Differentiability versus continuity: What good Calc 1 student may ask about

Krzysztof Chris Ciesielski

Department of Mathematics, West Virginia University
MIPG, Department of Radiology, University of Pennsylvania

Based on BAMS survey written with Juan B. Seoane–Sepúlveda

Colloquium of Math Department at WVU, December 4, 2019

Preamble

- All discussed notions should be known to any math major
- All presented results have “elementary” proofs
- The text of this presentation can be found on my page:

  https://math.wvu.edu/~kciesiel/presentations.html

Preamble

- All discussed notions should be known to any math major
- All presented results have "elementary" proofs
- The text of this presentation can be found on my page:

  https://math.wvu.edu/~kciesiel/presentations.html

Preamble

- All discussed notions should be known to any math major
- All presented results have “elementary” proofs
- The text of this presentation can be found on my page:

  https://math.wvu.edu/~kciesiel/presentations.html

Preamble

- All discussed notions should be known to any math major
- All presented results have "elementary" proofs
- The text of this presentation can be found on my page:

  https://math.wvu.edu/~kciesiel/presentations.html

Outline

1. What good properties differentiable functions MUST have?
2. How BAD differentiable functions can be?
3. Differentiability from continuity: differentiable restrictions
4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$
5. Differentiable extensions: Jarník and Whitney theorems

Outline

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

($F'(0) = 0$ by squeeze theorem, as $\left|\frac{F(x)-F(0)}{x-0}\right| \le \left|\frac{x^2-F(0)}{x-0}\right| = |x|$.)

True question: *To what extend $f = F'$ must be continuous?*

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

($F'(0) = 0$ by squeeze theorem, as $\left|\frac{F(x)-F(0)}{x-0}\right| \leq \left|\frac{x^2-F(0)}{x-0}\right| = |x|$.)

True question: *To what extend $f = F'$ must be continuous?*

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

($F'(0) = 0$ by squeeze theorem, as $\left|\frac{F(x)-F(0)}{x-0}\right| \leq \left|\frac{x^2-F(0)}{x-0}\right| = |x|$.)

True question: *To what extend $f = F'$ must be continuous?*

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

($F'(0) = 0$ by squeeze theorem, as $\left|\frac{F(x)-F(0)}{x-0}\right| \leq \left|\frac{x^2-F(0)}{x-0}\right| = |x|$.)

True question: *To what extend $f = F'$ must be continuous?*

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

($F'(0) = 0$ by squeeze theorem, as $\left|\frac{F(x)-F(0)}{x-0}\right| \leq \left|\frac{x^2-F(0)}{x-0}\right| = |x|$.)

True question: *To what extend $f = F'$ must be continuous?*

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

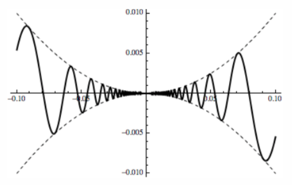
$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

($F'(0) = 0$ by squeeze theorem, as $\left|\frac{F(x)-F(0)}{x-0}\right| \le \left|\frac{x^2-F(0)}{x-0}\right| = |x|$.)

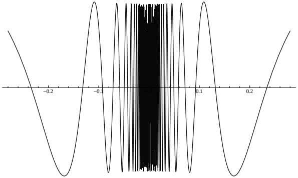
True question: *To what extend $f = F'$ must be continuous?*

About $F(x) = x^2 \sin\left(x^{-1}\right)$

This F appeared already in the 1881 paper of Vito Volterra (1860-1940)



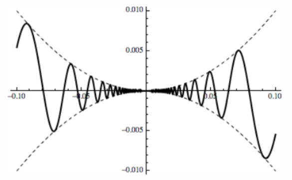
Graph of F



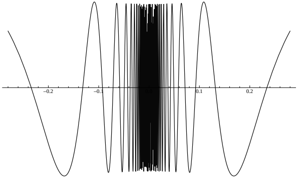
Graph of F'

About $F(x) = x^2 \sin\left(x^{-1}\right)$

This F appeared already in the 1881 paper of Vito Volterra (1860-1940)



Graph of F



Graph of F'

To what extend $f = F'$ must be continuous?

Jean-Gaston Darboux
(1842-1917)

Theorem (Darboux 1875)

Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called Darboux functions.

To what extend $f = F'$ must be continuous?

Jean-Gaston Darboux
(1842-1917)

Theorem (Darboux 1875)

Any derivative $f \colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called Darboux functions.

To what extend $f = F'$ must be continuous?

Jean-Gaston Darboux
(1842-1917)

Theorem (Darboux 1875)

Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called Darboux functions.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \leq y \leq f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi \colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \leq 0 \leq f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \leq y \leq f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi \colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \leq 0 \leq f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \leq y \leq f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \leq 0 \leq f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \leq y \leq f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \leq 0 \leq f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \leq y \leq f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \leq 0 \leq f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \le y \le f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \le 0 \le f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \le y \le f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \le 0 \le f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \le y \le f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \le 0 \le f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Easy proof that any $f = F'$ has IVP

Fix $a, b, y \in \mathbb{R}$ with $f(a) \le y \le f(b)$.

Can assume that $a < b$. Need $x \in [a, b]$ with $f(x) = y$.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ as $\varphi(t) := F(t) - yt$. So $\varphi'(t) = f(t) - y$

and $\varphi'(a) = f(a) - y \le 0 \le f(b) - y = \varphi'(b)$.

Need $x \in [a, b]$ with $\varphi'(x) = 0$. Can assume $\varphi'(a) < 0 < \varphi'(b)$.

Then, minimum of φ on $[a, b]$ is attained at an $x \in (a, b)$.

So, $\varphi'(x) = 0$, as needed.

Baire result

René-Louis Baire
(1874-1932)

Theorem (1899 dissertation of Baire)

The derivative of any differentiable $F\colon \mathbb{R} \to \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a ***dense*** *G_δ-set.*

Baire result

René-Louis Baire
(1874-1932)

Theorem (1899 dissertation of Baire)

*The derivative of any differentiable $F\colon \mathbb{R} \to \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a **dense** G_δ-set.*

Baire result

René-Louis Baire (1874-1932)

Theorem (1899 dissertation of Baire)

The derivative of any differentiable $F\colon \mathbb{R} \to \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a ***dense*** *G_δ-set.*

Quick peek at a proof and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R} \to \mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^{\infty} V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0} \{x \in \mathbb{R}\colon |g(s) - g(t)| < 1/n \text{ for all } s,t \in (x-\delta, x+\delta)\}.$$

If $g = \lim_{n\to\infty} g_n$, $g_n\colon \mathbb{R} \to \mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} U_N^n$, where each U_N^n is the interior of the closed set

$$\{x \in \mathbb{R}\colon |f_k(x) - f_m(x)| \le 1/n \text{ for all } m,k \ge N\}.$$

Theorem (Sets of points of continuity of derivatives)

Let $G \subset \mathbb{R}$.
There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

So, the complement of C_f can be dense.

Quick peek at a proof and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R}\to\mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^\infty V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0} \{x\in\mathbb{R}\colon |g(s)-g(t)|<1/n \text{ for all } s,t\in(x-\delta,x+\delta)\}.$$

If $g = \lim_{n\to\infty} g_n$, $g_n\colon \mathbb{R}\to\mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^\infty \bigcup_{N=1}^\infty U_N^n$, where each U_N^n is the interior of the closed set

$$\{x\in\mathbb{R}\colon |f_k(x)-f_m(x)|\le 1/n \text{ for all } m,k\ge N\}.$$

Theorem (Sets of points of continuity of derivatives)

Let $G\subset\mathbb{R}$.
There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

So, the complement of C_f can be dense.

Quick peek at a proof and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R}\to\mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^\infty V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0}\{x\in\mathbb{R}\colon |g(s)-g(t)|<1/n \text{ for all } s,t\in(x-\delta,x+\delta)\}.$$

If $g=\lim_{n\to\infty} g_n$, $g_n\colon\mathbb{R}\to\mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^\infty\bigcup_{N=1}^\infty U_N^n$, where each U_N^n is the interior of the closed set

$$\{x\in\mathbb{R}\colon |f_k(x)-f_m(x)|\le 1/n \text{ for all } m,k\ge N\}.$$

Theorem (Sets of points of continuity of derivatives)

Let $G\subset\mathbb{R}$.
There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

So, the complement of C_f can be dense.

Quick peek at a proof and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R} \to \mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^{\infty} V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0} \{x \in \mathbb{R}\colon |g(s) - g(t)| < 1/n \text{ for all } s,t \in (x-\delta, x+\delta)\}.$$

If $g = \lim_{n\to\infty} g_n$, $g_n\colon \mathbb{R} \to \mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} U_N^n$, where each U_N^n is the interior of the closed set

$$\{x \in \mathbb{R}\colon |f_k(x) - f_m(x)| \le 1/n \text{ for all } m,k \ge N\}.$$

Theorem (Sets of points of continuity of derivatives)

Let $G \subset \mathbb{R}$.
There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

So, the complement of C_f can be dense.

Quick peek at a proof and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R}\to\mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^{\infty} V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0} \{x\in\mathbb{R}\colon |g(s)-g(t)| < 1/n \text{ for all } s,t\in(x-\delta,x+\delta)\}.$$

If $g = \lim_{n\to\infty} g_n$, $g_n\colon \mathbb{R}\to\mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^{\infty}\bigcup_{N=1}^{\infty} U_N^n$, where each U_N^n is the interior of the closed set

$$\{x\in\mathbb{R}\colon |f_k(x)-f_m(x)| \le 1/n \text{ for all } m,k\ge N\}.$$

Theorem (Sets of points of continuity of derivatives)

Let $G\subset\mathbb{R}$.

There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

So, the complement of C_f can be dense.

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
  (True for any Darboux Baire class one map.)

- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point: an $x \in [0,1]$ with $f(x) = x$.
  $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
  So, there is $x \in [0,1]$ with $g(x) = 0$.

- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
  (True for any Darboux Baire class one map.)

- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point:
  an $x \in [0,1]$ with $f(x) = x$.
  $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
  So, there is $x \in [0,1]$ with $g(x) = 0$.

- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
(True for any Darboux Baire class one map.)

- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point:
an $x \in [0,1]$ with $f(x) = x$.
$g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
So, there is $x \in [0,1]$ with $g(x) = 0$.

- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
  (True for any Darboux Baire class one map.)
- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point:
  an $x \in [0,1]$ with $f(x) = x$.
  $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
  So, there is $x \in [0,1]$ with $g(x) = 0$.
- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
  (True for any Darboux Baire class one map.)
- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point:
  an $x \in [0,1]$ with $f(x) = x$.
  $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
  So, there is $x \in [0,1]$ with $g(x) = 0$.
- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
  (True for any Darboux Baire class one map.)
- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point:
  an $x \in [0,1]$ with $f(x) = x$.
  $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
  So, there is $x \in [0,1]$ with $g(x) = 0$.
- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
  (True for any Darboux Baire class one map.)
- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point:
  an $x \in [0,1]$ with $f(x) = x$.
  $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
  So, there is $x \in [0,1]$ with $g(x) = 0$.
- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph. (True for any Darboux Baire class one map.)
- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point: an $x \in [0,1]$ with $f(x) = x$. $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$. So, there is $x \in [0,1]$ with $g(x) = 0$.
- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Other properties that derivatives and continuous maps

- Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has a connected graph.
  (True for any Darboux Baire class one map.)
- Any derivative $f\colon [0,1] \to [0,1]$ has a fix point:
  an $x \in [0,1]$ with $f(x) = x$.
  $g(x) := f(x) - x$ is a derivative with $g(0) \geq 0 \geq g(1)$.
  So, there is $x \in [0,1]$ with $g(x) = 0$.
- (New result, from years 2000-2003) Finite composition f of derivatives from $I := [0,1]$ into I has a fix point.

Open Problem

Must f as above have connected graph?

Outline

Differentiable monster (# 1)

There are continuous nowhere monotone maps
(e.g. Weierstrass example we discuss latter).
Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; KC 2017; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Differentiable monster (# 1)

There are continuous nowhere monotone maps (e.g. Weierstrass example we discuss latter).

Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; KC 2017; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Differentiable monster (# 1)

There are continuous nowhere monotone maps (e.g. Weierstrass example we discuss latter).
Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; KC 2017; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Differentiable monster (# 1)

There are continuous nowhere monotone maps
(e.g. Weierstrass example we discuss latter).
Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; KC 2017; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Differentiable monster (# 1)

There are continuous nowhere monotone maps
(e.g. Weierstrass example we discuss latter).
Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; KC 2017; and many others)

There is differentiable $f \colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x \colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Differentiable monster (# 1)

There are continuous nowhere monotone maps (e.g. Weierstrass example we discuss latter).
Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; KC 2017; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Differentiable monster (# 1)

There are continuous nowhere monotone maps
(e.g. Weierstrass example we discuss latter).
Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; KC 2017; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Arnaud Denjoy and Dimitrie Pompeiu

Arnaud Denjoy (1884–1974)

Dimitrie Pompeiu (1873–1954)

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N\in\mathbb{N}} \bigcup_{n\ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n})-h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$, *with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N\in\mathbb{N}} \bigcup_{n\ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n})-h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$*.*

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$*.*

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N \in \mathbb{N}} \bigcup_{n \ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n}) - h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \leq i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$, *with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \geq 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N \in \mathbb{N}} \bigcup_{n \geq N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n}) - h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$*.*

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$*.*

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N\in\mathbb{N}} \bigcup_{n \ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n})-h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i : i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$*.*

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$*.*

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N\in\mathbb{N}} \bigcup_{n\ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n})-h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i : i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

(ii) $h = g^{-1} : \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} : h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N\in\mathbb{N}} \bigcup_{n\ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n})-h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$*.*

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$*.*

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N \in \mathbb{N}} \bigcup_{n \ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n}) - h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$, *with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity and "differentiability" of g is proved in the following slides. This easily implies the rest of (i).

(ii) Z is G_δ as $Z = \bigcap_{i,N\in\mathbb{N}} \bigcup_{n\ge N} h_n^{-1}(-1/i, 1/i)$, where $h_n(x) := \frac{h(x+2^{-n})-h(x)}{2^{-n}}$. The rest of (ii) & (iii) easily follow from (i). □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d\in D}((-d+Z) \cap (d-Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x-t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d\in D}((-d+Z)\cap(d-Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x-t) - h'(x)$.

$f' > 0$ on $t+D$: $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$, as $t+d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$, as $d-t \in Z$. □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x - t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D}((-d + Z) \cap (d - Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x - t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t + d) = h'(d) - h'(t + d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d - t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D}((-d + Z) \cap (d - Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x-t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d\in D}((-d+Z) \cap (d-Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x-t) - h'(x)$.

$f' > 0$ on $t+D$: $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$, as $t+d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$, as $d-t \in Z$. □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D} ((-d + Z) \cap (d - Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x-t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d\in D}((-d+Z) \cap (d-Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x-t) - h'(x)$.

$f' > 0$ on $t+D$: $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$, as $t+d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$, as $d-t \in Z$. □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x - t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D} \left((-d + Z) \cap (d - Z)\right)$ works.

Clearly f is differentiable with $f'(x) = h'(x - t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t + d) = h'(d) - h'(t + d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d - t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

$g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is continuous

g is continuous, since the series $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$

converges uniformly on every bounded set:

$|g(x)| \le \sum_{i=1}^{\infty} r^i (|x| + i + 1)$, as

$\left|(x - q_i)^{1/3}\right| \le (|x| + |q_i| + 1)^{1/3} \le |x| + |q_i| + 1 \le |x| + i + 1.$

$g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is "differentiable"

Let $\psi_i(x) := r^i (x - q_i)^{1/3}$. It is enough to show that

$$g'(x) = \sum_{i=1}^{\infty} \psi_i'(x). \qquad (1)$$

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) = \infty$, since, for every $y \neq x$,
$\frac{g(x)-g(y)}{x-y} = \sum_{i=1}^{\infty} \frac{\psi_i(x)-\psi_i(y)}{x-y} \geq \sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$,
and $\sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x-y|$.

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) < \infty$, since, for every $y \neq x$, we have $0 < \frac{\psi_i(x)-\psi_i(y)}{x-y} \leq 6\psi_i'(x)$. (Draw graph.)
Indeed, for $\varepsilon > 0$ and $n \in \mathbb{N}$ for which $\sum_{i=n+1}^{\infty} \psi_i'(x) < \varepsilon/14$,

$$\left| \frac{g(x)-g(y)}{x-y} - \sum_{i=1}^{\infty} \psi_i'(x) \right| \leq \sum_{i=1}^{n} \left| \frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x) \right| + 7 \left| \sum_{i=n+1}^{\infty} \psi_i'(x) \right|$$
$$\leq \sum_{i=1}^{n} \left| \frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x) \right| + \frac{\varepsilon}{2},$$

which is less than ε for y close enough to x.

$g(x)=\sum_{i=1}^{\infty} r^i(x-q_i)^{1/3}$ is “differentiable”

Let $\psi_i(x) := r^i(x-q_i)^{1/3}$. It is enough to show that

$$g'(x)=\sum_{i=1}^{\infty} \psi_i'(x). \qquad (1)$$

(1) holds when $\sum_{i=1}^{\infty}\psi_i'(x)=\infty$, since, for every $y\neq x$,
$\frac{g(x)-g(y)}{x-y}=\sum_{i=1}^{\infty}\frac{\psi_i(x)-\psi_i(y)}{x-y}\geq\sum_{i=1}^{n}\frac{\psi_i(x)-\psi_i(y)}{x-y}$,
and $\sum_{i=1}^{n}\frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x-y|$.

(1) holds when $\sum_{i=1}^{\infty}\psi_i'(x)<\infty$, since, for every $y\neq x$, we have $0<\frac{\psi_i(x)-\psi_i(y)}{x-y}\leq 6\psi_i'(x)$. (Draw graph.)
Indeed, for $\varepsilon>0$ and $n\in\mathbb{N}$ for which $\sum_{i=n+1}^{\infty}\psi_i'(x)<\varepsilon/14$,

$$\left|\frac{g(x)-g(y)}{x-y}-\sum_{i=1}^{\infty}\psi_i'(x)\right|\leq\sum_{i=1}^{n}\left|\frac{\psi_i(x)-\psi_i(y)}{x-y}-\psi_i'(x)\right|+7\left|\sum_{i=n+1}^{\infty}\psi_i'(x)\right|$$
$$\leq\sum_{i=1}^{n}\left|\frac{\psi_i(x)-\psi_i(y)}{x-y}-\psi_i'(x)\right|+\frac{\varepsilon}{2},$$

which is less than ε for y close enough to x.

$g(x)=\sum_{i=1}^{\infty} r^i(x-q_i)^{1/3}$ is "differentiable"

Let $\psi_i(x) := r^i(x-q_i)^{1/3}$. It is enough to show that

$$g'(x)=\sum_{i=1}^{\infty} \psi_i'(x). \qquad (1)$$

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x)=\infty$, since, for every $y\neq x$,

$\frac{g(x)-g(y)}{x-y}=\sum_{i=1}^{\infty}\frac{\psi_i(x)-\psi_i(y)}{x-y}\geq\sum_{i=1}^{n}\frac{\psi_i(x)-\psi_i(y)}{x-y}$,

and $\sum_{i=1}^{n}\frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x-y|$.

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x)<\infty$, since, for every $y\neq x$, we have $0<\frac{\psi_i(x)-\psi_i(y)}{x-y}\leq 6\psi_i'(x)$. (Draw graph.)

Indeed, for $\varepsilon>0$ and $n\in\mathbb{N}$ for which $\sum_{i=n+1}^{\infty}\psi_i'(x)<\varepsilon/14$,

$$\left|\frac{g(x)-g(y)}{x-y}-\sum_{i=1}^{\infty}\psi_i'(x)\right|\leq\sum_{i=1}^{n}\left|\frac{\psi_i(x)-\psi_i(y)}{x-y}-\psi_i'(x)\right|+7\left|\sum_{i=n+1}^{\infty}\psi_i'(x)\right|$$
$$\leq\sum_{i=1}^{n}\left|\frac{\psi_i(x)-\psi_i(y)}{x-y}-\psi_i'(x)\right|+\frac{\varepsilon}{2},$$

which is less than ε for y close enough to x.

$g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is "differentiable"

Let $\psi_i(x) := r^i (x - q_i)^{1/3}$. It is enough to show that

$$g'(x) = \sum_{i=1}^{\infty} \psi_i'(x). \tag{1}$$

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) = \infty$, since, for every $y \neq x$, $\frac{g(x)-g(y)}{x-y} = \sum_{i=1}^{\infty} \frac{\psi_i(x)-\psi_i(y)}{x-y} \geq \sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$, and $\sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x-y|$.

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) < \infty$, since, for every $y \neq x$, we have $0 < \frac{\psi_i(x)-\psi_i(y)}{x-y} \leq 6\psi_i'(x)$. (Draw graph.) Indeed, for $\varepsilon > 0$ and $n \in \mathbb{N}$ for which $\sum_{i=n+1}^{\infty} \psi_i'(x) < \varepsilon/14$,

$$\left|\frac{g(x)-g(y)}{x-y} - \sum_{i=1}^{\infty} \psi_i'(x)\right| \leq \sum_{i=1}^{n} \left|\frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + 7\left|\sum_{i=n+1}^{\infty} \psi_i'(x)\right|$$

$$\leq \sum_{i=1}^{n} \left|\frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + \frac{\varepsilon}{2},$$

which is less than ε for y close enough to x.

$g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is "differentiable"

Let $\psi_i(x) := r^i(x - q_i)^{1/3}$. It is enough to show that

$$g'(x) = \sum_{i=1}^{\infty} \psi_i'(x). \qquad (1)$$

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) = \infty$, since, for every $y \neq x$,
$\frac{g(x)-g(y)}{x-y} = \sum_{i=1}^{\infty} \frac{\psi_i(x)-\psi_i(y)}{x-y} \geq \sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$,
and $\sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x-y|$.

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) < \infty$, since, for every $y \neq x$, we have $0 < \frac{\psi_i(x)-\psi_i(y)}{x-y} \leq 6\psi_i'(x)$. (Draw graph.)
Indeed, for $\varepsilon > 0$ and $n \in \mathbb{N}$ for which $\sum_{i=n+1}^{\infty} \psi_i'(x) < \varepsilon/14$,

$$\left|\frac{g(x)-g(y)}{x-y} - \sum_{i=1}^{\infty} \psi_i'(x)\right| \leq \sum_{i=1}^{n} \left|\frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + 7\left|\sum_{i=n+1}^{\infty} \psi_i'(x)\right|$$

$$\leq \sum_{i=1}^{n} \left|\frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + \frac{\varepsilon}{2},$$

which is less than ε for y close enough to x.

$g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is "differentiable"

Let $\psi_i(x) := r^i (x - q_i)^{1/3}$. It is enough to show that

$$g'(x) = \sum_{i=1}^{\infty} \psi_i'(x). \tag{1}$$

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) = \infty$, since, for every $y \neq x$,
$\frac{g(x)-g(y)}{x-y} = \sum_{i=1}^{\infty} \frac{\psi_i(x)-\psi_i(y)}{x-y} \geq \sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$,
and $\sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x - y|$.

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) < \infty$, since, for every $y \neq x$, we have $0 < \frac{\psi_i(x)-\psi_i(y)}{x-y} \leq 6\psi_i'(x)$. (Draw graph.)
Indeed, for $\varepsilon > 0$ and $n \in \mathbb{N}$ for which $\sum_{i=n+1}^{\infty} \psi_i'(x) < \varepsilon/14$,

$$\left| \frac{g(x)-g(y)}{x-y} - \sum_{i=1}^{\infty} \psi_i'(x) \right| \leq \sum_{i=1}^{n} \left| \frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x) \right| + 7 \left| \sum_{i=n+1}^{\infty} \psi_i'(x) \right|$$
$$\leq \sum_{i=1}^{n} \left| \frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x) \right| + \frac{\varepsilon}{2},$$

which is less than ε for y close enough to x.

$g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is "differentiable"

Let $\psi_i(x) := r^i (x - q_i)^{1/3}$. It is enough to show that

$$g'(x) = \sum_{i=1}^{\infty} \psi_i'(x). \quad (1)$$

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) = \infty$, since, for every $y \neq x$,
$\frac{g(x)-g(y)}{x-y} = \sum_{i=1}^{\infty} \frac{\psi_i(x)-\psi_i(y)}{x-y} \geq \sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$,
and $\sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x-y|$.

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) < \infty$, since, for every $y \neq x$, we have $0 < \frac{\psi_i(x)-\psi_i(y)}{x-y} \leq 6\psi_i'(x)$. (Draw graph.)
Indeed, for $\varepsilon > 0$ and $n \in \mathbb{N}$ for which $\sum_{i=n+1}^{\infty} \psi_i'(x) < \varepsilon/14$,

$$\left|\frac{g(x)-g(y)}{x-y} - \sum_{i=1}^{\infty} \psi_i'(x)\right| \leq \sum_{i=1}^{n} \left|\frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + 7\left|\sum_{i=n+1}^{\infty} \psi_i'(x)\right|$$
$$\leq \sum_{i=1}^{n} \left|\frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x)\right| + \frac{\varepsilon}{2},$$

which is less than ε for y close enough to x.

$g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ is "differentiable"

Let $\psi_i(x) := r^i (x - q_i)^{1/3}$. It is enough to show that

$$g'(x) = \sum_{i=1}^{\infty} \psi_i'(x). \tag{1}$$

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) = \infty$, since, for every $y \neq x$,
$\frac{g(x)-g(y)}{x-y} = \sum_{i=1}^{\infty} \frac{\psi_i(x)-\psi_i(y)}{x-y} \geq \sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$,
and $\sum_{i=1}^{n} \frac{\psi_i(x)-\psi_i(y)}{x-y}$ is arbitrarily large for big n & small $|x-y|$.

(1) holds when $\sum_{i=1}^{\infty} \psi_i'(x) < \infty$, since, for every $y \neq x$, we have $0 < \frac{\psi_i(x)-\psi_i(y)}{x-y} \leq 6\psi_i'(x)$. (Draw graph.)
Indeed, for $\varepsilon > 0$ and $n \in \mathbb{N}$ for which $\sum_{i=n+1}^{\infty} \psi_i'(x) < \varepsilon/14$,

$$\begin{aligned} \left| \frac{g(x)-g(y)}{x-y} - \sum_{i=1}^{\infty} \psi_i'(x) \right| &\leq \sum_{i=1}^{n} \left| \frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x) \right| + 7 \left| \sum_{i=n+1}^{\infty} \psi_i'(x) \right| \\ &\leq \sum_{i=1}^{n} \left| \frac{\psi_i(x)-\psi_i(y)}{x-y} - \psi_i'(x) \right| + \frac{\varepsilon}{2}, \end{aligned}$$

which is less than ε for y close enough to x.

Outline

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F\colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

Bernard Bolzano (1781-1848)

Karl Weierstrass (1815–1897)

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F\colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

Bernard Bolzano (1781-1848)

Karl Weierstrass (1815–1897)

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F\colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

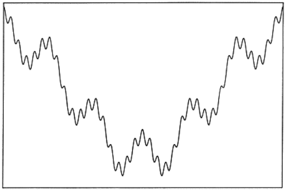
Bernard Bolzano (1781-1848)

Karl Weierstrass (1815–1897)

Weierstrass' Monster: $W(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x)$

Teiji Takagi (1875–1960)

Bartel van der Waerden (1903–1996)



$F(x) = \sum_{n=0}^{\infty} 4^n min\{|x - \frac{k}{8^n}| \colon k \in \mathbb{Z}\}$

Weierstrass' Monster of

Takagi from 1903, and

van der Waerden, from 1930

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f: \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f: \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2): [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

New proof of differentiable restriction theorem

Goal: If $f \colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Theorem (With new (2017/18) simple proof, by KC)

For every continuous increasing $f \colon [a, b] \to \mathbb{R}$ there is perfect P such that $f \restriction P$ is Lipschitz.

This reasonably easily implies the Goal.

New proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Theorem (With new (2017/18) simple proof, by KC)

For every continuous increasing $f\colon [a, b] \to \mathbb{R}$ there is perfect P such that $f \restriction P$ is Lipschitz.

This reasonably easily implies the Goal.

New proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Theorem (With new (2017/18) simple proof, by KC)

For every continuous increasing $f\colon [a, b] \to \mathbb{R}$ there is perfect P such that $f \restriction P$ is Lipschitz.

This reasonably easily implies the Goal.

Outline

1. What good properties differentiable functions MUST have?
2. How BAD differentiable functions can be?
3. Differentiability from continuity: differentiable restrictions
4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$
5. Differentiable extensions: Jarník and Whitney theorems

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$ ($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$) but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

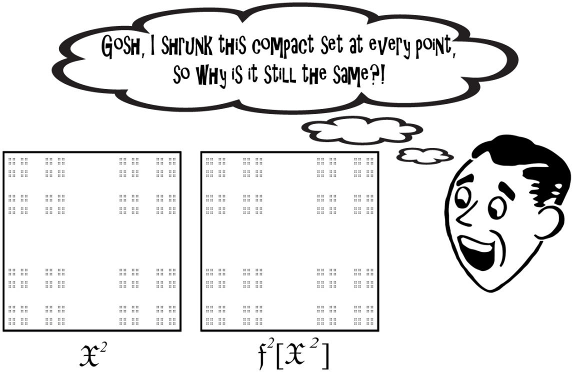


Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma\colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s = \langle 1,1,1,\ldots\rangle, \\ \langle 0,0,\ldots,0,1,s_{k+1},\ldots\rangle & \text{if } s = \langle 1,1,\ldots,1,0,s_{k+1},\ldots\rangle. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

where $N(s \restriction 0) = 1$ and, for $n > 0$,

$$N(s \restriction n) = \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n = (1(1-s_{n-1})s_{n-2}\ldots s_0)_2.$$

E.g. $N(101101) = (1001101)_2$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h \colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma \colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s = \langle 1,1,1,\ldots\rangle, \\ \langle 0,0,\ldots,0,1,s_{k+1},,\ldots\rangle & \text{if } s = \langle 1,1,\ldots,1,0,s_{k+1},\ldots\rangle. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

where $N(s \restriction 0) = 1$ and, for $n > 0$,

$$\begin{aligned} N(s \restriction n) &= \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n \\ &= (1(1 - s_{n-1})s_{n-2} \ldots s_0)_2. \end{aligned}$$

E.g. $N(101101) = (1001101)_2$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma\colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s = \langle 1,1,1,\ldots\rangle, \\ \langle 0,0,\ldots,0,1,s_{k+1},,\ldots\rangle & \text{if } s = \langle 1,1,\ldots,1,0,s_{k+1},\ldots\rangle. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

where $N(s \restriction 0) = 1$ and, for $n > 0$,

$$\begin{aligned} N(s \restriction n) &= \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n \\ &= (1(1 - s_{n-1})s_{n-2}\ldots s_0)_2. \end{aligned}$$

E.g. $N(101101) = (1001101)_2$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma\colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s = \langle 1,1,1,\ldots\rangle, \\ \langle 0,0,\ldots,0,1,s_{k+1},,\ldots\rangle & \text{if } s = \langle 1,1,\ldots,1,0,s_{k+1},\ldots\rangle. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

where $N(s \restriction 0) = 1$ and, for $n > 0$,

$$\begin{aligned} N(s \restriction n) &= \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n \\ &= (1(1 - s_{n-1})s_{n-2}\ldots s_0)_2. \end{aligned}$$

E.g. $N(101101) = (1001101)_2$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma\colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s = \langle 1,1,1,\ldots\rangle, \\ \langle 0,0,\ldots,0,1,s_{k+1},,\ldots\rangle & \text{if } s = \langle 1,1,\ldots,1,0,s_{k+1},\ldots\rangle. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

where $N(s \restriction 0) = 1$ and, for $n > 0$,

$$\begin{aligned} N(s \restriction n) &= \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n \\ &= (1(1 - s_{n-1})s_{n-2}\ldots s_0)_2. \end{aligned}$$

E.g. $N(101101) = (1001101)_2$

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega : s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega : s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \le |h(s) - h(t)| \le 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \le \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega\colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \le |h(s) - h(t)| \le 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega\colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \le \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots\rangle = \langle 1, \ldots, 1, 0, \ldots\rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,
Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega \ \exists k < \omega \ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Peek at a proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots\rangle = \langle 1, \ldots, 1, 0, \ldots\rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Outline

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.

Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.

Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.

Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

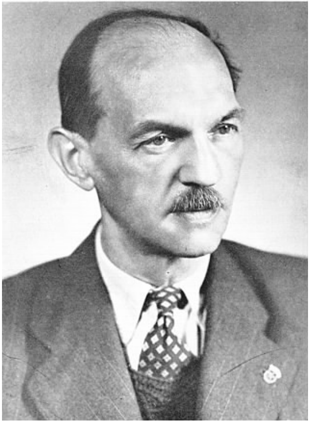
Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.

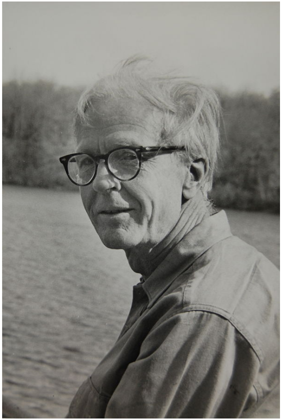
Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Vojtěch Jarník and Hassler Whitney



Vojtěch Jarník (1897–1970)



Hassler Whitney (1907-1989)

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f: Q \to \mathbb{R}$ has differentiable extension $F: \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \restriction \hat{Q}$ is continuously differentiable.

Here $\hat{Q}$ is a simple natural extension of Q.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f: \mathbb{R} \to \mathbb{R}$ there is C^1 map $g: \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \restriction \hat{Q}$ is continuously differentiable.

Here $\hat{Q}$ is a simple natural extension of Q.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \restriction \hat{Q}$ is continuously differentiable.

Here $\hat{Q}$ is a simple natural extension of Q.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \restriction \hat{Q}$ is continuously differentiable.

Here $\hat{Q}$ is a simple natural extension of Q.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \restriction \hat{Q}$ is continuously differentiable.

Here $\hat{Q}$ is a simple natural extension of Q.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

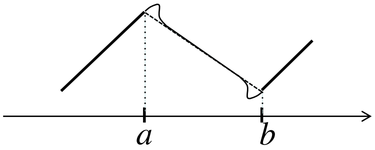


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

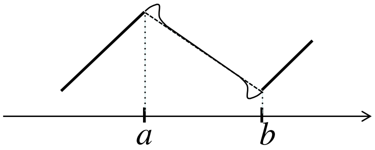


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

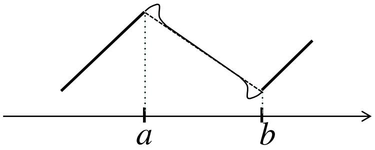


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

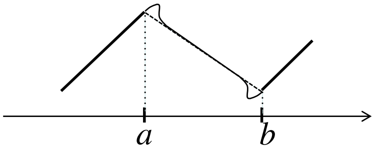


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

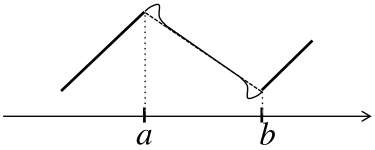


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F \colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f \colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f \colon P \to \mathbb{R}$, there exists a differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f\colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f\colon P \to \mathbb{R}$, there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f\colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f\colon P \to \mathbb{R}$, there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f\colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f\colon P \to \mathbb{R}$, there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f\colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f\colon P \to \mathbb{R}$, there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f\colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f\colon P \to \mathbb{R}$, there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1)
such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1) such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1) such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1) such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1) such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

BAMS survey contains a lot of other results

But this is all for today

Thank you for your attention!

BAMS survey contains a lot of other results

But this is all for today

Thank you for your attention!

BAMS survey contains a lot of other results

But this is all for today

Thank you for your attention!