Linearly continuous maps discontinuous on the graphs of twice differentiable functions

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Outline

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1. Separate and linear continuity – definitions and background
2. Sets of points of discontinuity: characterizations
3. Tangent lines and characterization of $\mathcal{D}^n_L$
4. Proof of Main Theorem
5. Comments and open problem

Definitions of separate and linear continuities

An $f\colon \mathbb{R}^n \to \mathbb{R}$, with $n = 2, 3, 4, \ldots$, is

- separately continuous, SC, iff the mapping $t \mapsto f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$ is continuous for every $\langle x_1, \ldots, x_n \rangle \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$;
- equivalently, when $f \restriction \ell$ is continuous for every line ℓ in $\mathbb{R}^n$ parallel to one of the coordinate axis;
- **linearly continuous**, LC, iff $f \restriction \ell$ is continuous for every line ℓ in $\mathbb{R}^n$.

Example (Genocchi and Peano 1884 calculus text)

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{for } \langle x, y \rangle \neq \langle 0, 0 \rangle \\ 0 & \text{for } \langle x, y \rangle = \langle 0, 0 \rangle \end{cases} \tag{1}$$

is linearly continuous but discontinuous (on $\{(y^2, y)\colon y \in \mathbb{R}\}$).

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Implications between these continuities

Clearly we have the following irreversible implications

$$f \text{ is cont.} \overset{(1)}{\Longrightarrow} f \text{ linearly cont.} \overset{\frac{xy}{x^2+y^2}}{\Longrightarrow} f \text{ separately cont.}$$

But Cauchy, in his 1821 book *Cours d'analyse*, has a theorem:

A separately continuous function of real variables is continuous.

Q. Was Cauchy mistaken?

A. Perhaps, but not necessarily!
Cauchy worked with non-Archimedean reals and for such reals the result can be interpreted as correct, see

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Baire classification

Theorem ([Baire 1899] for $n = 2$, [Lebesgue 1905] for all n)

Every separately continuous $f : \mathbb{R}^n \to \mathbb{R}$ is Baire class $n - 1$ and need not be of lower Baire class.

On the other hand

Theorem ([Zajíček 2019] and [Banakh-Maslyuchenko 2020])

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Sets of discontinuity points for SC functions

$D(f)$ denotes the set of points of discontinuity of f

Theorem (Kershner 1943, characterization of $\{D(f) : f \in SC\}$)

For any set $D \subset \mathbb{R}^n$

- *$D = D(f)$ for some separately continuous f on $\mathbb{R}^n$ iff*
- *D is an F_σ set and every orthogonal projection of D onto a coordinate hyperplane has first category image.*

Problem (Kronrod 1944, still not satisfactorily answered)

Find a characterization of the classes

$$\mathcal{D}_L^n := \{D(f) : f : \mathbb{R}^n \to \mathbb{R} \text{ is linearly continuous}\}$$

for $n = 2, 3, \ldots$.

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A result of Slobodnik

For a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ let $\mathbb{E}(\mathcal{F})$ be the closure of $\mathcal{F}$ under countable unions and isometrical images.

Notice that $\mathcal{D}^n_L = \mathbb{E}(\mathcal{D}^n_L)$.

Theorem (Slobodnik 1976)

For every $n \geq 2$

$$\mathcal{D}^n_L \subset \mathbb{E}(\mathrm{Lip}_{nwd}),$$

where Lip_{nwd} is the family of all restrictions of Lipschitz functions $g\colon \mathbb{R}^{n-1} \to \mathbb{R}$ to compact nowhere dense $K \subset \mathbb{R}^{n-1}$.

In particular, any $D \in \mathcal{D}^n_L$ has Lebesgue measure 0,

while there is a separately continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ having positive Lebesgue measure.

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Revised problem of Kronrod

(P) For $n \geq 2$ find a family $\mathcal{F} \subset \mathrm{Lip}_{nwd}$ such that $\mathbb{E}(\mathcal{F}) = \mathcal{D}^n_L$.

Let Conv, D^k, and C^k be the classes of all $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ that are, respectively, convex, k-times differentiable, and continuously k-times differentiable.

Theorem (KC and T. Glatzer 2013)

- $\mathbb{E}(\mathrm{Conv}_{nwd}) \subset \mathcal{D}^n_L$.
- $\mathbb{E}(C^2_{nwd}) \subset \mathcal{D}^n_L$ for $n = 2$.
- $\mathbb{E}(D^1_{nwd}) \not\subset \mathcal{D}^n_L$ for $n = 2$.

Problem (KC and T. Glatzer 2013)

For $n = 2$

- *is $\mathbb{E}(C^1_{nwd}) \subset \mathcal{D}^n_L$?*
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- Is $\mathbb{E}(C^1_{nwd}) \subset \mathcal{D}^2_L$? What about $\mathbb{E}(D^2_{nwd}) \subset \mathcal{D}^2_L$?

Theorem (Zajíček, 2022 preprint)

$\mathbb{E}(C^1_{nwd}) \not\subset \mathcal{D}^2_L$.

Theorem (Main result of the talk)

For every $f \in C^1$ with nowhere monotone derivative f' there exists a nowhere dense perfect $P \subset \mathbb{R}$ such that $f \restriction P \notin \mathcal{D}^2_L$.

Corollary

$\mathbb{E}(D^2_{nwd}) \not\subset \mathcal{D}^2_L$.

Proof of Corollary.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be differentiable nowhere monotone.
Use Main Theorem with $f(x) := \int_0^x h(t)\,dt$. □

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Outline

LC maps $g\colon \mathbb{R}^2 \to \mathbb{R}$ with $D(g) = f \restriction P$: tangents of f

Choose a C^1 map $\eta\colon \mathbb{R} \to [0,\infty)$ with $\eta^{-1}(0) = P$ and a set $C := \{c_i \in \mathbb{R}^2 \colon i \in \mathbb{N}\}$ contained in the envelope $E := \{\langle x, y\rangle \colon f(x) < y < f(x) + \eta(x)\}$ with $C' = f \restriction P$.

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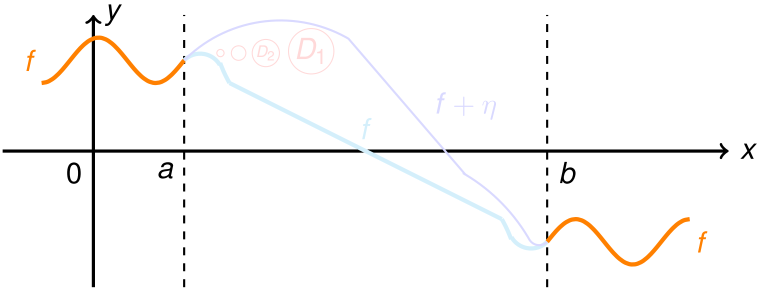

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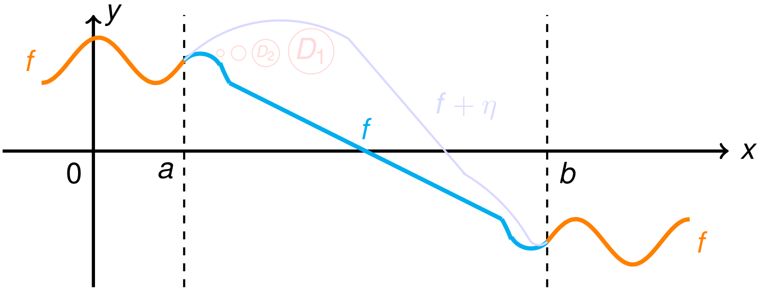

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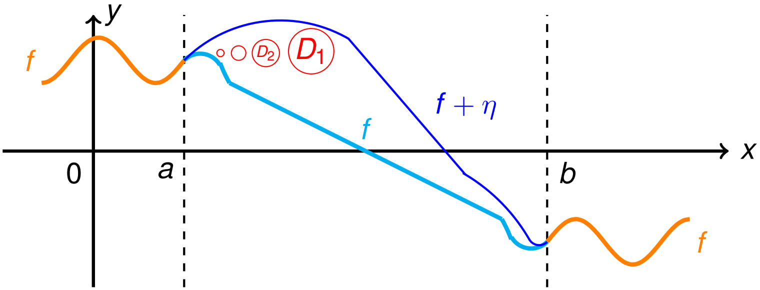

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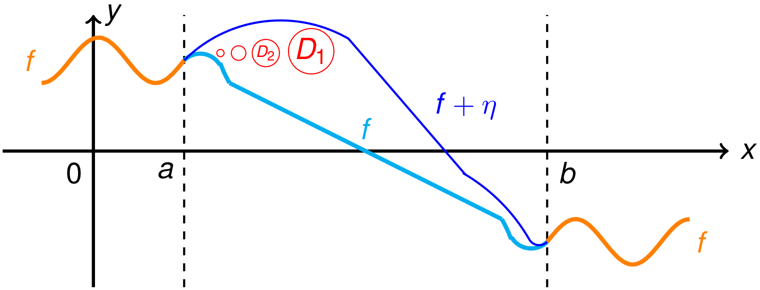

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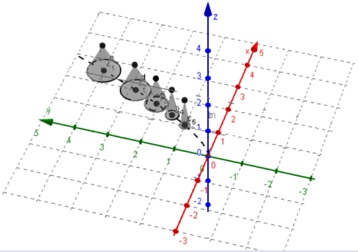

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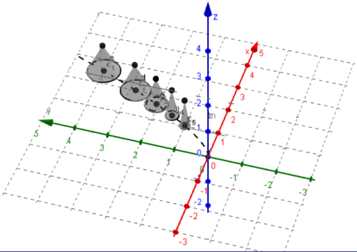

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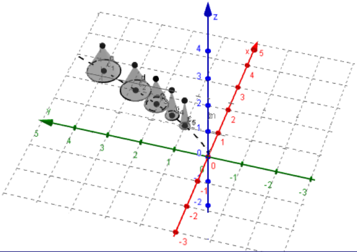

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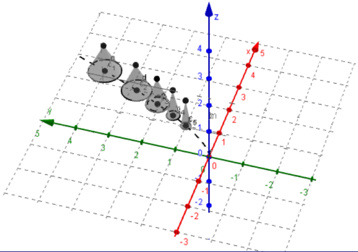

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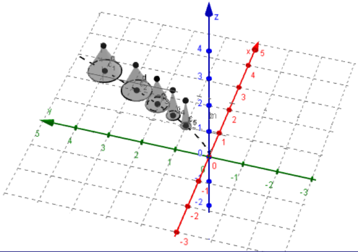

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If $f \in C^2$, then $T_{f,P}$ — the union of all lines tangent to f at $x \in P$ — is nowhere dense in $\mathbb{R}^2$. (Requires some argument.) So, we can choose disks D_i disjoint with $T_{f,P}$. □

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Theorem (Banakh & Maslyuchenko 2020)

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Outline

1. Separate and linear continuity – definitions and background
2. Sets of points of discontinuity: characterizations
3. Tangent lines and characterization of $\mathcal{D}_L^n$
4. Proof of Main Theorem
5. Comments and open problem

The main lemma

Thm: For every $f \in C^1$ with nowhere monotone f' there exists a nowhere dense perfect $P \subset \mathbb{R}$ such that $f \restriction P \notin \mathcal{D}^2_L$.

Lemma (Main Lemma)

For every $a < b$ and $f \in C^1$ with nowhere monotone f' there are $d \in (a, b)$ and perfect nowhere dense $N_d \subset (a, d)$ such that $\langle d, f(d)\rangle \in \operatorname{int}(T_{f,N_d})$.

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Construction of nowhere dense $P \subset \mathbb{R}$ with $f \restriction P \notin \mathcal{D}^2_L$

Construct a sequence $\langle \langle I_s, d_s, N_s \rangle : s \in 2^{<\omega} \rangle$ s.t.

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Desired contradiction

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A result used to prove the main lemma

Main Lemma: For every $a < b$ and $f \in C^1$ with nowhere monotone f' there are $d \in (a, b)$ and perfect nowhere dense $N_d \subset (a, d)$ such that $\langle d, f(d) \rangle \in \text{int}(T_{f,N_d})$.

Fact

Let $f \in C^1$ be s.t. f' is nowhere monotone. If $Z \subset (-\infty, a]$ and $\emptyset \neq (r, s) \subset Z$, then there is $\emptyset \neq (u, v) \subset (r, s)$ s.t.

$$T_{f,Z\setminus(u,v)} \cap ((a, \infty) \times \mathbb{R}) = T_{f,Z} \cap ((a, \infty) \times \mathbb{R}).$$

If Z is compact, then there is nowhere dense $N \subset Z$ s.t.

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Outline

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Thank you for your attention!