The Minimum Barrier Distance Transform

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Outline

1. Minimum Barrier Distance, $\hat{\rho}$, in the discrete setting
2. How to compute $\hat{\rho}$?
3. Minimum Barrier Distance, ρ, in the continuous setting
4. Experiments: comparison with other distance measures
5. Newest result: fast algorithm for computing $\hat{\rho}$

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Image, scene, and the associated graph

$\widehat{f}\colon \widehat{D} \to \mathbb{R}$ is a digital image, where

$\widehat{D} = \mathbb{Z}^k \cap \prod_{i=1}^{k} [a_i, b_i]$ ($a_i, b_i \in \mathbb{R}$) is a digital scene

with $x, y \in \widehat{D}$ adjacent provided $\sum_i |x(i) - y(i)| = 1$.

We will treat also this structure,

$G = \langle \widehat{D}, \{\{x, y\} \colon x, y \text{ adjacent}\}, \widehat{f} \rangle$,

as a vertex weighted graph $G = \langle V(G), E(G), \widehat{w} \rangle$.

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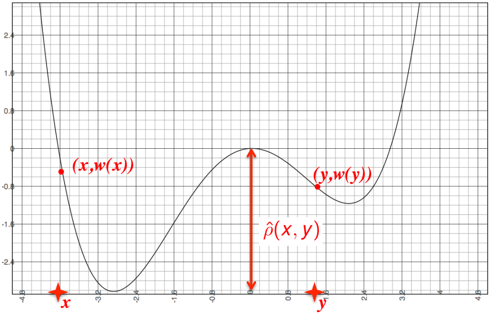
Minimum Barrier Distance in discrete setting

For a path $p = \langle c_1, \ldots, c_k \rangle$ in $G = \langle \widehat{D}, E, \widehat{w} \rangle$

$$c_b(p) = \max_i \widehat{w}(c_i) - \min_i \widehat{w}(c_i)$$

is the barrier cost of p.



The barrier distance between x and y in $\widehat{D}$ is given by:

$$\hat{\rho}(x, y) = \min\{c_b(p) \colon p \text{ is a path in } G \text{ from } x \text{ to } y\}$$

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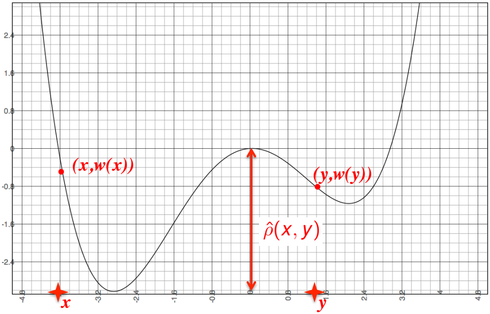

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MBD vs geodesic distance

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$\hat{\rho}(x,y)$ is, in a way,

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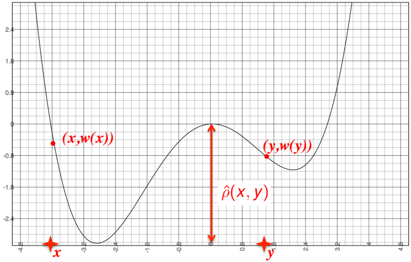

Theorem

$\hat{\rho}$ is a pseudo-metric:
it is symmetric and it satisfies the triangle inequality.
(However, $\hat{\rho}(x,y)$ can be equal 0 for $x \neq y$.)

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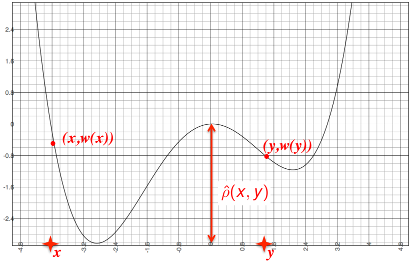

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MBD as a measure of connectivity

$\hat{\rho}(x, y) = \min\{c_b(p) \colon p \text{ is a path in } G \text{ from } x \text{ to } y\}$

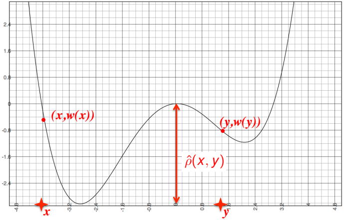
$\beta(x, y) = \exp(-\hat{\rho}(x, y))$

has some similarity to the

FC connectivity measure for the

object-feature base affinity with

average intensity value $m = \widehat{w}(s)$:



$\beta(x, s)$ is small when $|\widehat{w}(x) - m|$ is large.

$\hat{\rho}(x, y)$ can be used to define RFC-like object:

$$P(s, t) = \{c \in \widehat{D} \colon \hat{\rho}(c, s) < \hat{\rho}(c, t)\}.$$

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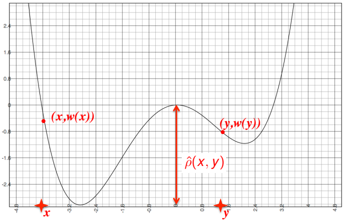

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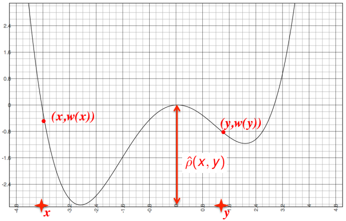

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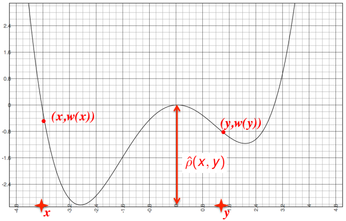

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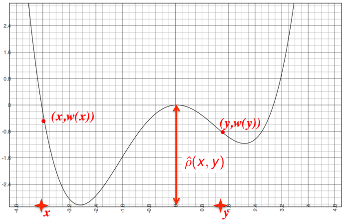

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Can Dijkstra-like algorithm find $\hat{\rho}(x, y)$?

Natural Algorithm:

Input: A seed x in the image/graph $G = \langle \widehat{D}, E, \widehat{w} \rangle$.

Output: $L(y), U(y) \in \mathbb{R}$, a path p_y from x to y with the range in $[L(y), U(y)]$ s.t. (hopefully) $\hat{\rho}(x, y) = U(y) - L(y)$.

Initialization: Push x to queue Q ordered via $U(y) - L(y)$.

Put $L(y) = -\infty$, $U(y) = \infty$ for $y \neq x$, $L(x) = U(x) = \widehat{w}(x)$;
while Q is not empty **do**
    Pop z from Q;
    for every y adjacent to z **do**
        Put $L = \min\{L(z), \widehat{w}(y)\}$ and $U = \max\{U(z), \widehat{w}(y)\}$;
        if $U(y) - L(y) > U - L$ **then**
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Failure of Natural Algorithm!

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0.41	0.5
0.5	0.6

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Let y be the upper right spel, with $\widehat{w} = 0.8$.

The optimal path is $p_o = \langle .5; .6; .5; .8 \rangle$,

giving $\hat{\rho}(x, y) = .3$

However, the Natural Algorithm returns

the path $p = \langle .5; .41; .5; .8 \rangle$

with $c_b(p) = .39 > \hat{\rho}(x, y)$.

Question (We do not know the answer)

Do the numbers $U(y) - L(y)$ returned by Natural Algorithm approximate $\hat{\rho}(x, y)$ in any reasonable sense?

So, how do we effectively compute the numbers $\hat{\rho}(x, y)$?

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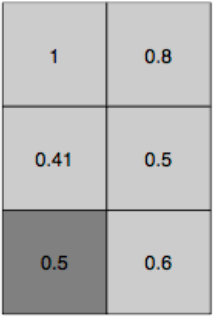

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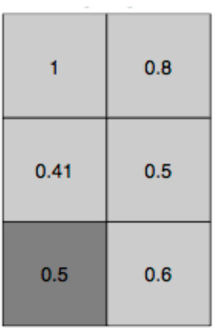

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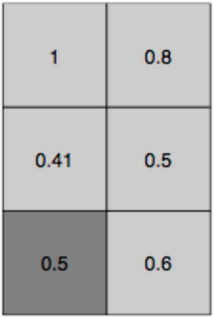

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Let $\hat{\Pi}_{x,y} = \{p \colon p \text{ is a path in } G \text{ from } x \text{ to } y\}$ and, for $p \in \hat{\Pi}_{x,y}$,

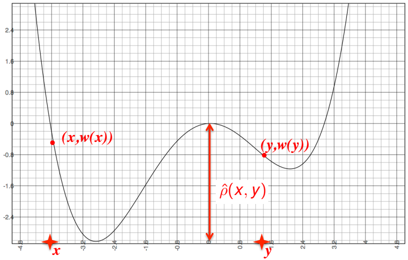
let $c_{\min}(p) = \min_t \widehat{w}(p(t))$ and $c_{\max}(p) = \max_t \widehat{w}(p(t))$.

Let $\hat{\varphi}(x, y) = \min_{p \in \hat{\Pi}_{x,y}} c_{\max}(p) - \max_{p \in \hat{\Pi}_{x,y}} c_{\min}(p)$

Clearly

$\hat{\varphi}(x, y) \leq \hat{\rho}(x, y)$

Is $\hat{\varphi}(x, y) = \hat{\rho}(x, y)$?



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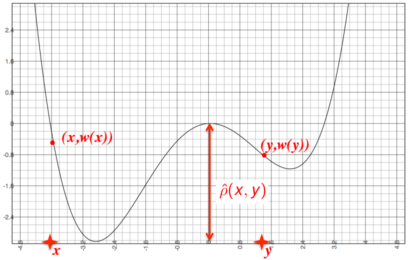

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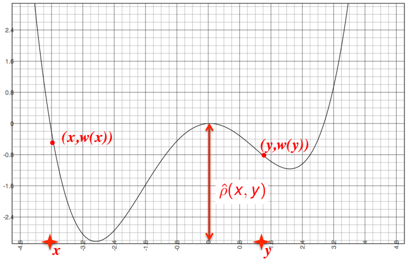

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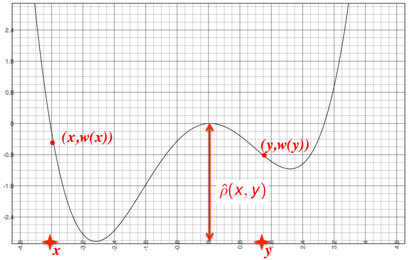

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Let $\hat{\Pi}_{x,y} = \{p\colon p \text{ is a path in } G \text{ from } x \text{ to } y\}$ and, for $p \in \hat{\Pi}_{x,y}$,

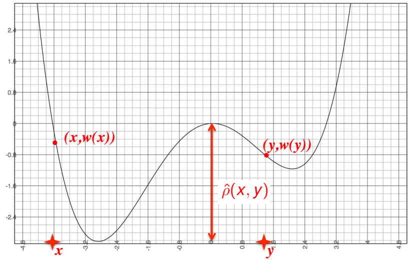
let $c_{\min}(p) = \min_t \widehat{w}(p(t))$ and $c_{\max}(p) = \max_t \widehat{w}(p(t))$.

Let $\hat{\varphi}(x, y) = \min_{p\in\hat{\Pi}_{x,y}} c_{\max}(p) - \max_{p\in\hat{\Pi}_{x,y}} c_{\min}(p)$

Clearly

$\hat{\varphi}(x, y) \leq \hat{\rho}(x, y)$

Is $\hat{\varphi}(x, y) = \hat{\rho}(x, y)$?



$\hat{\varphi}(x, y) \neq \hat{\rho}(x, y)$ but $\hat{\varphi}(x, y) \approx \hat{\rho}(x, y)$

y – upper right

0.41	0.5
0.5	0.6

x – lower left

$\min_{p \in \hat{\Pi}_{x,y}} c_{\max}(p) = c_{\max}(0.5, 0.41, 0.5) = 0.5$

$\max_{p \in \hat{\Pi}_{x,y}} c_{\min}(p) = c_{\min}(0.5, 0.6, 0.5) = 0.5$

But $\hat{\varphi}(x, y) = 0 \neq 0.09 = \hat{\rho}(x, y)$.

Nevertheless,

- $\hat{\varphi}(x, y) \approx \hat{\rho}(x, y)$, as we prove via continuous MBD.
- There is a very fast algorithm calculating $\hat{\varphi}(x, y)$.

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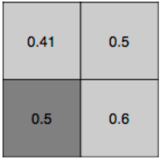

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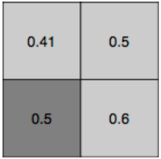

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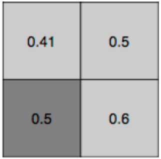

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Fast algorithm $A_{\hat{\varphi}}$ calculating $\hat{\varphi}(x,\cdot)$

Let $DA(G,\widehat{w},x)$ – Dijkstra algorithm returning p_y's with

$\hat{\rho}_{\max}(x,y) = c_{\max}(p_y)$.

Then $DA(G,-\widehat{w},x)$ returns p_y's with $\hat{\rho}_{\min}(x,y) = -c_{\max}(p_y)$.

Algorithm $A_{\hat{\varphi}}$:

Input: A seed x in the image/graph $G = \langle \widehat{D}, E, \widehat{w} \rangle$.

Output: A map $\hat{\varphi}(x,\cdot)$.

1: Run $DA(G,\widehat{w},x)$ and record $C^{+}(y) = c_{\max}(p_y)$ for $y \in \widehat{D}$;
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Outline

1. Minimum Barrier Distance, $\hat{\rho}$, in the discrete setting
2. How to compute $\hat{\rho}$?
3. Minimum Barrier Distance, ρ, in the continuous setting
4. Experiments: comparison with other distance measures
5. Newest result: fast algorithm for computing $\hat{\rho}$

Image, barrier cost of a path, and barrier distance

Input: Continuous function $f\colon D \to \mathbb{R}$, considered as an image,

where $D = \prod_{i=1}^{k} [a_i, b_i]$ $(a_i, b_i \in \mathbb{R})$.

For a (continuous) path $p\colon [0,1] \to D$ its barrier cost is

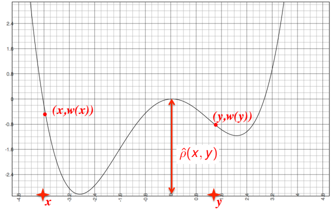
$$c_b(p) = \max_t w(p(t)) - \min_t w(p(t)), \quad \text{here } w = f.$$

(Note that max and min are attained, as $w \circ p$ is continuous.)

The continuous barrier distance

between $x, y \in D$ is given by:

$$\rho(x,y) = \inf\{c_b(p) \colon p \text{ from } x \text{ to } y\}$$



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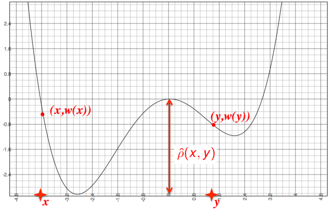

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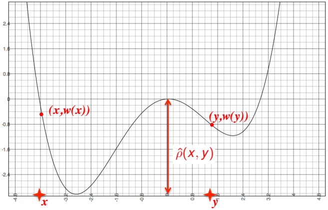

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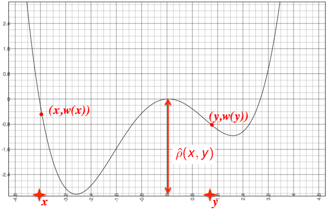
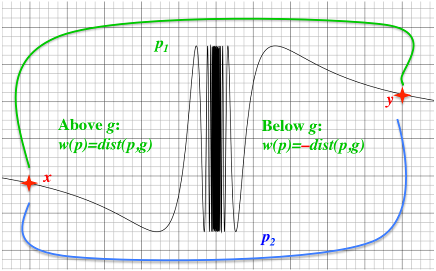


Topologists sine curve example

$g(t) = \sin(1/t)$ for $t \neq 0$, $g(0) = 0$



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In $\rho(x, y)$, operation inf cannot be replaced with min !

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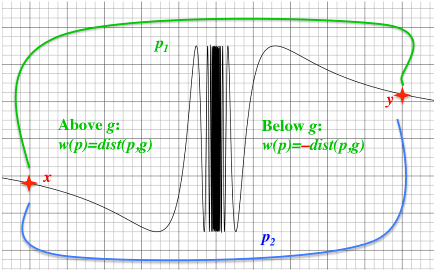

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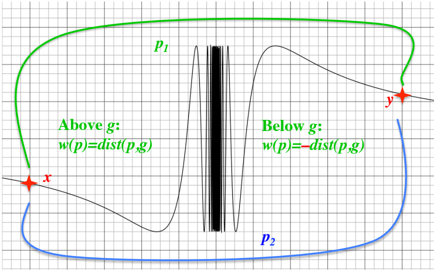

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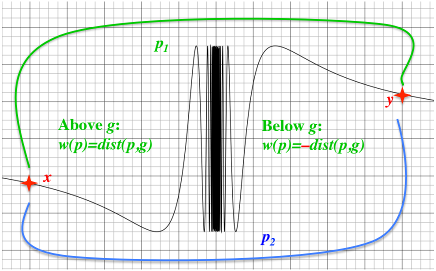

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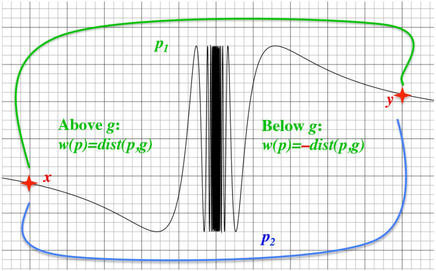

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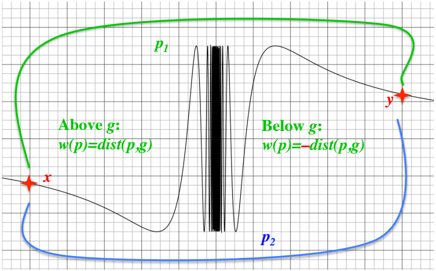

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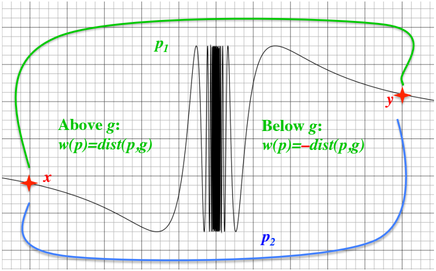

$\rho(x, y) = \inf\{c_b(p) \colon p \in \Pi_{x,y}\}$

Put $c_{\min}(p) = \min_t w(p(t))$

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$c_{\min}(p_1) = 0 < c_{\max}(p_1)$

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For $\varphi(x, y) = \min_{p \in \Pi_{x,y}} c_{\max}(p) - \max_{p \in \Pi_{x,y}} c_{\min}(p)$

$c_{\max}(p_2) - c_{\min}(p_1) = 0 = \varphi(x, y) = \rho(x, y) < c_b(p)$
for any $p \in \Pi_{x,y}$.

In $\rho(x, y)$, operation inf cannot be replaced with min !

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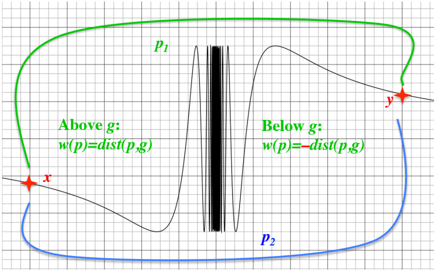

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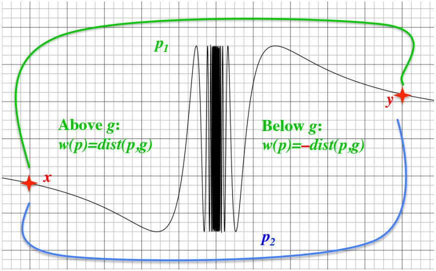

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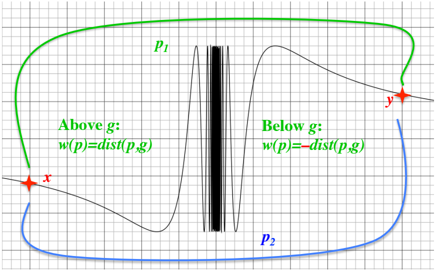

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Continuous barrier distance to the rescue of $A_{\hat{\varphi}}$

Theorem (Deep result on simple connected domains)

If there are $p_1, p_2 \in \Pi_{x,y}$ with $a < c_{\min}(p_1)$ and $c_{\max}(p_2) < b$, then there is a single $p \in \Pi_{x,y}$ with the range in (a, b).

Corollary (continuous case)

$\varphi(x, y) = \rho(x, y)$ *for a w on a simple connected domain D.*

Theorem ($\hat{\varphi}(x, y) \to \varphi(x, y) = \rho(x, y)$ when $\hat{w} \to w$)

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$\varepsilon = \max\{|w(x) - w(y)| : x, y \in \widehat{D}\ \&\ \max_i |x(i) - y(i)| \leq 1\}$.
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Proof: (1) Extend $\hat{w}$ to w via k-linear interpolation. (2) Find p for w with $c_b(p) \approx \varphi(x, y) = \rho(x, y)$. (3) Digitize p.

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Outline

1. Minimum Barrier Distance, $\hat{\rho}$, in the discrete setting
2. How to compute $\hat{\rho}$?
3. Minimum Barrier Distance, ρ, in the continuous setting
4. Experiments: comparison with other distance measures
5. Newest result: fast algorithm for computing $\hat{\rho}$

Experiments

We compared the output $\hat{\varphi}(x, y)$ of $A_{\hat{\varphi}}$ (approximating $\hat{\rho}(x, y)$) with the distances minimizing costs $c(p)$, $p = \langle p_1, p_2, \ldots, p_m \rangle$:

- fuzzy d_F; $c(p) = \sum_{i=1}^{m-1} \frac{f_A(p_i)+f_A(p_{i+1})}{2} \cdot \|p_i - p_{i+1}\|$;
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FC: max-arc $d_{\max}$; $c(p) = \max_{i=1,\ldots,m-1} |f_A(p_i) - f_A(p_{i+1})|$;

We compared the distances with respect to:

(A) the ratios between inter-object and intra-object distances & the influence by the seed points position: should be low;

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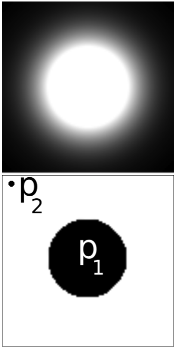
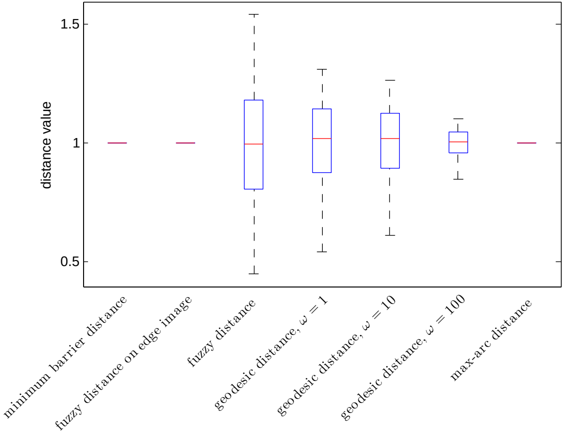


p_2 fixed;
p_1 is chosen randomly 1000 times



Boxes: 25th to the 75th percentile; central mark: the median.

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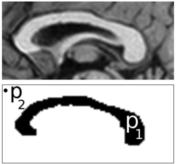

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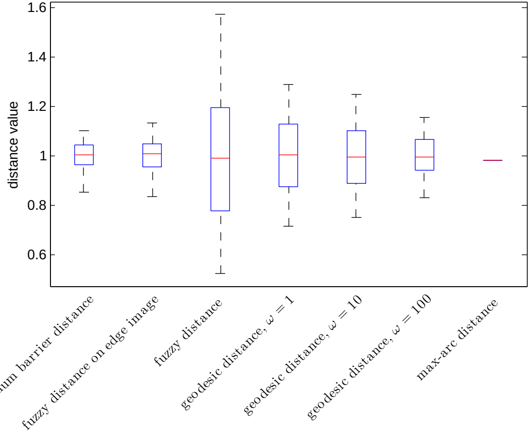

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- max-arc (Fuzzy Connectedness) distance is the most robust;
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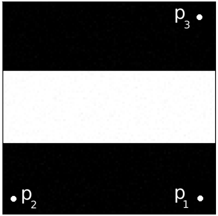
Stability w.r.t. Gaussian noise and smoothing

Test image



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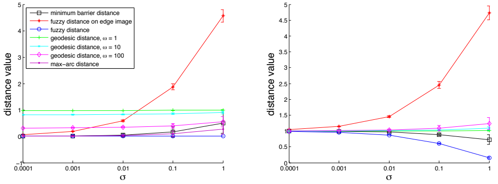
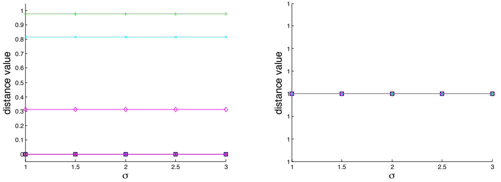
p_1 is randomly chosen 1000 times.

Gaussian noise – Distance values as function of sigma



Intra-object distance $(d(p_1, p_2))$ Inter-object distance $(d(p_1, p_3))$

Gaussian smoothing – Distance values as function of sigma



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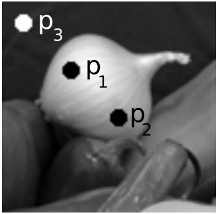

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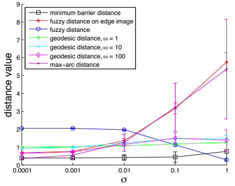

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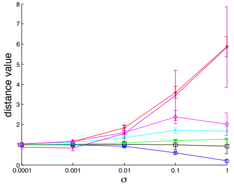
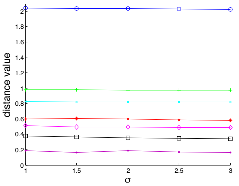


Inter-object distance $(d(p_1, p_3))$

Gaussian smoothing – Distance values as function of sigma



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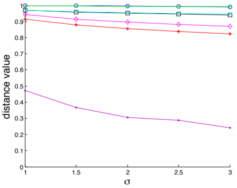

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Interpretation:

Sensitivity to noise and blur:

- MBD has low sensitivity;
- fuzzy distance on edge image and max-arc are sensitive;
- fuzzy distance: performs well for the image with a high boundary gradient; not so well for the image with a low boundary gradient.

Separation of the object from the background (ratio):

- all considered distances perform reasonably well;
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Outline

1. Minimum Barrier Distance, $\hat{\rho}$, in the discrete setting
2. How to compute $\hat{\rho}$?
3. Minimum Barrier Distance, ρ, in the continuous setting
4. Experiments: comparison with other distance measures
5. Newest result: fast algorithm for computing $\hat{\rho}$

Newest fast algorithm for computing $\hat{\rho}$

$DA(G, \widehat{w}, x)$ – returns p_y's with $\hat{\rho}_{\max}(x, y) = c_{\max}(p_y)$.

Define $w_a(x) = w(x)$ for $w(x) \geq a$ and $w_a(x) = \infty$ otherwise.

Algorithm $A_{\hat{\rho}}$:

Input: Graph $G = \langle \widehat{D}, E, \widehat{w} \rangle$ and the vertex x in G.
Output: A path $\hat{p}_y$ in G from x to y with $c_b(\hat{p}_y) = \hat{\rho}(x, y)$.
Auxiliary: Current value C_b of $c_b(\hat{p}_y)$;

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List R = {w(c) ≤ w(x): c ∈ D̂} with no repetition;
for every a ∈ R do
    Run DA(G, w_a, x);
    if c_b(p_y) < C_b then
        Put C_b = c_b(p_y) and p̂_y = p_y;
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4: **if** $c_b(p_y) < C_b$ **then**
5: Put $C_b = c_b(p_y)$ and $\hat{p}_y = p_y$;
6: **end if**
7: **end for**
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On the algorithm $A_{\hat{\rho}}$

Theorem (KC: Proved about two weeks ago)

$A_{\hat{\rho}}$ returns paths $\hat{p}_y$ with the exact values $c_b(\hat{p}_y) = \hat{\rho}(x, y)$.

$A_{\hat{\rho}}$ requires $O(k(n + k))$ operations
n – the size of $\widehat{D}$, and k – the size of $\{w(c) \leq w(x) \colon c \in \hat{D}\}$.

This estimate reduces to $O(n)$, when $k \ll n$,

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Minimum Barrier Distance $\hat{\rho}$ — novel quasi distance function:

- Can be effectively computed.
- Is quite stable with respect to:
  change of seed position and introduction of noise or blur.
  (Comparing to fuzzy, geodesic, and max-arc distances.)
- $\hat{\rho}(x, y)$ measures:
  - homogeneity, for $|\hat{w}(x) - \hat{w}(y)|$ small;
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Thank you for your attention!