Linearly continuous maps discontinuous on the graphs of twice differentiable functions

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Outline

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Definitions of separate and linear continuities

An $f\colon \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3, 4, \ldots$, is

- separately continuous, SC, iff the mapping $t \mapsto f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ is continuous for every $\langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$;
- equivalently, when $f \restriction \ell$ is continuous for every line ℓ in $\mathbb{R}^n$ parallel to one of the coordinate axis;
- **linearly continuous**, LC, iff $f \restriction \ell$ is continuous for every line ℓ in $\mathbb{R}^n$.

Example (Genocchi and Peano 1884 calculus text)

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{for } \langle x,y\rangle \neq \langle 0,0\rangle \\ 0 & \text{for } \langle x,y\rangle = \langle 0,0\rangle \end{cases} \qquad (1)$$

is linearly continuous but discontinuous (on $\{(y^2, y)\colon y \in \mathbb{R}\}$).

Implications between these continuities

Clearly we have the following irreversible implications

$$f \text{ is cont.} \overset{(1)}{\Longrightarrow} f \text{ linearly cont.} \overset{\frac{xy}{x^2+y^2}}{\Longrightarrow} f \text{ separately cont.}$$

But Cauchy, in his 1821 book *Cours d'analyse*, has a theorem:

A separately continuous function of real variables is continuous.

Q. Was Cauchy mistaken?

A. Perhaps, but not necessarily!
Cauchy worked with non-Archimedean reals and for such reals the result can be interpreted as correct, see

K. Ciesielski & D. Miller, A continuous tale . . ., RAEx 2016

Baire classification

Theorem ([Baire 1899] for $n = 2$, [Lebesgue 1905] for all n)

Every separately continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ is Baire class $n-1$ and need not be of lower Baire class.

On the other hand

Theorem ([Zajíček 2019] and [Banakh-Maslyuchenko 2020])

Every linearly continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ is Baire class 1.

Outline

Sets of discontinuity points for SC functions

$D(f)$ denotes the set of points of discontinuity of f

Theorem (Kershner 1943, characterization of $\{D(f)\colon f \in \mathrm{SC}\}$)

For any set $D \subset \mathbb{R}^n$

- $D = D(f)$ *for some separately continuous f on $\mathbb{R}^n$ iff*
- D *is an F_σ set and every orthogonal projection of D onto a coordinate hyperplane has first category image.*

Problem (Kronrod 1944, still not satisfactorily answered)

Find a characterization of the classes

$$\mathcal{D}^n_L := \{D(f)\colon f\colon \mathbb{R}^n \to \mathbb{R} \text{ is linearly continuous}\}$$

for $n = 2, 3, \ldots$.

A result of Slobodnik

For a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ let $\mathbb{E}(\mathcal{F})$ be the closure of $\mathcal{F}$ under countable unions and isometrical images.

Notice that $\mathcal{D}^n_L = \mathbb{E}(\mathcal{D}^n_L)$.

Theorem (Slobodnik 1976)

For every $n \geq 2$

$$\mathcal{D}^n_L \subset \mathbb{E}(\mathrm{Lip}_{nwd}),$$

where Lip_{nwd} is the family of all restrictions of Lipschitz functions $g\colon \mathbb{R}^{n-1} \to \mathbb{R}$ to compact nowhere dense $K \subset \mathbb{R}^{n-1}$.

In particular, any $D \in \mathcal{D}^n_L$ has Lebesgue measure 0,

while there is a separately continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ having positive Lebesgue measure.

Revised problem of Kronrod

(P) For $n \geq 2$ find a family $\mathcal{F} \subset \mathrm{Lip}_{nwd}$ such that $\mathbb{E}(\mathcal{F}) = \mathcal{D}_L^n$.

Let Conv, D^k, and C^k be the classes of all $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ that are, respectively, convex, k-times differentiable, and continuously k-times differentiable.

Theorem (KC and T. Glatzer 2013)

- $\mathbb{E}(\mathrm{Conv}_{nwd}) \subset \mathcal{D}_L^n$.
- $\mathbb{E}(C^2_{nwd}) \subset \mathcal{D}_L^n$ *for* $n = 2$.
- $\mathbb{E}(D^1_{nwd}) \not\subset \mathcal{D}_L^n$ *for* $n = 2$.

Problem (KC and T. Glatzer 2013)

For $n = 2$

- *is* $\mathbb{E}(C^1_{nwd}) \subset \mathcal{D}_L^n$?
- *what about* $\mathbb{E}(D^2_{nwd}) \subset \mathcal{D}_L^n$?

Main results of this talk

- Is $\mathbb{E}(C^1_{nwd}) \subset \mathcal{D}^2_L$? What about $\mathbb{E}(D^2_{nwd}) \subset \mathcal{D}^2_L$?

Theorem (Zajíček, 2022 preprint)

$\mathbb{E}(C^1_{nwd}) \not\subset \mathcal{D}^2_L$.

Theorem (Main result of the talk)

For every $f \in C^1$ with nowhere monotone derivative f' there exists a nowhere dense perfect $P \subset \mathbb{R}$ such that $f \restriction P \notin \mathcal{D}^2_L$.

Corollary

$\mathbb{E}(D^2_{nwd}) \not\subset \mathcal{D}^2_L$.

Proof of Corollary.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be differentiable nowhere monotone.
Use Main Theorem with $f(x) := \int_0^x h(t)\, dt$. □

Outline

LC maps $g\colon \mathbb{R}^2 \to \mathbb{R}$ with $D(g) = f \restriction P$: tangents of f

Choose a C^1 map $\eta\colon \mathbb{R} \to [0,\infty)$ with $\eta^{-1}(0) = P$ and a set $C := \{c_i \in \mathbb{R}^2\colon i \in \mathbb{N}\}$ contained in the envelope $E := \{\langle x,y\rangle\colon f(x) < y < f(x)+\eta(x)\}$ with $C' = f \restriction P$.

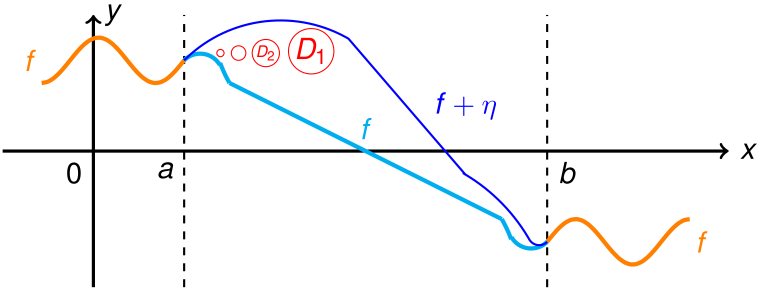
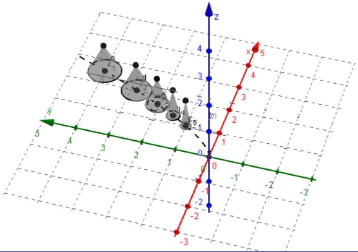


Figure: (a,b) is a component of $\mathbb{R}\setminus P$; each D_i is centered in c_i

LC maps $g\colon \mathbb{R}^2 \to \mathbb{R}$ with $D(g) = f \restriction P$: tangents of f

Choose pairwise disjoint open disks $D_i := B(c_i, \varepsilon_i) \subset E$ and put

$$g(p) := \sum_{i\in\mathbb{N}} \frac{\mathrm{dist}(p, D_i^c)}{\varepsilon_i}$$ see sketch of its graph.



Lemma

$D(g) = f \restriction P$ and $g \restriction \ell$ is continuous, except possibly when ℓ intersects infinitely many D_i's and is a "tangent line" to f at $x \in P$.

$\mathbb{E}(\mathrm{Conv}_{nwd}) \subset \mathcal{D}^n_L$ and $\mathbb{E}(C^2_{nwd}) \subset \mathcal{D}^n_L$ for $n = 2$.

$D(g) = f \restriction P$ and $g \restriction \ell$ is continuous, except possibly when ℓ intersects infinitely many D_i's and is a "tangent line" to f at $x \in P$.

Proof of $\mathbb{E}(\mathrm{Conv}_{nwd}) \subset \mathcal{D}^n_L$.

Any ℓ that intersects infinitely many D_i's is below convex f, while all disks D_i are above f. □

Proof of $\mathbb{E}(C^2_{nwd}) \subset \mathcal{D}^n_L$.

If $f \in C^2$, then $T_{f,P}$ — the union of all lines tangent to f at $x \in P$ — is nowhere dense in $\mathbb{R}^2$. (Requires some argument.) So, we can choose disks D_i disjoint with $T_{f,P}$. □

Banakh-Maslyuchenko characterizations of $\mathcal{D}^n_L$

Theorem (Banakh & Maslyuchenko 2020)

$M \in \mathcal{D}^n_L$ iff M is a countable union of closed ℓ-miserable sets $K \subset \mathbb{R}^n$, that is, such that there exists a closed set $L \subset \mathbb{R}^n$ containing K with the properties:

(i) *L is an ℓ-neighborhood of K: for any line ℓ in $\mathbb{R}^n$ and any $\bar{p} \in \ell \cap K$ there is an open J in ℓ such that $\bar{p} \in J \subset L$;*

(ii) *$K \subset \mathrm{cl}(\mathbb{R}^2 \setminus L)$.*

- For LC map $g(p) := \sum_{i\in\mathbb{N}} \frac{\mathrm{dist}(p, D_i^c)}{\varepsilon_i}$ defined above $K := f \restriction P$ is ℓ-miserable with $L := \mathbb{R}^2 \setminus \bigcup_{i\in\mathbb{N}} D_i$.
- This characterization is still hard to grasp and/or use.

Outline

The main lemma

Thm: For every $f \in C^1$ with nowhere monotone f' there exists a nowhere dense perfect $P \subset \mathbb{R}$ such that $f \restriction P \notin \mathcal{D}^2_L$.

Lemma (Main Lemma)

For every $a < b$ and $f \in C^1$ with nowhere monotone f' there are $d \in (a, b)$ and perfect nowhere dense $N_d \subset (a, d)$ such that $\langle d, f(d)\rangle \in \mathrm{int}(T_{f,N_d})$.

Proof of Lemma is based on several simpler facts.

Construction of nowhere dense $P \subset \mathbb{R}$ with $f \restriction P \notin \mathcal{D}^2_L$

Construct a sequence $\langle\langle I_s, d_s, N_s\rangle : s \in 2^{<\omega}\rangle$ s.t.

(A_n) $\mathcal{I}_n = \{I_s : s \in 2^n\}$ consists of pairwise disjoint non-trivial closed intervals each of length $|I_s| \leq (\frac{2}{3})^n$.

(B_n) If $s, t \in 2^{\leq n}$ and $s \subset t$, then $I_t \subset I_s$ and $N_t \cup \{d_t\} \subset \bigcup \mathcal{I}_n$.

(C_n) If $I_s = [a_s, b_s]$, then $d_s \in (a_s, b_s)$, $N_s \subset (a_s, d_s)$ is nowhere dense, and $\langle d_s, f(d_s)\rangle \in \operatorname{int}(T_{f,N_s})$.

Construction: If M_s is the middle third of I_s, $s \in 2^n$,

- choose d_s and N_s in I_s as in Main Lemma;
- pick open interval $\emptyset \neq J_s \subset M_s \setminus \bigcup_{t \in 2^{\leq n}} (N_t \cup \{d_t\})$ and define $\{I_u : u \in 2^{n+1} \ \&\ s \subset u\}$ as two components of $I_s \setminus J_s$.

Define $P := \bigcap_{n<\omega} \bigcup \mathcal{I}_n$.

There is no LC $g\colon \mathbb{R}^2 \to \mathbb{R}$ with $D(g) = f \restriction P$

Otherwise, by Baire Category and Banakh-Maslyuchenko thms

- there is an $s_0 \in 2^{<\omega}$ s.t. $K_0 := f \restriction (P \cap I_{s_0})$ is ℓ-miserable, i.e., $\exists$ closed ℓ-nbhd L of K_0 with K_0 in closure of $U := L^c$.

Construct $\langle\langle s_n, c_n, \varepsilon_n\rangle \in 2^{<\omega} \times U \times \mathbb{R}^+ : n < \omega\rangle$ s.t.

(a_n) $c_n \in U \cap \operatorname{int}(T_{f,N_{s_n}})$ and $\|c_n - \langle d_{s_n}, f(d_{s_n})\rangle\| \leq 2^{-n}$;

(b_n) $\varepsilon_n \in (0, 2^{-n})$ and $B(c_n, \varepsilon_n) \subset U \cap \operatorname{int}(T_{f,N_{s_n}})$;

(c_n) $s_{n+1} \supset s_n$ and $T_{f,p} \cap B(c_n, \varepsilon_n) \neq \emptyset$ for every $p \in I_{s_{n+1}}$.

Construction: Given s_n,

- there are ε_n and c_n as $\langle d_{s_n}, f(d_{s_n})\rangle \in \mathrm{cl}(U) \cap \operatorname{int}(T_{f,N_{s_n}})$;
- to find s_{n+1} choose: $x \in N_{s_n} \subset I_{s_n}$ s.t. $T_{f,x} \cap B(c_n, \varepsilon_n) \neq \emptyset$; $\delta > 0$ s.t. $T_{f,p} \cap B(c_n, \varepsilon_n) \neq \emptyset$ for every $p \in (x_n - \delta, x_n + \delta)$; $s_{n+1} \supset s_n$ s.t. $I_{s_{n+1}} \subset (x_n - \delta, x_n + \delta)$.

Desired contradiction

We have $\langle\langle s_n, c_n, \varepsilon_n\rangle \in 2^{<\omega} \times U \times \mathbb{R}^+ \colon n < \omega\rangle$ s.t.

(a_n) $c_n \in U \cap \text{int}(T_{f,N_{s_n}})$ and $\|c_n - \langle d_{s_n}, f(d_{s_n})\rangle\| \le 2^{-n}$;

(b_n) $\varepsilon_n \in (0, 2^{-n})$ and $B(c_n, \varepsilon_n) \subset U \cap \text{int}(T_{f,N_{s_n}})$;

(c_n) $s_{n+1} \supset s_n$ and $T_{f,p} \cap B(c_n, \varepsilon_n) \neq \emptyset$ for every $p \in I_{s_{n+1}}$.

Let $\{p\} = \bigcap_{n<\omega} I_{s_n}$, $\bar{p} := \langle p, f(p)\rangle$, and $\ell := T_{f,p}$.

Then for every $n < \omega$ there is $p_n \in \ell \cap B(c_n, \varepsilon_n) \subset \ell \cap U$.

As $p_n \to_n \bar{p}$, there is no open J in ℓ with $\bar{p} \in J \subset \mathbb{R}^2 \setminus U = L$.

So, L is NOT ℓ-nbhd L of $K_0 \ni \bar{p}$, a contradiction.

A result used to prove the main lemma

Main Lemma: For every $a < b$ and $f \in C^1$ with nowhere monotone f' there are $d \in (a, b)$ and perfect nowhere dense $N_d \subset (a, d)$ such that $\langle d, f(d)\rangle \in \mathrm{int}(T_{f,N_d})$.

Fact

Let $f \in C^1$ be s.t. f' is nowhere monotone. If $Z \subset (-\infty, a]$ and $\emptyset \neq (r, s) \subset Z$, then there is $\emptyset \neq (u, v) \subset (r, s)$ s.t.

$$T_{f,Z\setminus(u,v)} \cap ((a, \infty) \times \mathbb{R}) = T_{f,Z} \cap ((a, \infty) \times \mathbb{R}).$$

If Z is compact, then there is nowhere dense $N \subset Z$ s.t.

$$T_{f,N} \cap ((a, \infty) \times \mathbb{R}) = T_{f,Z} \cap ((a, \infty) \times \mathbb{R}).$$

Outline

Remark and open problem

Remark

$\mathbb{E}(D^2_{nwd}) \not\subset \mathcal{D}^2_L$ implies that $\mathbb{E}(D^2_{nwd}) \not\subset \mathcal{D}^n_L$ for all $n \geq 2$.

Problem

Is the inclusion $\mathbb{E}(C^2_{nwd}) \subset \mathcal{D}^n_L$ *true for* $n > 2$*?*

Thank you for your attention!