorist have an interest in finding natural examples of objects such as ideals of compact sets which are complex.

2 Preliminaries

Let X be a set. We let $|X|$ denote the cardinality of X. Given a cardinal κ we let $[X]^{<\kappa}$, $[X]^{\leq\kappa}$ and, $[X]^{\kappa}$ denote the subsets of X of cardinality strickly less than κ, less or equal to κ, and equal to κ, respectively. Given a function $f \colon X \to Y$ and $A \subseteq X$ we let $f|_A$ denote the restriction of f to A. Given a product of two sets $X \times Y$ we let π_X and π_Y denote the usual projections onto X or Y, respectively.

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Suppose X is a Polish space with metric d. For a set $A \subseteq X$ we write $\mathrm{cl}_X(A)$, $\mathrm{int}_X(A)$, $\mathrm{bd}_X(A)$ for the closure, interior, and boundary of A in X, respectively. When it is understood what space we are referring to the subscript will be dropped. Given sets $A, B \subseteq X$ we define $\mathrm{dist}(A, B) = \inf(\{d(x, y) \colon x \in A \ \&\ y \in B\})$. Given sets $A, B \subseteq X$ we define the Hausdorff distance between A and B to be $H_d(A, B) = \max(\sup(\{\mathrm{dist}(\{x\}, B) \colon x \in A\}), \sup(\{\mathrm{dist}(A, \{y\}) \colon y \in B\}))$. When H_d is restricted to the compact subsets of X it is a metric known as the Hausdorff metric. The diameter of a nonempty set $A \subseteq X$ is defined by $\mathrm{diam}(A) = \sup\{d(x, y) \colon x, y \in A\}$, if $A = \emptyset$ we let $\mathrm{diam}(A) = 0$. It is known that if X is Polish, then $\mathrm{J}(X)$ is Polish as well [6, 4.25].

By a Cantor set we mean a compact totally disconnected metric space with no isolated points.

Let X be Polish. By $\mathcal{B}(X)$ we denote the Borel subsets of X as defined in [6, 11.A]. For $0 < \alpha < \omega_1$ let $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$, $\Delta^0_\alpha(X)$ stand for the subclasses of $\mathcal{B}(X)$ defined as in [6, 11.B] (e.g., Π^0_2 is G_δ and Σ^0_2 is F_σ). The analytic subsets of X and the coanalytic subsets of X as defined in [6] will be denoted by $\Sigma^1_1(X)$ and $\Pi^1_1(X)$, respectively. A set $A \subseteq X$ is said to be coanalytic hard provided that for any zero-dimensional Polish space Y and coanalytic $B \subseteq Y$ there is a continuous function $f \colon Y \to X$ such that $f^{-1}(A) = B$. To say that A is coanalytic hard is essentially saying that A is at least as complex as any coanalytic set. In particular, if A is coanalytic hard, then A is neither Borel nor analytic.

If a function $f \colon X \to Y$ has the property that for every open set $U \subseteq Y$ the set $f^{-1}(U) \in \Sigma^0_2(X)$, then we say f is a Borel class one function. Let $\mathcal{B}_1$ denote the Borel class one functions. If a function $f \colon X \to Y$ has the property that for every open set $U \subseteq Y$ the set $f^{-1}(U) \in \mathcal{B}(X)$, then we say f is a Borel function. We let $\mathcal{B}$ denote the Borel functions. For a function $f \colon X \to Y$ and $S \subseteq X$ we let $\mathrm{osc}(f, S) = \sup\{\mathrm{dist}(f(x), f(y)) \colon x, y \in S\}$. For a function $f \colon X \to Y$ we let $D(f)$ denote the set of discontinuity points of f.

We say $f \colon X \to Y$ is a discrete limit of a sequence of functions $\{f_n\}_{n\in\omega}$ provided that for every $x \in X$ there is an $n_x \in \omega$ such that $f_k(x) = f(x)$ for all $k \geq n_x$. For more facts about discrete limits see [4].

3 Results

If a function $f \colon X \to Y$ has the property that for every $x \in X$ there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $x \in U$, $f(x) \in V$, and $f|_{\mathrm{cl}(f^{-1}(V)\cap U)}$ is continuous then we say $f \in \mathrm{T}_0$. If a function $f \colon X \to Y$ has the property that for every $x \in X$ there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $f(x) \in V$, $x \in U$, and $f|_{f^{-1}(V)\cap U}$ is continuous, then we say $f \in \mathrm{T}_1$.

We may now state our theorems.

Theorem 1 *If X and Y are Polish spaces and $f \colon X \to Y$ is a function, then f is continuous if and only if $\mathcal{C}(f) \in \Pi^0_2(\mathrm{J}(X))$.*

Theorem 2 *If X and Y are Polish spaces and $f\colon X \to Y$, then the following are equivalent:*

(i) $f \in \mathrm{T}_0$

(ii) $\mathcal{C}(f) \in \Sigma^0_2(\mathrm{J}(X))$

(iii) *there is a $\subseteq$-increasing sequence $\{T_n\}_{n\in\omega}$ of closed subsets of X such that $\mathcal{C}(f) = \bigcup_{n\in\omega} \mathrm{J}(T_n)$*

(iv) $\mathcal{C}(f) \in \Delta^0_3(\mathrm{J}(X))$.

Moreover, if $Y = \mathbb{R}$ the conditions (i)-(iv) are equivalent to :

(v) *f is open in* $\mathrm{cl}(f)$

(vi) *f is the discrete limit of continuous functions $\{f_n\}_{n\in\omega}$ such that $\mathcal{C}(f) = \{C \in \mathrm{J}(X)\colon \{f_n|_C\}_{n\in\omega}$ is eventually constant$\}$.*

Theorem 3 *If X and Y are Polish spaces and $f\colon X \to Y$, then $(i) \Rightarrow (ii) \Rightarrow (iii)$ where :*

(i) $\mathcal{C}(f) \in \Pi^0_3(\mathrm{J}(X))$

(ii) $f \in \mathcal{B}_1$

(iii) $\mathcal{C}(f) \in \Pi^0_4(\mathrm{J}(X))$

and none of the implications may be reversed. Moreover, there is a $\mathcal{B}_1$ function f such that $\mathcal{C}(f) \notin \Sigma^0_4(\mathrm{J}(X))$.

Theorem 4 *If X and Y are Polish spaces, and $f\colon X \to Y$, then the following are equivalent:*

(i) $\mathcal{C}(f) \in \Sigma^0_3(\mathrm{J}(X))$

(ii) *$f \in \mathrm{T}_1$ and f has G_δ-graph.*

The following theorem shows the importance of the assumption in Theorem 4 (ii) that f has G_δ-graph:

Theorem 5 *If X and Y are Polish and $f\colon X \to Y$ is Borel, then the following are equivalent:*

(i) *$\mathcal{C}(f)$ is Borel,*

(ii) *f has G_δ graph, and*

(iii) *$\mathcal{C}(f|_A)$ is coanalytic hard for no $A \in \mathrm{J}(X)$.*

In particular, let g be the characteristic function of the rationals. Clearly, $g \in \mathrm{T}_1$ but does not have G_δ-graph, so $\mathcal{C}(g) \notin \Sigma^0_3(X)$.

We note the following propositions which will be used repeatedly:

Proposition 6 *([6, 23.1]) The set*

$$\{\sigma \in 2^{\omega\times\omega} \colon (\forall m \in \omega)(\exists k \in \omega)(\forall n \geq k)(\sigma(\langle m,n\rangle)) = 0\}.$$

is in $\Pi^0_3(2^{\omega\times\omega}) \setminus \Sigma^0_3(2^{\omega\times\omega})$.

We will let H denote the subset of $2^{\omega\times\omega}$ described in Proposition 6.

Proposition 7 *[6, 23.6] The set*

$$\{\sigma \in 2^{\omega\times\omega} \colon (\exists l \in \omega)(\forall m \geq l)(\exists k \in \omega)(\forall n \geq k)(\sigma(\langle m,n\rangle) = 0)\}.$$

is in $\Sigma^0_4(2^{\omega\times\omega}) \setminus \Pi^0_4(2^{\omega\times\omega})$.

We will let I denote the subset of $2^{\omega\times\omega}$ described in Proposition 7.

4 Proof of Theorem 1

If $f\colon X \to Y$ is continuous, then $\mathcal{C}(f) = \mathrm{J}(X) \in \Pi^0_2(\mathrm{J}(X))$. Suppose now that $f\colon X \to Y$ is not continuous. There exist $x \in X$ and $x_n \in X$ such that $\lim_{n\to\infty} x_n = x$ and no subsequence of $\{f(x_n)\}_{n\in\omega}$ converges to $f(x)$. Let $A = \{x_n \colon n \in \omega\} \cup \{x\}$. Notice that $B = \{Y \in \mathrm{J}(A) \colon x \in Y\}$ is compact in $\mathrm{J}(X)$ and has no isolated points. Clearly, a compact set $K \in \mathcal{C}(f) \cap B$ if and only if K is finite. Since the finite members of B form a countable dense subset of B, we have $B \cap \mathcal{C}(f) \in \Sigma^0_2(\mathrm{J}(X)) \setminus \Pi^0_2(\mathrm{J}(X))$. Thus, $\mathcal{C}(f) \notin \Pi^0_2(\mathrm{J}(X))$. ■

5 Proof of Theorem 5

We begin with two lemmas the first being a version of the Blumberg Theorem [3] the proof of which is similar to the method used in [1].

Lemma 8 *Let X and Y be separable metric spaces with $|X| > 1$. If $f\colon X \to Y$ has no isolated points, then there is a nonempty set $D \subseteq X$ such that D has no isolated points and $f|_D$ is continuous.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be countable bases for X and Y, respectively. We may assume that both bases are closed under the operation $W_1 \setminus \mathrm{cl}(W_2)$ where $W_1, W_2 \in \mathcal{U}$ or $W_1, W_2 \in \mathcal{V}$. Let $\mathcal{R}$ denote the rational rectangles, i.e., sets of the form $U \times V$ where $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Let $X_1 \subseteq X$ be a countable set such that $f|_{X_1}$ is dense in f. Notice that $f|_{X_1}$ has no isolated points. Let $A \subseteq X_1$ and $|A| > 1$. We define the mesh of A to be $\mathrm{mesh}(A) = \sup\{\mathrm{dist}(x, A\setminus\{x\}) \colon x \in A\}$. Let P be the collection of all pairs $(A, S) \in [X_1]^{<\omega} \times [\mathcal{R}]^{<\omega}$ such that

(0) $|A| > 1$,

(1) $\pi_X[R_1] \cap \pi_X[R_2] = \emptyset$ for all distinct $R_1, R_2 \in S$, and

(2) $f|_A \subseteq \cup S$.

We say $(A_1, S_1) \leq (A_2, S_2)$ provided $A_2 \subseteq A_1$, $\mathrm{mesh}(A_1) \leq \mathrm{mesh}(A_2)$, and $\cup S_1 \subseteq \cup S_2$. Now $(P, \leq)$ is a reflexive and transitive ordering.

For each $x \in X_1$ and $n > 0$, let

$$E_n^x = \{(A, S) \in P\colon \text{if } \langle x, f(x)\rangle \in T \in S, \text{ then } \mathrm{diam}(\pi_Y[T]) < 1/n\}.$$

We show E_n^x is dense in P. Let $(A, S) \in P$. If $\langle x, f(x)\rangle \notin \cup S$, then $(A, S) \in E_n^x$ by failure of hypothesis. So we may asume that $\langle x, f(x)\rangle \in T$ for some $T \in S$. Pick $V \in \mathcal{V}$ so that $\mathrm{diam}(V) < 1/n$ and $f(x) \in V$. If $x \in A$, then pick U open such that $\mathrm{cl}(U) \subseteq \pi_X(T)$ and $\{x\} = A \cap U = A \cap \mathrm{cl}(U)$. If $x \notin A$, then pick an open set U such that $\mathrm{cl}(U) \subseteq \pi_X[T]$ and $A \cap U = \emptyset$. Let $S^* = (S \setminus \{T\}) \cup \{U \times V, (\pi_X[T] \setminus \mathrm{cl}(U)) \times (\pi_Y[T] \setminus \mathrm{cl}(V))\}$. Now $(A, S^*) \leq (A, S)$ and $(A, S^*) \in E_n^x$. So, E_n^x is dense for all $x \in X_1$ and $n \in \omega$.

For each $n > 0$, let

$$F_n = \{(A, S)\colon \mathrm{dist}(\{x\}, A \setminus \{x\}) < 1/n \text{ for all } x \in A\}.$$

We show F_n is dense in P. Let $(A, S) \in P$. Fix $x \in A$. Since $(A, S) \in P$, there is a $T \in S$ such that $\langle x, f(x)\rangle \in T$. Since T is open and $f|_{X_1}$ has no isolated points we can find an $x^* \in X_1 \setminus A$ such that $\langle x^*, f(x^*)\rangle \in T$ and $\mathrm{dist}(x, x^*) < \min\{\mathrm{mesh}(A), 1/n\}$. Let $A^* = A \cup \{x^*\colon x \in A\}$. Now $(A^*, S) \leq (A, S)$ and $(A^*, S) \in F_n$. So, F_n is dense in P for all $n > 0$.

Since $|\{E_n^x\colon x \in X_1 \;\&\; n > 0\} \cup \{F_n\colon n > 0\}| \leq \omega$ we may find a filter $G \subseteq P$ such that G has nonempty intersection with each of the dense sets defined. Let $D = \bigcup\{A\colon (A, S) \in G\}$. For every $(A, S) \in G$ we have $f|_D \subseteq \cup S$. To see it let $x \in D$ and $(A, S) \in G$. By definition of D, there is an $(A_1, S_1) \in G$ such that $x \in A_1$. Pick $(A_2, S_2) \in G$ such that $(A_2, S_2) \leq (A, S)$ and $(A_2, S_2) \leq (A_1, S_1)$. Since $x \in A_2$ there is a $T \in S_2$ such that $\langle x, f(x)\rangle \in T$. Thus, $\langle x, f(x)\rangle \in T \subseteq \cup S_2 \subseteq \cup S$.

We show that $f|_D$ is continuous. Let $x \in D$ and $\epsilon > 0$. Pick $n > 0$ such that $1/n < \epsilon$ and pick $(A, S) \in G \cap E_n^x$. Since $x \in D$, there is an $(A_1, S_1) \in G$ such that $x \in A_1$. Pick $(B, M) \in G$ such that $(B, M) \leq (A_1, S_1)$ and $(B, M) \leq (A, S)$. Now $x \in B$ so there is a $N \in M$ such that $\langle x, f(x)\rangle \in N$. Since $(B, M) \leq (A, S)$, we have $N \subseteq \cup M \subseteq \cup S$. So, $\langle x, f(x)\rangle \in \cup S$. Hence, there is a $T \in S$ such that $\langle x, f(x)\rangle \in T$ and $\pi_Y[T] < 1/n$. Since $f|_{D \cap \pi_X[T]} \subseteq T$, there is an open neighborhood U of x such that $\mathrm{dist}(f(x), f(w)) < 1/n < \epsilon$ for all $w \in U \cap D$. Therefore, $F|_D$ is continuous.

We now show that D has no isolated points. Let $x \in D$ and $\epsilon > 0$. Pick $n > 0$ such that $1/n < \epsilon$. There is an $(A, S) \in G$ such that $x \in A$. Pick $(A_1, S_1) \in G \cap F_n$. Pick $(A_2, S_2) \in G$ such that $(A_2, S_2) \leq (A, S)$ and $(A_2, S_2) \leq (A_1, S_1)$. Now $x \in A_2$. Since $(A_2, S_2) \leq (A_1, S_1)$ and $\mathrm{mesh}(A_1) < 1/n$, we have $\mathrm{mesh}(A_2) < 1/n$. Thus, there is a $w \in A_2 \subseteq D$ such that $\mathrm{dist}(x, w) < 1/n < \epsilon$. Thus, D has no isolated points. ∎

Lemma 9 *Let C be a Cantor set and $D \subseteq C$ be countable and dense. If $\mathcal{S}$ is the collection of all $K \in \mathrm{J}(C)$ such that $K \cap D$ and $K \cap (C \setminus D)$ are both compact, then $\mathcal{S}$ is coanalytic hard.*

PROOF. Let $N \subseteq 2^\omega$ be the set all binary sequences τ such that $\tau^{-1}(1)$ is finite. Notice that N is countable and dense in 2^ω. It is well known [6, 33.B] that $I = \{K \in \mathrm{J}(2^\omega) \colon K \subseteq N\}$ is a coanalytic hard set. For $K \subseteq 2^\omega$ and $n \in \omega$ let $K|_n = \{\sigma|_n \colon \sigma \in K\}$.

Define $\Theta \colon \mathrm{J}(2^\omega) \to \mathrm{J}(2^\omega)$ by

$$\Theta(K) = \mathrm{cl}(\bigcup_{n \in \omega} \{\sigma \in 2^\omega \colon \sigma|_n \in K|_n \ \&\ (\forall k \geq n)(\sigma(k) = 0)\})$$

for every $K \in \mathrm{J}(2^\omega)$. It is easily seen that Θ is continuous. It should be clear that $\Theta(K) \subseteq N$ if $K \subseteq N$. On the other hand, suppose $K \setminus N \neq \emptyset$. Let $k \in K \setminus N$. Now $k \in \Theta(K)$ and there exist $\{n_l \in N\}_{l \in \omega}$ such that $\lim_{l \to \infty} n_l = k$. So, in this case $\Theta(K) \cap N$ is not compact. Thus, $\Theta^{-1}(\mathcal{S}) = I$. Therefore, $\mathcal{S}$ is coanalytic hard. ■

Lemma 10 *Let X be Polish. If $G \in \Pi^0_2(X)$ is dense and D is a dense set disjoint from G, then there is a countable $E \subseteq D$ such that E is dense in X and $G \cup E \in \Pi^0_2(X)$.*

PROOF.

Let $X \setminus G = \bigcup_{n \in \omega} F_n$ where each F_n is closed. We may assume that $\lim_{n \to \infty} \mathrm{diam}(F_n) = 0$ and that $F_m \setminus \bigcup_{n<m} F_n \neq \emptyset$ for every $m \in \omega$. Fix $m \in \omega$. If $(F_m \setminus \bigcup_{n<m} F_n) \cap D \neq \emptyset$, pick $e_m \in (F_m \setminus \bigcup_{n<m} F_n) \cap D$. Let $E = \{e_m \colon D \cap (F_m \setminus \bigcup_{n<m} F_n) \neq \emptyset\}$. Clearly, $E \in \Sigma^0_2(X)$ and $E \subseteq D$.

Let $U \subseteq X$ be a nonempty open set. Since F_n is nowhere dense for every $n \in \omega$ and D is dense, there is no $N \in \omega$ such that $D \cap U \subseteq \bigcup_{n \leq N} F_n$. Thus, there is a $d \in D$ and a $k \in \omega$ such that $(F_k \setminus \bigcup_{n<k} F_n) \cap D \neq \emptyset$ and $F_k \subseteq U$. Now $e_k \in U$. Thus, E is dense in X.

Since $E \cup G = X \setminus (\bigcup_{m \in \omega} F_m \setminus E)$ and $F_m \cap E$ is finite for every $m \in \omega$, we have $E \cup G \in \Pi^0_2(X)$. ■

PROOF OF THEOREM 5 Clearly, (i) implies (iii).

We show that (ii) implies (i). First notice that the set $\mathrm{J}(f)$ is a G_δ-subset of $\mathrm{J}(X \times Y)$ and that $\mathcal{C}(f)$ is an injective continuous image of $\mathrm{J}(f)$ by the function $\Theta \colon \mathrm{J}(f) \to \mathrm{J}(X)$ defined by $\Theta(K) = \pi_X[K]$. Since $\mathcal{C}(f)$ is an injective image of a Borel set, $\mathcal{C}(f)$ is Borel by [6, 15.1].

We now show that (iii) implies (ii). Suppose $f \colon X \to Y$ is Borel and f does not have G_δ-graph. Since f is Borel we have that f is a Π^1_1-subset of $X \times Y$. By a theorem of Hurewicz [6, 21.18], there is a relatively closed subset B of f such that B is homeomorphic to the rational numbers. Let $B_1 = \pi_X(B)$. Since $f|_{B_1}$ has no isolated points, by Lemma 8, there is a $B_2 \subseteq B_1$ such that $f|_{B_2}$ is continuous and B_2 has no isolated points. Let $C \subseteq X$ be a Cantor set such that B_2 is dense in C. Since f is Borel, there is a dense G_δ-subset D of C such that $f|_D$ is continuous. By Lemma 10 there is a dense subset B_3 of B_2 such that $D \cup B_3 \in \Pi^0_2(C)$. Notice B_3 has no isolated points. Let $d \in D$. Since B is relatively closed in f, there exist $\epsilon, \delta > 0$ such that for any $x \in \mathrm{B}_\delta(d) \cap D$ and $w \in \mathrm{B}_\delta(d) \cap B_3$ we have $|f(x) - f(w)| > \epsilon$. Let $C_1 \subseteq \mathrm{B}_\delta(d) \cap (D \cup B_3)$ be a Cantor set such that $B_3 \cap C_1$ is countable and dense in C_1. It is clear that $\mathcal{C}(f|_{C_1})$ is exactly the compact subsets P of C_1 with the property that both $B_3 \cap P$ and $(C_1 \setminus B_3) \cap P$ are both compact. By Lemma 9, $\mathcal{C}(f|_{C_1})$ is coanalytic hard. ■

6 Proof of Theorem 4

Suppose X and Y are Polish spaces and $f\colon X \to Y$.

We show that (i) implies (ii).

Lemma 11 *If $\mathcal{C}(f) \in \Sigma^0_3(\mathrm{J}(X))$, then $f \in \mathrm{T}_1$.*

Proof. Suppose $f \notin \mathrm{T}_1$. Let $x \in X$ be such that for every pair of open sets $U \subseteq X$ and $V \subseteq Y$ with $x \in U$ and $f(x) \in V$ we have $f|_{f^{-1}(V)\cap U}$ not continuous. Let $\{V_n\}_{n\in\omega}$ be a decreasing sequence of open subsets of Y such that $H_d(V_n, f(x)) < 1/2^n$ and $f(x) \in V_n$ for every $n \in \omega$. Let $\{U_n\}_{n\in\omega}$ be a decreasing sequence of open subsets of X such that $H_d(U_n, x) < 1/2^n$ and $x \in U_n$ for every $n \in \omega$. For each $n \in \omega$ pick $x_n \in D(f|_{f^{-1}(V_n)\cap U_n})$. For each $n \in \omega$ we may find $\{w_{n,k} \in f^{-1}(V_n) \cap U_n\}_{k\in\omega}$ such that $\lim_{k\to\infty} w_{n,k} = x_n$ and no subsequence of $\{f(w_{n,k})\}_{k\in\omega}$ converges to $f(x_n)$, we may also assume that $\mathrm{cl}(\{\langle w_{n,k}, f(w_{n,k})\rangle : k \in \omega\}) \cap \mathrm{cl}(\{\langle w_{m,k}, f(w_{m,k})\rangle : k \in \omega\}) = \emptyset$ for all $n, m \in \omega$ such that $n \neq m$. Let $C = \mathrm{cl}(\{w_{n,k} : n, k \in \omega\})$. Define $h\colon 2^{\omega\times\omega} \to \mathrm{J}(C)$ by $h(\sigma) = \{w_{n,k} : \sigma(\langle n, k\rangle) = 1\} \cup \{x_n : n \in \omega\} \cup \{x\}$. Notice that h is continuous. It is straight forward to check that $f|_{h(\sigma)}$ is continuous if and only if

$$(\forall m \in \omega)(\exists k \in \omega)(\forall n \geq k)(\sigma(\langle m, n\rangle)) = 0.$$

Thus, $h^{-1}(\mathcal{C}(f) \cap \mathrm{J}(C)) = H$. By Proposition 6 and the continuity of h, we have $\mathcal{C}(f) \cap \mathrm{J}(C) \notin \Sigma^0_3(\mathrm{J}(X))$. Since $\mathrm{J}(C)$ is closed, $\mathcal{C}(f) \notin \Sigma^0_3(\mathrm{J}(X))$. ■

We now show that (ii) implies (i). We first define an operation M on collections of subsets of product spaces. Given a collection $\mathcal{A}$ of subsets of $X \times Y$. Define

$$M(\mathcal{A}) = \bigcup_{x\in X} \left(\pi_X^{-1}(\{x\}) \cap \bigcap\{A \in \mathcal{A} : x \in \pi_X[A]\}\right).$$

Lemma 12 *If $f\colon X \to Y$ is a function and $\mathcal{A}$ is a finite collection of subsets of $X \times Y$ such that $\pi_X[A]$ is closed for every $A \in \mathcal{A}$ and $f|_{\pi_X[A\cap f]}$ is continuous for each $A \in \mathcal{A}$, then $f|_{\pi_X[M(\mathcal{A})\cap f]}$ is continuous.*

Proof. Let $\{x_n\}_{n\in\omega}$ be a sequence of points in $\pi_X[M(\mathcal{A})\cap f]$ which converges to some $x \in \pi_X[M(\mathcal{A})\cap f]$. Since $\mathcal{A}$ is finite, we may assume that there is an $A \in \mathcal{A}$

such that $x_n \in \pi_X[A \cap f]$ for every $n \in \omega$. Since A has closed X-projection, $x \in \pi_X[A]$. Since $x \in \pi_X[M(\mathcal{A}) \cap f]$ and $x \in \pi_X[A]$, we have $\langle x, f(x)\rangle \in A$. In particular, $\{x_n \colon n \in \omega\} \cup \{x\} \subseteq \pi_X[A \cap f]$. Thus, $\lim_{n\in\omega} f(x_n) = f(x)$. Therefore, $f|_{\pi_X[M(\mathcal{A})\cap f]}$ is continuous. ∎

Lemma 13 *If $\mathcal{A}$ is a finite collection of closed subsets of $X \times Y$ such that $\pi_X[A]$ is closed for every $A \in \mathcal{A}$, then $M(\mathcal{A}) \in \Pi^0_2(X \times Y)$.*

PROOF. Notice that for every $A \in \mathcal{A}$ we have

$$A \cup ((\pi_X[\cup\mathcal{A}] \setminus \pi_X[A]) \times Y) \in \Pi^0_2(X \times Y).$$

It is easily checked that

$$M(\mathcal{A}) = \bigcap_{A\in\mathcal{A}} \left(A \cup ((\pi_X[\cup\mathcal{A}] \setminus \pi_X[A]) \times Y)\right).$$

Thus, $M(\mathcal{A}) \in \Pi^0_2(X \times Y)$. ∎

Lemma 14 *Let $f \in \mathrm{T}_1$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ be countable bases for X and Y, respectively. If $x \in A \subseteq X$ and $f|_A$ is continuous, then there exist $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that $f|_{f^{-1}(\mathrm{cl}(B_2))\cap \mathrm{cl}(B_1)}$ is continuous and $f[A \cap \mathrm{cl}(B_1)] \subseteq \mathrm{cl}(B_2)$.*

PROOF. Since $f \in \mathrm{T}_1$, there exist open sets $U \subseteq X$ and, $V \subseteq Y$ such that $x \in U$, $f(x) \in V$, and $f|_{f^{-1}(V)\cap U}$ is continuous. Pick $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ so that $\mathrm{cl}(B_1) \subseteq U$, $\mathrm{cl}(B_2) \subseteq V$, $x \in B_1$, and $f(x) \in B_2$. Since $f^{-1}(\mathrm{cl}(B_2)) \cap \mathrm{cl}(B_1) \subseteq f^{-1}(V) \cap U$, we have that $f|_{f^{-1}(\mathrm{cl}(B_2))\cap\mathrm{cl}(B_1)}$ is continuous. Since $f|_A$ is continuous we may assume B_1 is small enough that $f[A \cap \mathrm{cl}(B_1)] \subseteq \mathrm{cl}(B_2)$. ∎

Lemma 15 *If $f \in \mathrm{T}_1$ and f has G_δ-graph, then $\mathcal{C}(f) \in \Sigma^0_3(X)$.*

PROOF. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be countable bases for X and Y respectively. Let $\mathcal{W}$ be the collection of all finite collections $Z = \{W_0, \ldots W_n\}$ of sets of the form $W_i = \mathrm{cl}(B_1) \times \mathrm{cl}(B_2)$ (where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$) such that $f|_{\pi_X[M(Z)\cap f]}$ is continuous. Let $Z \in \mathcal{W}$. By Lemma 13 and the assumption that f has G_δ-graph, $M(Z) \cap f \in \Pi^0_2(X \times Y)$. Since $f|_{\pi_X[M(Z)\cap f]}$ is continuous, $\pi_X[M(Z) \cap f] \in \Pi^0_2(X)$. Thus, $\mathcal{T} = \bigcup\{\mathrm{J}(\pi_X[M(Z) \cap f]) \colon Z \in \mathcal{W}\} \in \Sigma^0_3(X)$.

The proof will be complete if we show that $\mathcal{C}(f) = \mathcal{T}$. The containment $\mathcal{T} \subseteq \mathcal{C}(f)$ is obvious. We work for the opposite containment. Let $C \in \mathcal{C}(f)$. We will construct a finite collection $W = \{W_1, W_2, \ldots W_n\}$ of sets of the form $W_i = \mathrm{cl}(B_1) \times \mathrm{cl}(B_2)$ where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that

(a) $f|_C \subseteq \bigcup W$,

(b) $f|_{\pi_X[f\cap W_i]}$ is continuous for every $1 \leq i \leq n$, and

(c) $f|_{C\cap\pi_X(W_i)} \subseteq W_i$ for every $1 \leq i \leq n$.

By Lemma 14, for every $x \in C$ there exist $B_1^x \in \mathcal{B}_1$ and $B_2^x \in \mathcal{B}_2$ such that $x \in B_1^x$, $f(x) \in B_2^x$, $f|_{f^{-1}(\mathrm{cl}(B_2^x)) \cap \mathrm{cl}(B_2^x)}$ is continuous, and $f[\mathrm{cl}(B_1^x) \cap C] \subseteq \mathrm{cl}(B_2^x)$. Since $f|_C$ is compact, we we may find a finite subcover $W^* = \{W_1^*, \dots W_n^*\}$ of $\{B_1^x \times B_2^x \colon x \in C\}$. For each $1 \leq i \leq n$ let $W_i = \mathrm{cl}(W_i^*)$. The collection $W = \{W_1 \dots W_n\}$ clearly satisfies conditions (a), (b), and (c).

By (b), and Lemma 12, $f|_{\pi_X[f \cap M(W)]}$ is continuous. So $W \in \mathcal{W}$. We will be done if we show that $C \subseteq \pi_X[M(W) \cap f]$. Let $x \in C$. By (a), there is some $W_i \in W$ such that $\langle x, f(x)\rangle \in W_i$. By (c), for any $W_k \in W$ if $x \in \pi_X(W_k)$, then $\langle x, f(x)\rangle \in W_k$. Thus, $\langle x, f(x)\rangle \in M(W) \cap f$. So, $x \in \pi_X[M(W) \cap f]$. Therefore, $C \subseteq \pi_X[M(W) \cap f]$. ■

7 Proof of Theorem 2

Suppose X and Y are Polish spaces and $f \colon X \to Y$.

Lemma 16 *If $\mathcal{C}(f) \in \Delta_3^0(\mathrm{J}(X))$, then $f \in \mathrm{T}_0$.*

Proof. By way of contradiction assume $f \notin \mathrm{T}_0$. Let $x \in X$ be such that for every pair of open sets $U \subseteq X$ and $V \subseteq Y$ with $x \in U$ and $f(x) \in V$ we have $f|_{\mathrm{cl}(f^{-1}(V) \cap U)}$ not continuous. Let $\{V_n\}_{n \in \omega}$ be a decreasing sequence of open subsets of Y such that $H_d(V_n, f(x)) < 1/2^n$ and $f(x) \in V_n$ for every $n \in \omega$. Let $\{U_n\}_{n \in \omega}$ be a decreasing sequence of open subsets of X such that $H_d(U_n, x) < 1/2^n$ and $x \in U_n$ for every $n \in \omega$. By Lemma 11, we may assume that $f|_{f^{-1}(V_0) \cap U_0}$ is continuous.

Fix $n > 0$. Since $f|_{\mathrm{cl}(f^{-1}(V_n) \cap U_n)}$ is not continuous and $f|_{f^{-1}(V_0) \cap U_0}$ is continuous, we may find an $x_n \in \mathrm{cl}(f^{-1}(V_n) \cap U_n) \setminus (f^{-1}(V_0) \cap U_0)$. There exist $\{w_{n,m} \in f^{-1}(V_n) \cap U_n\}_{m \in \omega}$ such that $\lim_{m \to \infty} w_{n,m} = x_n$. Since $x_n \notin f^{-1}(V_0)$, $\lim_{m \to \infty} f(w_{n,m}) \neq f(x_n)$.

Since $x_n \neq x$ for all $n \in \omega$, we may assume that $\mathrm{cl}(\{w_{n+1,m} \colon m \in \omega\}) \cap \mathrm{cl}(\{w_{l+1,m} \colon m \in \omega\}) = \emptyset$ for distinct $n, l \in \omega$. Let $C = \mathrm{cl}(\{w_{n+1,m} \colon n, m \in \omega\})$. We will have a contradiction if we show that $\mathcal{C}(f) \cap \mathrm{J}(C) \notin \Delta_3^0(\mathrm{J}(C))$. Define $h \colon 2^{\omega \times \omega} \to \mathrm{J}(C)$ by the formula $h(\sigma) = \{x\} \cup (\bigcup_{n \in \omega} L_n)$ where

$$L_n = \begin{cases} \emptyset & \text{if } \{m \colon \sigma(\langle n+1, m\rangle) = 1\} = \emptyset \\ \{x_{n+1}\} & \text{if } \{m \colon \sigma(\langle n+1, m\rangle) = 1\} \text{ is infinite;} \\ \{w_{n+1, \max\{m \colon \sigma(\langle n,m\rangle)=1\}}\} & \text{otherwise.} \end{cases}$$

We claim that $h \in \mathcal{B}_1$. For each $l \in \omega$ define $h_l \colon 2^{\omega \times \omega} \to \mathrm{J}$ by the formula $h_l(\sigma) = \{x\} \cup (\bigcup_{n \in \omega} L_{n,l})$ where

$$L_{n,l} = \begin{cases} \{w_{n+1, \max\{m \leq l \colon \sigma(\langle n,m\rangle)=1\}}\} & \text{if } \{m \leq l \colon \sigma(\langle n+1, m\rangle) = 1\} \neq \emptyset; \\ \emptyset & \text{if } \{m \leq l \colon \sigma(\langle n+1, m\rangle) = 1\} = \emptyset. \end{cases}$$

Notice that h_l is continuous for all $l \in \omega$ and that $h_l(\sigma) \to h(\sigma)$ for every $\sigma \in 2^{\omega \times \omega}$. So $h \in \mathcal{B}_1$. Notice that $h(\sigma) \in \mathcal{C}(f)$ if and only if $\sigma \in I$ where I is the set from Proposition 7. In particular, $h^{-1}(\mathcal{C}(f)) \notin \Pi^0_4(2^{\omega\times\omega})$. Since $h \in \mathcal{B}_1$, we must have $\mathcal{C}(f) \notin \Pi^0_3(\mathrm{J}(C))$. Hence, $\mathcal{C}(f) \notin \Delta^0_3(\mathrm{J}(C))$. Thus, we have the desired contradiction. ■

Lemma 17 *If $f \in \mathrm{T}_0$, then there exist $\{\langle U_n, V_n\rangle\}_{n\in\omega}$ such that for every $n \in \omega$ we have: $\langle U_n, V_n\rangle \in \Sigma^0_1(X) \times \Sigma^0_1(Y)$, $f|_{\mathrm{cl}(f^{-1}(\mathrm{cl}(V_n))\cap \mathrm{cl}(U_n))}$ is continuous, and $f \subseteq \bigcup_{n\in\omega} U_n \times V_n$.*

Proof. Let $x \in X$. Let $U^*_x \in \Sigma^0_1(X)$ and $V^*_x \in \Sigma^0_1(Y)$ be such that $x \in U^*_x$, $f(x) \in V^*_x$, and $f|_{\mathrm{cl}(f^{-1}(V^*_x)\cap U^*_x)}$ is continuous. Pick $U_x \in \Sigma^0_1(X)$ and $V_x \in \Sigma^0_1(Y)$ such that $\mathrm{cl}(U_x) \subseteq U^*_x$ and $\mathrm{cl}(V_x) \subseteq V^*_x$ and $x \in U_x$ and $f(x) \in V_x$. Since $\mathrm{cl}(f^{-1}(\mathrm{cl}(V_x)) \cap \mathrm{cl}(U_x)) \subseteq \mathrm{cl}(f^{-1}(V^*_x) \cap U^*_x)$, we have that $f|_{\mathrm{cl}(f^{-1}(\mathrm{cl}(V_x)\cap\mathrm{cl}(U_x))}$ is continuous. Since the graph of f is second countable and $f \subseteq \bigcup_{x\in X} U_x \times V_x$, we may find the desired countable collection. ■

Lemma 18 *If $f \in \mathrm{T}_0$, then there exists a $\subseteq$-increasing sequence $\{W_n\}_{n\in\omega}$ of closed subsets of X such that $\mathcal{C}(f) = \bigcup_{n\in\omega} \mathrm{J}(W_n)$. In particular, $\mathcal{C}(f) \in \Sigma^0_2(\mathrm{J}(X))$.*

Proof. Let $\mathcal{U} = \{U_n \times V_n\}_{n\in\omega}$ be as in Lemma 17. For each $n \in \omega$ let $W_n = \bigcup_{k\le n} \mathrm{cl}(f^{-1}(\mathrm{cl}(V_k)) \cap \mathrm{cl}(U_k))$. We show that $\mathcal{C}(f) = \bigcup_{n\in\omega} \mathrm{J}(W_n)$. Fix $n \in \omega$. Since $f|_{\mathrm{cl}(f^{-1}(\mathrm{cl}(V_k))\cap\mathrm{cl}(U_k))}$ is continuous for every $k \le n$, we have that $f|_{W_n}$ is continuous. Thus, $\bigcup_{n\in\omega} \mathrm{J}(W_n) \subseteq \mathcal{C}(f)$. We now show the reverse inequality. Suppose C is compact and $f|_C$ is continuous. Since $f|_C$ is compact, $f|_C$ is contained in a finite number of members of $\mathcal{U}$. So $C \subseteq W_n$ for some $n \in \omega$. Thus, $\mathcal{C}(f) \subseteq \bigcup_{n\in\omega} \mathrm{J}(W_n)$. ■

Lemma 16 and Lemma 18 show that (i) (ii) (iii), and (iv) of Theorem 2 are equivalent when X and Y are Polish spaces.

We now assume that $Y = \mathbb{R}$ and X is Polish.

Lemma 19 *If for a function $f\colon X \to \mathbb{R}$ there exists a $\subseteq$-increasing sequence $\{W_n\}_{n\in\omega}$ of closed subsets of X such that $\mathcal{C}(f) = \bigcup_{n\in\omega} \mathrm{J}(W_n)$, then f is a discrete limit of continuous functions $\{f_n\}_{n\in\omega}$ such that*

$$\mathcal{C}(f) = \{C \in \mathrm{J}(X)\colon \{f_n|_C\}_{n\in\omega} \textit{ is eventually constant }\}.$$

Proof. Fix $n \in \omega$. Since $\mathrm{J}(W_n) \subseteq \mathcal{C}(f)$, we have that $f|_{W_n}$ is continuous. By the Tietze Extension Theorem there is a continuous $f_n\colon X \to \mathbb{R}$ such that $f_n|_{W_n} = f|_{W_n}$. Clearly, $\{f_n\}_{n\in\omega}$ converges discretely to f. We show $\{f_n\}_{n\in\omega}$ is as desired.

Suppose $C \in \mathcal{C}(f)$. By assumption $C \subseteq W_n$ for some $n \in \omega$. In particular, $f_m|_C \subseteq f_n|_{W_n}$ for all $m \ge n$. Thus, $\{f_n|_C\}_{n\in\omega}$ is eventually constant.

Suppose $C \in \mathrm{J}(X)$ and $\{f_n|_C\}_{n\in\omega}$ is eventually constant. There is an $n \in \omega$ such that $f|_C = f_n|_C$. Thus, $C \in \mathcal{C}(f)$. ■

Lemma 20 *If $f\colon X \to \mathbb{R}$ is a discrete limit of continuous functions $\{f_n\}_{n\in\omega}$ such that $\mathcal{C}(f) = \{C \in \mathrm{J}(X)\colon \{f_n|_C\}_{n\in\omega}$ is eventually constant $\}$, then $\mathcal{C}(f) \in \Sigma^0_2(\mathrm{J}(X))$.*

PROOF. For each $n \in \omega$ let $Z_n = \{x \in X\colon (\forall m \geq n)(f_n(x) = f_m(x))\}$. It is easily checked that Z_n is closed for every $n \in \omega$. Now for every $n \in \omega$ we have that $\mathrm{J}(Z_n) = \{C \in \mathrm{J}(X)\colon (\forall m \geq n)(f_n|_C = f_m|_C)\}$ is closed in $\mathrm{J}(X)$. Therefore, $\mathcal{C}(f) = \{C \in 2^X\colon \{f_n|_C\}_{n\in\omega}$ is eventually constant $\} \in \Sigma^0_2(\mathrm{J}(X)$. ■

Lemma 19 and Lemma 20 show that (v) is equivalent to (i), (ii), and (iii) when $Y = \mathbb{R}$.

Lemma 21 *Let $f\colon X \to Y$. If $\mathcal{C}(f) \in \Sigma^0_2(\mathrm{J}(X))$, then f is open in $\mathrm{cl}(f)$.*

PROOF. Suppose f is not open in $\mathrm{cl}(f)$. There exists an $x \in X$ and $\{\langle x_n, y_n\rangle\}_{n\in\omega}$ such that $\langle x_n, y_n\rangle \in \mathrm{cl}(f) \setminus f$ for every $n \in \omega$ and $\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x, f(x)\rangle$. For each $n \in \omega$ we may find a sequence $\{w_{n,k}\}_{k\in\omega}$ of points in X such that $\lim_{n\to\infty}\langle w_{n,k}, f(w_{n,k})\rangle = \langle x_n, y_n\rangle$. Notice that $\langle x_n, y_n\rangle \neq \langle x_n, f(x_n)\rangle$. We may assume that $\mathrm{cl}(\{\langle w_{n,k}, f(w_{n,k})\rangle\colon k \in \omega\}) \cap \mathrm{cl}(\{\langle w_{m,k}, f(w_{m,k})\rangle\colon k \in \omega\}) = \emptyset$ for all distinct $n, m \in \omega$. Let $C = \mathrm{cl}(\{w_{n,k}\colon n, k \in \omega\})$. We will have a contradiction if we show that $\mathcal{C}(f) \cap \mathrm{J}(C) \notin \Sigma^0_2(\mathrm{J}(C))$. Define $g\colon 2^{\omega\times\omega} \to \mathrm{J}(C)$ by $h(\sigma) = \mathrm{cl}(\{w_{n,k}\colon \sigma(n,k) = 1\}) \cup \{x\}$. We claim that $h \in \mathcal{B}_1$. For each $m \in \omega$ define $h_m\colon 2^{\omega\times\omega} \to \mathrm{J}(C)$ by $h_m(\sigma) = \{w_{n,k}\colon \sigma(n,k) = 1 \text{ and } k \leq m\} \cup \{x\}$. Notice that h_m is continuous for all $m \in \omega$ and that $h_m(\sigma) \to h(\sigma)$ for every $\sigma \in 2^{\omega\times\omega}$. So $h \in \mathcal{B}_1$. It is also easy to see that $h(\sigma) \in \mathcal{C}(f)$ if and only if $\sigma \in X_0$. In particular, $h^{-1}(\mathcal{C}(f)) \notin \Sigma^0_3(2^{\omega\times\omega})$. Since $h \in \mathcal{B}_1$, we must have $\mathcal{C}(f) \notin \Sigma^0_2(\mathrm{J}(C))$. Thus, we have the desired contradiction. ■

Lemma 22 *Let $f\colon X \to \mathbb{R}$. If f is open in $\mathrm{cl}(f)$, then $f \in \mathrm{T}_0$.*

PROOF. Let $x \in X$. Since f is open in $\mathrm{cl}(f)$, we may find an open set $U \subseteq X$ and a bounded open interval $V \subseteq \mathbb{R}$ such that $x \in U$ and $f(x) \in V$ and $\mathrm{cl}(f) \cap (U \times V) = f \cap (U \times V)$. Pick open sets $U_1 \subseteq U$ and $V_1 \subseteq V$ contianing x and $f(x)$, respectively such that $\mathrm{cl}(U_1) \subseteq U_1$ and $\mathrm{cl}(V_1) \subseteq V_1$. Now $f \cap (\mathrm{cl}(U_1) \times \mathrm{cl}(V_1)) = \mathrm{cl}(f) \cap (\mathrm{cl}(U_1) \times \mathrm{cl}(V_1))$. By way of contradiction, assume that $f|_{\mathrm{cl}(f^{-1}(V_1)\cap U_1)}$ is not continuous. Let $\{w_n\}_{n\in\omega}$ be a sequence of points in $\mathrm{cl}(f^{-1}(V_1) \cap U_1)$ and $w \in \mathrm{cl}(f^{-1}(V_1) \cap U_1)$ be such that $\lim_{n\in\omega} w_n = w$ and $\lim_{n\in\omega} f(w_n) \neq f(w)$. Without loss of generality, we may assume that no subsequence $\{f(w_n)\}_{n\in\omega}$ converges to $f(w)$. Since $\mathrm{cl}(V_1)$ is compact, there is a $r \in \mathrm{cl}(V_1)$ such that $\lim_{n\in\omega} f(w_n) = r$. However, $f \cap (\mathrm{cl}(U_1) \times \mathrm{cl}(V_1))$ is closed so $f(w) = r$ which contradicts our choice of $\{w_n\}_{n\in\omega}$. ■

Lemma 21 and Lemma 22 show that (vi) is equivalent to (i), (ii), and (iii) when $Y = \mathbb{R}$. Which completes the proof of Theorem 2.

8 Proof of Theorem 3

We show that (i) implies (ii).

Lemma 23 *Let K be a Cantor set with a countable dense subset D. If $\mathcal{S} \subseteq \mathrm{J}(K)$ is the collection of compact sets C with the property that $C \cap D$ is finite and $C \setminus D$ is compact, then $\mathcal{S} \in \Sigma^0_3(\mathrm{J}(K)) \setminus \Pi^0_3(\mathrm{J}(K))$.*

PROOF. First we show that $\mathcal{S} \in \Sigma^0_3(\mathrm{J}(K))$. Let $D = \{d_n : n \in \omega\}$ be an enumeration of D. Define $f : K \to \mathbb{R}$ so that $f(d_n) = n+1$ for every $n \in \omega$ and $f(x) = 0$ for $x \in K \setminus D$. Notice that $\mathcal{C}(f) = \mathcal{S}$. Since f has G_δ-graph and $f \in \mathrm{T}_1$, Theorem 4 guarantees that $\mathcal{S} = \mathcal{C}(f) \in \Sigma^0_3(\mathrm{J}(K))$.

We now work to show that $\mathcal{S} \notin \Pi^0_3(\mathrm{J}(K))$. In what follows we let $\omega+1$ denote $\omega \cup \{\omega\}$ topologized to be a convergent sequence of isolated points with limit point ω. Let $L = \{\tau \in (\mathrm{J}(\omega+1))^\omega : (\forall n \in \omega)(\omega \in \tau(n))\}$ and $E \subseteq L$ be the collection of all $\tau \in L$ such that for some $n \in \omega$ we have $|\tau(k)| < \omega$ for all $k < n$ and $\tau(k) = \omega+1$ for all $k \geq n$. Since L is a Cantor set and E is countable and dense in L, we may assume that $K = L$ and $D = E$.

Define $\Theta : 2^{\omega\times\omega} \to L$ by setting $\Theta(\sigma)(n) = \{k \in \omega : \sigma(\langle n, k\rangle) = 1\} \cup \{\omega\}$ for every $\sigma \in 2^{\omega\times\omega}$ and $n \in \omega$. Notice that Θ is continuous.

Define $\Psi : L \to \mathrm{J}(L)$ by letting $\Psi(\tau)$ be the closure of the collection of all $\rho \in L$ such that for some $n \in \omega$ we have $\rho|_n = \tau|_n$ and $\rho(k) = \omega+1$ for all $k \geq n$. If for infinitely many $n \in \omega$ we have $\tau(n) \neq \omega+1$, then $\Psi(\tau)$ is a convergent of sequence points in L with limit point τ. If there is an $n \in \omega$ such that for all $k \geq n$ we have $\tau(k) = \omega+1$, then $\Psi(\tau)$ is a finite subset of L containing τ.

We claim that Ψ is continuous. Suppose $\{\tau_k\}_{n\in\omega}$ is a sequence points in L converging to some $\tau \in L$. We show that $\lim_{k\in\omega} \Psi(\tau_k) = \Psi(\tau)$.

Suppose there exist an infinite $A \subseteq \omega$ such that $\rho_k \in \Psi(\tau_k)$ for every $k \in A$ and $\lim_{k\in A} \rho_k = \rho$. We claim that $\rho \in \Psi(\tau)$. We will consider two exhaustive cases. First, suppose that there is an $N \in \omega$ such that for infinitely many $k \in A$ we have $\rho_k(l) = \omega+1$ for all $l \geq N$. We may assume that N is minimal with respect to this property. Let A^* be the set of all $k \in A$ such that $\rho_k(l) = \omega+1$ for all $l \geq N$. By minimality, there are only finitely many $k \in A^*$ such that $\rho_k(N-1) = \omega+1$. So, for almost all $k \in A^*$ we have $\rho_k|_N = \tau_k|_N$. Thus, we have $\rho|_N = \tau|_N$ and $\rho(l) = \omega+1$ for all $l \geq N$, so $\rho \in \Psi(\tau)$. For the second case, suppose that for every $N \in \omega$ there are only finitely many $k \in A$ such that $\rho_k(l) = \omega+1$ for all $l \geq N$. In this case we have $\lim_{k\in A} \rho_k(j) = \lim_{k\in A} \tau_k(j) = \tau(j)$ for every $j \in \omega$. Thus, $\rho = \tau \in \Psi(\tau)$. By cases, we have the claim.

We show that for every $\rho \in \Psi(\tau)$ there is a sequence $\{\rho_k\}_{k\in\omega}$ such that $\rho_k \in \Psi(\tau_k)$ and $\lim_{k\to\infty} \rho_k = \rho$. If $\rho = \tau$, then we can let $\rho_k = \tau_k$ for every $k \in \omega$ and have $\lim_{k\to\infty} \rho_k = \rho$. If there is an $n \in \omega$ such that $\rho|_n = \tau|_n$ and $\rho(l) = \omega+1$ for all $l \geq n$, then we pick $\rho_k \in \Psi(\tau_k)$ such that $\rho_k|_n = \tau_k|_n$ and $\rho_k(l) = \omega+1$ for all $l \geq n$ to get $\lim_{k\to\infty} \rho_k = \rho$.

By the proceeding two paragraphs, $\lim_{n\in\omega} \Psi(\tau_k) = \Psi(\tau)$. Thus, Ψ is continuous.

Let $\Gamma\colon 2^{\omega\times\omega} \to \mathrm{J}(L)$ be defined by $\Gamma(\sigma) = \Psi(\Theta(\sigma))$. Clearly, Γ is continuous. We claim $\Gamma^{-1}(\mathcal{S}) = 2^{\omega\times\omega} \setminus H$ where H is the set from Proposition 6. Suppose $\sigma \in 2^{\omega\times\omega}\setminus H$. By definition of H there is a smallest $n \in \omega$ such that $|\Theta(\sigma)(n)| = \omega$. It follows that at most n elements of $\Psi(\Theta(\sigma))$ are in E. We will show that $\Psi(\Theta(\sigma)) \setminus E$ is compact. If $\Theta(\sigma) \notin E$, then either $\Psi(\Theta(\sigma))$ is finite or $\Psi(\Theta(\sigma))$ is a convergent sequence with limit point not in E. If $\Theta(\sigma) \in E$, then $\Psi(\Theta(\sigma))$ is finite. In any of the three cases above $\Psi(\Theta(\sigma))\setminus E$ is compact. Thus, $2^{\omega\times\omega} \setminus H \subseteq \Gamma^{-1}(\mathcal{S})$. Suppose $\sigma \in H$. By definition of H, $|\Theta(\sigma)(n)| < \omega$ for every $n \in \omega$. Thus, $\Psi(\Theta(\sigma))$ is a convergent sequence of elements of E with limit point $\Theta(\sigma) \notin E$. So, $\Psi(\Theta(\sigma)) \notin \mathcal{S}$. Hence, $\Gamma^{-1}(\mathcal{S}) \subseteq 2^{\omega\times\omega} \setminus H$. Since $\Gamma^{-1}(\mathcal{S}) = 2^{\omega\times\omega} \setminus H$ and $H \notin \Sigma^0_3(2^{\omega\times\omega})$, we have $\mathcal{S} \notin \Pi^0_3(\mathrm{J}(K))$. ∎

Lemma 24 *If X is Polish and $G \in \Pi^0_2(X)$ is countable, the the set I of isolated points of G is a dense open subset of G.*

Proof. Clearly, G is a countable dense G_δ-subset of $\mathrm{cl}(G)$. Since $\mathrm{cl}(G)$ is countable and closed, the set J of isolated points of $\mathrm{cl}(G)$ form a dense open subset of $\mathrm{cl}(G)$. Clearly, $I = J$. So, $I = G \cap J$ is dense and open in G. ∎

Proof that $\mathcal{C}(f) \in \Pi^0_3(\mathrm{J}(X))$ implies $f \in \mathcal{B}_1$. Let $f \notin \mathcal{B}_1$. If f does not have G_δ-graph, then, by Theorem 5, $\mathcal{C}(f)$ is not Borel and so $\mathcal{C}(f) \notin \Pi^0_3(\mathrm{J}(X))$. So, we may assume that f has G_δ-graph.

Since $f \notin \mathcal{B}_1$, there is a Cantor set C such that $f|_C$ is nowhere continuous. We may assume that there is a $K > 0$ such that

$$\mathrm{osc}(f|_C, x) > 3K \tag{1}$$

for every $x \in C$. Since f is Borel, there is a G_δ-set G such that G is a dense subset of C and $f|_G$ is continuous.

Let U be a nonempty open subset of C. Let $x \in G\cap U$. There is an open set $V \subseteq U$ such that $x \in V$ and $\mathrm{diam}(f[V \cap G]) < K$. By (1) there is a $d \in V$ such that $H_d(f(d), f[V \cap G]) > K$. Since U was arbitrary we may find a countable dense subset D of C such that for every $d \in D$ there is an open set V_d such that $d \in V_d$ and $H_d(f(d), f[V_d \cap G]) > K$. By Lemma 10, there is a countable dense subset E of D such that $G_1 = G\cup E$ is a G_δ-subset of C. Since $f|_G$ and $f|_E$ are disjoint open subsets of $f|_{G_1}$ which is a G_δ-subset of $X\times Y$, we have that $f|_E$ is a countable G_δ-set. By Lemma 24, the collection J of isolated points of $f|_E$ is dense in $f|_E$. So, we may find a countable dense $E_1 \subseteq E$ such that $f|_{E_1} = J$ is the collection of isolated points in $f|_{G_1}$. Find a compact perfect set $K \subseteq G\cup E_1$ such that $E_1\cap K$ is dense in K. Letting $Q = E_1\cap K$ and $H = K\setminus Q$ it should be clear that $\mathcal{C}(f|_K)$ is the collection of compact sets $L \in \mathrm{J}(K)$ with the property that $L\cap Q$ is finite and $L\cap H$ is compact. By Lemma 23, $\mathcal{C}(f|_K) \notin \Pi^0_3(\mathrm{J}(K))$. Since $\mathcal{C}(f) \cap \mathrm{J}(K) = \mathcal{C}(f|_K)$, we have that $\mathcal{C}(f) \notin \Pi^0_3(J(X))$. ∎

Proof that $f \in \mathcal{B}_1$ implies $\mathcal{C}(f) \in \Pi^0_4(\mathrm{J}(X))$. Since Y is Polish, we can consider Y as subset of $[0,1]^\omega$ with the usual product topology and f to be a function from X into $[0,1]^\omega$. Since every $\mathcal{B}_1$ function into $[0,1]$ is a pointwise limit of continuous functions, we have that $f\colon X \to [0,1]^\omega$ is a pointwise limit of continuous functions $f_i\colon X \to [0,1]^\omega$.

For each $n, k, l \in \omega$ let

$$A_{k,l,n} = \{P \in \mathrm{J}(X)\colon (\exists x, w \in P)(\forall i \geq n)(d(x,w) \leq \frac{1}{2^l})(d(f_i(x), f_i(w)) \geq \frac{1}{2^k})\}.$$

We show that $A_{k,l,n}$ is closed. Let $P_j \in A_{k,l,n}$ and $P_j \to P$. For each $j \in \omega$ there are $x_j, w_j \in P_j$ such that $d(x_j, w_j) \leq 1/2^l$ and for all $i \geq n$ we have $d(f_i(x_j), f_i(w_j)) \geq 1/2^k$. Taking a subsequence if necessary we may assume that there exist $x, w \in P$ such that $\lim_{j\in\omega}\{x_j, w_j\} = \{x, w\}$ in $\mathrm{J}(X)$. Clearly, $d(x,w) \leq 1/2^l$. For $i \geq n$ fixed the continuity of f_i implies that $d(f_i(x), f_i(w)) \geq 1/2^k$. Hence, $P \in A_{k,l,n}$. So, $A_{k,l,n}$ is closed.

Let $E = \bigcup_{k\in\omega}\bigcap_{l\in\omega}\bigcup_{n\in\omega} A_{k,l,n}$. Clearly, $E \in \Sigma^0_4(\mathrm{J}(X))$. We will be done if we show that $\mathcal{C}(f) = \mathrm{J}(X) \setminus E$.

Suppose $P \in E$. There is a $k \in \omega$ such that $P \in \bigcap_{l\in\omega}\bigcup_{n\in\omega} A_{k,l,n}$. So for every $l \in \omega$ there exist $x_l, w_l \in P$ such that $d(x_l, w_l) \leq 1/2^l$ and $d(f_i(x_l), f_i(w_l)) \geq 1/2^k$ for all sufficently large $i \in \omega$. It follows that $d(f(x_l), f(w_l)) \geq 1/2^k$. Since P is compact, there is a $p \in P$ such that $\lim_{l\in\omega}\{x_l, w_l\} = \{p\}$ in $\mathrm{J}(X)$. Clearly, the oscillation of $f|_P$ at x is at least $1/2^k$. Hence, $P \notin \mathcal{C}(f)$.

Suppose $P \notin \mathcal{C}(f)$. There is a $p \in P$ and a $k \in \omega$ and a sequence $(p_l)_{l\in\omega}$ of elements of P such that $d(p_l, p) \leq 1/2^l$ for every $l \in \omega$ and

$$d(f(p_l), f(p)) > 1/2^k. \tag{2}$$

Since $(f_i)_{i\in\omega}$ converges to f pointwise, (2) implies we may find for each $l \in \omega$ a $n_l \in \omega$ such that for all $i \geq n_l$ we have $d(f_i(p_l), f_i(p)) \geq 1/2^k$. Thus, $P \in E$. ■

We now show that none the implications of Theorem 3 may be reversed.

Let $\{q_n\colon n \in \omega\}$ be an enumeration of the rational numbers in $\mathbb{R}$. Define $f\colon \mathbb{R} \to \omega$ by

$$f(x) = \begin{cases} n+1 & \text{if } x = q_n; \\ 0 & \text{otherwise.} \end{cases}$$

Now $f \notin \mathcal{B}_1$ since it has no point of continuity. However, $\mathcal{C}(f) \in \Sigma^0_3(\mathrm{J}(X)) \subseteq \Pi^0_4(\mathrm{J}(X))$ since f is T_1 and has G_δ-graph. So, the implication $(ii) \Rightarrow (iii)$ may not be reversed.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the characteristic function of a convergent sequence without its limit point. Clearly, $f \in \mathcal{B}_1$. However, f is not T_0 so $\mathcal{C}(f) \notin \Delta^0_3(\mathrm{J}(X))$. Since f is T_1 and has G_δ-graph, we have $\mathcal{C}(f) \in \Sigma^0_3(\mathrm{J}(X))$. Thus, $\mathcal{C}(f) \notin \Pi^0_3(\mathrm{J}(X))$. So, the implication $(i) \Rightarrow (ii)$ may not be reversed.

We construct a $\mathcal{B}_1$ function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\mathcal{C}(f) \in \Pi^0_4(\mathrm{J}(X)) \setminus \Sigma^0_4(\mathrm{J}(X))$. For each $n \in \omega$ pick an increasing sequence $(w_{n,m})_{m\in\omega}$ in $(1-1/2^n, 1-1/2^{n+1}]$ which converges to $1-1/2^{n+1}$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } x = w_{n,m}; \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $f \in \mathcal{B}_1$. For each $n, m \in \omega$ let $(z_{n,m,l})_{l\in\omega}$ be an increasing sequence in $(1 - 1/2^n, w_{n,0}]$ if $m = 0$ or $(w_{n,m-1}, w_{n,m}]$ if $m \neq 0$, in either case let $\lim_{l\in\omega} z_{n,m,l} = w_{n,m}$. Let I be the set from Proposition 7. Define $J = \Pi_{i\in\omega} I$. By [6, 23.3], $J \in \Pi^0_5((2^{\omega\times\omega})^\omega) \setminus \Sigma^0_5((2^{\omega\times\omega})^\omega)$. Define $h\colon (2^{\omega\times\omega})^\omega \to \mathrm{J}(\mathbb{R})$ by

$$h(\sigma) = \{1\} \cup \left\{1 - \frac{1}{2^{n+1}} : n \in \omega\right\} \bigcup_{n,m\in\omega} L_{n,m},$$

where $L_{n,m}$ is defined by

$$L_{n,m} = \begin{cases} \emptyset & \text{if } \{l\colon \sigma(n)(m,l) = 1\} = \emptyset \\ \{w_{m,n}\} & \text{if } \{l\colon \sigma(n)(m,l) = 1\} \text{ is infinite;} \\ \{z_{n,m,\max\{l\colon \sigma(n)(m,l)=1\}}\} & \text{otherwise.} \end{cases}$$

By an argument similar to the one used in the proof of Lemma 11, one can show that h is in $\mathcal{B}_1$. It is easy to verify that $h^{-1}(\mathcal{C}(f)) = J$. Since $h \in \mathcal{B}_1$, we have $\mathcal{C}(f) \notin \Sigma^0_4(\mathrm{J}(X))$.

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