Continuous, differentiable, and twice differentiable functions: How big are the gaps between these classes?

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Project scope: understanding the hierarchy

$$\mathcal{A} \subset \mathcal{C}^\infty \subset \cdots \subset \mathcal{C}^2 \subset D^2 \subset \mathcal{C}^1 \subset D^1 \subset \mathcal{C} \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_\alpha \subset \cdots$$

- D^n – n times differentiable functions
- $\mathcal{C}^n$ – continuously n times differentiable functions
- $\mathcal{B}_\alpha$ – Baire class α functions, $\alpha < \omega_1$
- $\mathcal{A}$ – analytic functions

All for functions $f\colon X \to Y$, where the classes are defined.

Scope: Understanding this hierarchy by

Finding natural properties that distinguish between these classes.

The simplest example, and (partially) Calc 1 puzzle

Theorem (Tietze Extension Thm)

For every closed subset X of $\mathbb{R}$ and $f\colon X \to \mathbb{R}$ with $f \in \mathcal{C}$ there is an $F\colon \mathbb{R} \to \mathbb{R}$ extending f such that $F \in \mathcal{C}$.

Question (To ponder during the talk)

Does Tietze Extension Thm hold if the class $\mathcal{C}$ of continuous functions is replaced with the class of:

- $\mathcal{C}^1$ functions?
- D^1 functions?

What happens with these questions, if $X \subset \mathbb{R}^n$ and we like to extend f to $\mathbb{R}^n$?
What about other, more general spaces than $\mathbb{R}^n$?

It makes sense to assume here that X has no isolated points.

Baire class functions: $\mathcal{C} \subsetneq \mathcal{B}_1$

- The derivatives $\Delta = \{f' \colon f \colon \mathbb{R} \to \mathbb{R},\ f \in D^1\}$, are $\mathcal{B}_1$, need not be in $\mathcal{C}$:

$$\Delta \subset \mathcal{B}_1,\ \Delta \not\subset \mathcal{C}$$

- The same for the class Appr of approximately continuous functions $f \colon \mathbb{R} \to \mathbb{R}$, that is, such that for every $a < b$, every $x \in f^{-1}((a,b))$ is a density point of $f^{-1}((a,b))$:

$$\mathrm{Appr} \subset \mathcal{B}_1,\ \mathrm{Appr} \not\subset \mathcal{C}$$

Any other natural examples here that I missed?

Baire class functions: $\mathcal{B}_1 \subsetneq \mathcal{B}_2$

The following classes of generalized continuity functions $f\colon \mathbb{R} \to \mathbb{R}$:

extendable Ext, almost continuous AC, connectivity Conn, and Darboux Darb,

coincide within $\mathcal{B}_1$ class [Brown, Humke, Laczkovich, 1988]:

$$\mathrm{Ext} \cap \mathcal{B}_1 = \mathrm{AC} \cap \mathcal{B}_1 = \mathrm{Conn} \cap \mathcal{B}_1 = \mathrm{Darb} \cap \mathcal{B}_1,$$

but are all distinct within the Baire class 2
[Brown 1974], [Jastrzębski 1989], [Ciesielski, Jastrzębski 2000]:

$$\mathrm{Ext} \cap \mathcal{B}_2 \subsetneq \mathrm{AC} \cap \mathcal{B}_2 \subsetneq \mathrm{Conn} \cap \mathcal{B}_2 \subsetneq \mathrm{Darb} \cap \mathcal{B}_2.$$

(The situation is drastically different for these classes and functions $f\colon \mathbb{R}^n \to \mathbb{R}$, $n > 1$.)

Any other natural examples for $\mathcal{B}_1 \subsetneq \mathcal{B}_2$?

Baire class functions: $\mathcal{B}_{n-1} \subsetneq \mathcal{B}_n$, $n \geq 1$

A function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $n \geq 2$, is separately continuous if it is continuous w.r.t. each variable.

For the class SC_{n+1} of separately continuous functions on $\mathbb{R}^{n+1}$ we have

Theorem ([Baire 1899] for $n = 1$, [Lebesgue 1905] for all n)

Every f from SC_{n+1} is of Baire calss n, but need not be of Baire class $n-1$:

$$\mathrm{SC}_{n+1} \subset \mathcal{B}_n, \ \mathrm{SC}_{n+1} \not\subset \mathcal{B}_{n-1}$$

Separately continuous function $f\colon \mathbb{R}^{\omega} \to \mathbb{R}$ need not be Borel!

Problems for: $\mathcal{B}_\alpha \subsetneq \mathcal{B}_\beta$, $\alpha < \beta < \omega_1$

Question

Are there any natural properties distinguishing the classes $\mathcal{B}_\alpha \subsetneq \mathcal{B}_\beta$ for $\omega \leq \alpha < \beta < \omega_1$?

Question

Are there any natural classes of functions from $\mathbb{R}$ to $\mathbb{R}$ that distinguish classes $\mathcal{B}_n \subsetneq \mathcal{B}_{n+1}$ for $n \geq 2$?

Any progress on Calc 1 puzzle, for the $\mathcal{C}^1$ case?

Question (Reminder)

If $X \subset \mathbb{R}$ is perfect and $f\colon X \to \mathbb{R}$ is f is $\mathcal{C}^1$, must there exist a $\mathcal{C}^1$ extension $F\colon \mathbb{R} \to \mathbb{R}$ of f?

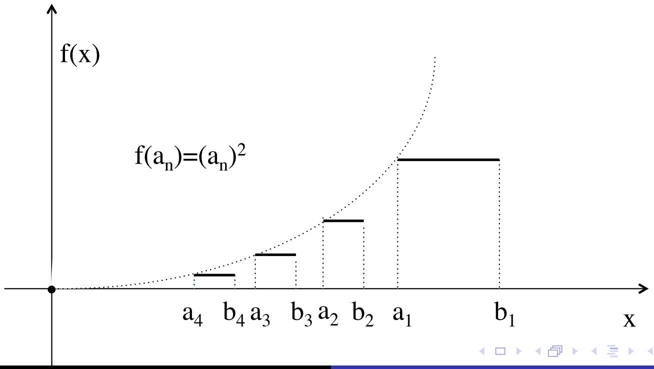
YES? NO?

Solution for Calc 1 puzzle, the $\mathcal{C}^1$ case:

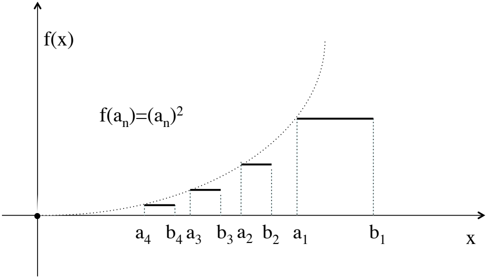
Question

If $X \subset \mathbb{R}$ is perfect and $f\colon X \to \mathbb{R}$ is f is $\mathcal{C}^1$, must there exist a $\mathcal{C}^1$ extension $F\colon \mathbb{R} \to \mathbb{R}$ of f?

Answer: NO $\quad X = \{0\} \cup \bigcup_n [a_n, b_n], \quad f'(x) = 0$ for all $x \in X$.



Truly Calc 1 problem:



How to choose the intervals to insure there is no $\mathcal{C}^1$ extension?

1. Insure that $\lim_{n\to\infty} \frac{f(a_n)-f(b_{n+1})}{a_n-b_{n+1}} > 0$.
2. Apply Mean Value Theorem to notice that no D^1 extension of f can have continuous derivative at 0.

Differentiable functions: the $\mathcal{C}^1 \subsetneq D^1$ case

Tietze Extension Theorem does not hold for $\mathcal{C}^1$ functions.

However, it does hold for D^1 functions (from $X \subset \mathbb{R}$ into $\mathbb{R}$):

Theorem ([Petruska, Laczkovich, 1974], possibly earlier Jarnik)

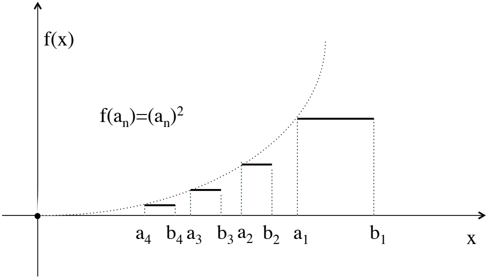
For every closed subset X of $\mathbb{R}$ and $f\colon X \to \mathbb{R}$ with $f \in D^1$ there is an $F\colon \mathbb{R} \to \mathbb{R}$ extending f such that $F \in D^1$.

Question (may be easy)

Does the above theorem hold for functions of n-variables, $n > 1$?

Tietze-type Extension Theorems: Summary

- $\mathcal{A} \not\rightsquigarrow \mathcal{A}$ — log function
- $\mathcal{A} \rightsquigarrow \mathcal{C}^\infty$ — easy to see
- $\mathcal{C}^\infty \not\rightsquigarrow \mathcal{C}^1$ — our Calc 1 example:



- $D^1 \rightsquigarrow D^1$ — Petruska, Laczkovich; possibly Jarnik
- $\mathcal{C} \rightsquigarrow \mathcal{C}$ — Tietze

$D^2 \subsetneq \mathcal{C}^1$: Interpolation property

For perfect $P \subset \mathbb{R}$ and $\mathcal{F} \subset \mathcal{G} \subset \mathcal{C}$

$Int_P(\mathcal{G}, \mathcal{F})$: $\forall g \in \mathcal{G}\ \exists f \in \mathcal{F}$ such that $P \cap [f = g]$ is uncountable.

- [Zahorski 1947], answering question of Ulam: $\neg Int_{\mathbb{R}}(\mathcal{C}, \mathcal{A})$; [Zahorski 1947] asked for $Int_{\mathbb{R}}(\mathcal{C}, \mathcal{C}^\infty)$
- [Agronsky, Bruckner, Laczkovich, Preiss 1985], $Int_P(\mathcal{C}, \mathcal{C}^1)$
- [Olevskiĭ 1994]: $\neg Int_{\mathbb{R}}(\mathcal{C}, \mathcal{C}^2)$
- $\neg Int_{\mathbb{R}}(\mathcal{C}, D^2)$, by [Morayne, 1985]: $Int_P(D^n, \mathcal{C}^n)$ for all n

Related results [Olevskiĭ 1994]: discripancy between $\mathcal{C}^2$ and $\mathcal{C}^1$

- $Int_P(\mathcal{C}^1, \mathcal{C}^2)$, but $\neg Int_P(\mathcal{C}^1, \mathcal{C}^3)$, so also $\neg Int_P(\mathcal{C}^1, D^3)$
- $\neg Int_P(\mathcal{C}^n, \mathcal{C}^{n+1})$, so also $\neg Int_P(\mathcal{C}^n, D^{n+1})$, for $n \geq 2$

$D^2 \subsetneq \mathcal{C}^1$: path continuity — generalization of SC

Theorem ([Rosenthal, 1955], earlier results: Lebesgue; others)

Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be such that for every $f\colon \mathbb{R} \to \mathbb{R}$ from $\mathcal{C}^1$ its restriction to $f \cup f^{-1} = \bigcup_{x\in\mathbb{R}}\{\langle x, f(x)\rangle, \langle f(x), x\rangle\}$ is continuous. Then F is continuous.
However, there are discontinuous $F\colon \mathbb{R}^2 \to \mathbb{R}$ with continuous restrictions to $f \cup f^{-1}$ for every $f \in D^2$

Theorem ([Ciesielski, Glatzer, 2012])

There is a $F\colon \mathbb{R}^2 \to \mathbb{R}$ which has continuous restrictions to $f \cup f^{-1}$ for every $f \in D^2$ and for which the set of points of discontinuities has positive Hausdorff 1-measure.
This is the best possible result in this direction.

$D^2 \subsetneq \mathcal{C}^1$: set-theoretical angle

Covering the plane by few graphs of functions

Theorem ([Ciesielski, Pawlikowski, 2005], generalizing [Steprāns 1999])

It is consistent with the standard axioms of set theory ZFC (follows from the Covering Property Axiom CPA), and independent from the ZFC axioms, that the plane $\mathbb{R}^2$ can be covered by less that continuum many ($<$ card$(\mathbb{R})$) sets $f \cup f^{-1}$ with $f \in \mathcal{C}^1$.

However, $\mathbb{R}^2$ cannot be covered by less that continuum many sets $f \cup f^{-1}$ with $f \in D^2$.

Open problem on covering $\mathbb{R}^n$ by graphs of functions

For $f\colon \mathbb{R}^2 \to \mathbb{R}$ let

$$Gr(f) = \bigcup_{\langle x,y\rangle\in\mathbb{R}^2} \{\langle x,y,f(x,y)\rangle, \langle x,f(x,y),y\rangle, \langle f(x,y),x,y\rangle\}$$

Theorem ([Sikorski ?], generalizing Sierpiński)

$\mathbb{R}^3$ can be covered by $\le\kappa$ many sets $Gr(f)$, with $f\colon \mathbb{R}^2 \to \mathbb{R}$ if, and only if, $card(\mathbb{R}) \le \kappa^{++}$

Question (probably difficult)

Is it consistent with ZFC that $card(\mathbb{R}) = \kappa^{++}$ and $\mathbb{R}^3$ can be covered by κ many sets $Gr(f)$, with $f \in \mathcal{C}^1$? $f \in \mathcal{C}$?

$\mathcal{A} \subsetneq \mathcal{C}^\infty$: set-theoretical angle

$[\mathbb{R}]^{\mathfrak{c}}$ all $S \subset \mathbb{R}$ of cardinality continuum; $\mathcal{F} \subset \mathcal{C}$.

Im$(\mathcal{F})$: $\forall S \in [\mathbb{R}]^{\mathfrak{c}}\ \exists f \in \mathcal{F}$ such that $f[S]$ contains a perfect set.

Theorem ([A. Miller 1983])

It is consistent with ZFC that Im$(\mathcal{C})$ *holds.*
However, Im$(\mathcal{C})$ *fails under the Continuum Hypothesis.*
So, Im$(\mathcal{C})$ *is independent from the ZFC axioms.*

Theorem ([Ciesielski, Pawlikowski, 2003])

Im$(\mathcal{C})$ *follows from the Covering Property Axiom CPA.*

Theorem ([Ciesielski, Nishura, 2012])

Im$(\mathcal{C}^\infty)$ *is equivalent to Im*$(\mathcal{C})$*, so it follows from CPA.*
However, Im$(\mathcal{A})$ *is false.*

$D^1 \subsetneq \mathcal{C}$: Peano curve part of talk

For $P \subset \mathbb{R}$ and $\mathcal{F} \subset \mathcal{C}(P) = \mathcal{C}(P, \mathbb{R}^2)$ let

Peano$(P, \mathcal{F})$: $\exists f \in \mathcal{F}$ s.t. $f[P] = P^2$.

- *Peano*$([a, b], \mathcal{C})$ holds — classic result of Peano
- *Peano*$([0, 1], D^1)$ is false — noticed by Morayne, 1985, as

  $f[P]$ has planar Lebesgue measure zero for differentiable f

Interesting:

Fact: $\exists f \in \mathcal{C}$ from $[0, 1]$ onto $[0, 1]^2$ s.t. $f[0, b]$ convex for all b

Open: Does there exist such f with $f[a, b]$ convex for all $a \leq b$?

True Peano Curve?

Remarkable Portraits Made with a Single Sewing Thread Wrapped through Nails, by Kumi Yamashita

www.thisiscolossal.com/2012/06/

KC: General Peano curve project

For $\mathcal{F}$ being either $\mathcal{C}^n$ or D^n, $n = 0, 1, 2, \ldots$, let

$$Peano(\mathcal{F}) = \{P \in \mathrm{Perf}\colon \exists f \in \mathcal{F} \text{ s.t. } f[P] = P^2\},$$

where $\mathrm{Perf} = \{P \subset \mathbb{R}\colon P \text{ closed in } \mathbb{R}, \text{ no isolated points}\}$.

In this notation: $[0,1] \in Peano(\mathcal{C}) \setminus Peano(D^1)$.

Assumption $P \in \mathrm{Perf}$ can be weakened to

- arbitrary subsets of $\mathbb{R}$ for $\mathcal{F} = \mathcal{C}$
- subsets with no isolated points for $\mathcal{F} = \mathcal{C}^n, D^n$ with $n \geq 1$.

Peano project scope

To describe classes

- $Peano(\mathcal{C}^n)$, $n = 0, 1, 2, \ldots, \infty$
- $Peano(D^m)$, $m = 1, 2, 3, \ldots$

Warning: By Tietze-type extension theorems

- $Peano(\mathcal{C}(\mathbb{R})) = Peano(\mathcal{C}(P))$ and $Peano(D^1(\mathbb{R})) = Peano(D^1(P))$

- But no reason for $Peano(\mathcal{C}^n(\mathbb{R})) = Peano(\mathcal{C}^n(P))$ for $n > 0$ though, clearly $Peano(\mathcal{C}^n(\mathbb{R})) \subset Peano(\mathcal{C}^n(P))$.

  Similarly for classes D^n, $n > 1$.

Peano curve project for class $\mathcal{C}$

Easy examples

- $[0,1]$ and $[0,1] \cup [2,3] \cup \cdots$ are in *Peano*$(\mathcal{C})$, but
- $P = [0,1] \cup [2,3]$ is not in *Peano*$(\mathcal{C})$!

  Two components of P cannot be map onto four of P^2
- If P is compact and $P \cap [a,b]$ is homeomorphic to the Cantor set, then $P \in$ *Peano*$(\mathcal{C})$

  as any compact set is an image of the Cantor set.
- If $P = \{0\} \cup \bigcup_{n=1}^{\infty} [1/(2n+1), 1/2n]$, then $P \notin$ *Peano*$(\mathcal{C})$ ($\{0\}$ would need to be mapped to $\{0\} \times P$.)

Problem: Characterize *Peano*$(\mathcal{C})$ (at least for compact sets)

Something to do with # and distribution of components

Fact: If $P \in Peano(D^1)$, then P has Lebesgue measure zero:
Morayne's argument, via Banach condition (T_2)

Theorem (KC, unpublished)

$Peano(\mathcal{C}^1)$ contains no compact set.

So, the result holds also for $Peano(\mathcal{C}^1(\mathbb{R})) \subseteq Peano(\mathcal{C}^1)$

Actually, the proof is considerably easier for $Peano(\mathcal{C}^1(\mathbb{R}))$.

Question

Can $Peano(D^1)$ contain a compact set?
If so, can such a set has positive Hausdorff dimension?

As *Peano*$(\mathcal{C}^1)$ contains no compact set, is *Peano*$(\mathcal{C}^1) \neq \emptyset$?

Theorem (KC, unpublished)

Peano$(\mathcal{C}^\infty) \neq \emptyset$

Question

With $\to$ denoting $\subset$ and $P(\mathcal{F}) = \mathit{Peano}(\mathcal{F})$,
can any inclusion in the chart below be reversed?

$$\begin{array}{ccccccccc} P(\mathcal{C}^\infty(\mathbb{R})) & \to \cdots \to & P(\mathcal{C}^2(\mathbb{R})) & \to & P(D^2(\mathbb{R})) & \to & P(\mathcal{C}^1(\mathbb{R})) & \to & P(D^1(\mathbb{R})) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \updownarrow \\ P(\mathcal{C}^\infty) & \to \cdots \to & P(\mathcal{C}^2) & \to & P(D^2) & \to & P(\mathcal{C}^1) & \to & P(D^1) \end{array}$$

Chart for compact sets

When restricted to compact sets, the chart

$$\begin{array}{ccccccccc}
P(\mathcal{C}^\infty(\mathbb{R})) & \to \cdots \to & P(\mathcal{C}^2(\mathbb{R})) & \to & P(D^2(\mathbb{R})) & \to & P(\mathcal{C}^1(\mathbb{R})) & \to & P(D^1(\mathbb{R})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \updownarrow \\
P(\mathcal{C}^\infty) & \to \cdots \to & P(\mathcal{C}^2) & \to & P(D^2) & \to & P(\mathcal{C}^1) & \to & P(D^1)
\end{array}$$

reduces to:

$$\begin{array}{ccccccccc}
\emptyset & \to \cdots \to & \emptyset & \to & \emptyset & \to & \emptyset & \to & P(D^1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \updownarrow \\
\emptyset & \to \cdots \to & \emptyset & \to & \emptyset & \to & \emptyset & \to & P(D^1)
\end{array}$$

Summary on $\mathcal{A} \subset \mathcal{C}^\infty \subset \mathcal{C}^2 \subset \mathcal{C}^1 \subset \mathcal{C} \subset \mathcal{B}_1 \subset \mathcal{B}_2 \cdots$

$\mathcal{B}_{n-1} \subsetneq \mathcal{B}_n$: separately continuous functions $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$

$\mathcal{B}_1 \subsetneq \mathcal{B}_2$: Ext, AC, Conn, and Darb

$\mathcal{C} \subsetneq \mathcal{B}_1$: derivatives Δ;
approximately continuous functions

$D^1 \subsetneq \mathcal{C}$: Peano curve for sets of positive messure

$\mathcal{C}^1 \subsetneq D^1$: Tietze Extension Thm

$D^2 \subsetneq \mathcal{C}^1$: Interpolation property;
Path continuity;
Covering plane by graphs of functions

$\mathcal{A} \subsetneq \mathcal{C}^\infty$: Image of sets $X \in [\mathbb{R}]^{\mathfrak{c}}$ contain perfect set;
Tietze Extension Thm $\mathcal{A} \rightsquigarrow \mathcal{C}^\infty$

Thank you for your attention!