Differentiability versus continuity: Restriction and extension theorems and monstrous examples

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Based on BAMS survey written with Juan B. Seoane–Sepúlveda

Function Theory on Infinite Dimensional Spaces XVI,
Madrid, November 19, 2019

Preamble: Whats ahead

- All discussed notions should be known to any math major

- All results presented have proofs (often very new) that require no Lebesgue measure theory

- The text of this presentation can be found on my page:

  https://math.wvu.edu/˜kciesiel/presentations.html

Outline

1. Continuity from differentiability: classical results

2. Continuity from differentiability: newer results

3. Differentiability from continuity: differentiable restrictions

4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$

5. Differentiable extensions: Jarník and Whitney theorems

6. Summary

Outline

1. Continuity from differentiability: classical results

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Continuity from differentiability: What is it to ask?

Clearly, if $F \colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G \colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)
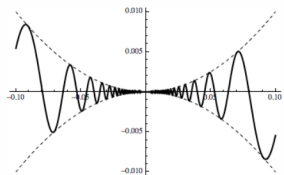
However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

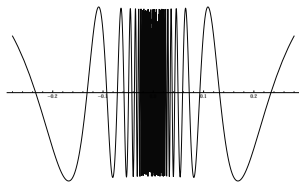**True question:** *To what extend $f = F'$ must be continuous?*

About $F(x) = x^2 \sin\left(x^{-1}\right)$

This F appeared already in the
1881 paper of Vito Volterra
(1860-1940)

Graph of F

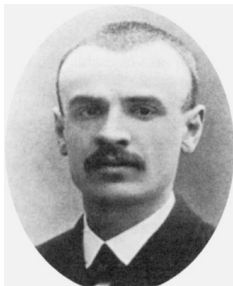Graph of F'

To what extend $f = F'$ must be continuous?

Jean-Gaston Darboux
(1842-1917)

Theorem (Darboux 1875)

Any derivative $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called Darboux functions.

Baire result

René-Louis Baire
(1874-1932)

Theorem (1899 dissertation of Baire)

The derivative of any differentiable $F \colon \mathbb{R} \to \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a dense G_δ-set.

Proof of previous theorem and a characterization

$F'(x) = \lim_{n \to \infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g \colon \mathbb{R} \to \mathbb{R}$, $C_g := \{x \colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^{\infty} V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta > 0} \{x \in \mathbb{R} \colon |g(s) - g(t)| < 1/n \text{ for all } s, t \in (x - \delta, x + \delta)\}.$$

If $g = \lim_{n \to \infty} g_n$, $g_n \colon \mathbb{R} \to \mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} U_N^n$, where each U_N^n is the interior of the closed set

$$\{x \in \mathbb{R} \colon |f_k(x) - f_m(x)| \leq 1/n \text{ for all } m, k \geq N\}.$$

Theorem (Sets of points of continuity of derivatives)

Let $G \subset \mathbb{R}$.
There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

Outline

Fixed point property

Theorem (Relatively new)

If $f = f_n \circ \cdots \circ f_1$, where each $f_i\colon [0,1] \to [0,1]$ is a derivative, then f has a fixed point.

For $n = 1$: easy exercise, as $h(x) = f(x) - x$ is Darboux.

For $n = 2$: proved independently in 2001
by Csörnyei, O'Neil & Preiss and by Elekes, Keleti & Prokaj.

For arbitrary n: Szuca 2003.

Open Problem

Must f as in the theorem have connected graph?

Yes for $n = 1$. Positive answer would imply the theorem.

Baire classification of composition of the derivatives.

Let $f = f_n \circ \cdots \circ f_1$, where each f_i is a derivative.

Then f is Darboux.

Any Darboux Baire class one map has connected graph.

A natural question: must f be of Baire class 1? NO

Theorem (*Andy* Bruckner and K. Ciesielski 2018)

There exist derivatives $\varphi, \gamma \colon [-1, 1] \to [-1, 1]$ such that their composition $\psi := \varphi \circ \gamma$ is not of Baire class one.

We use $\gamma(x) := \cos(x^{-1})$ and φ Pompeiu's map (see below).

Differentiable monster (# 1)

There are continuous nowhere monotone maps.
Can such maps be differentiable?

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson &
Stromberg 1974; Weil 1976; Aron, Gurariy &
Seoane-Sepúlveda 2005; and many others)

There is differentiable $f \colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x \colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Arnaud Denjoy and Dimitrie Pompeiu

Arnaud Denjoy (1884–1974)

Dimitrie Pompeiu (1873–1954)

A variant of Pompeiu function, of 1907

Fix $r \in (0, 1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \leq i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto $\mathbb{R}$, with $g'(q) = \infty$ for all $q \in \mathbb{Q}$.*

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with $h' \geq 0$ and $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ being a dense G_δ-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in $\mathbb{R}$.*

Pr. (i) Continuity follows from $|g(x)| \leq \sum_{i=1}^{\infty} r^i (|x| + i + 1)$.

Differentiability requires $g'(x) = \sum_{i=1}^{\infty} r^i \frac{1}{3} \frac{1}{(x - q_i)^{2/3}}$. Easy when series $= \infty$. Other case follows from $0 < \frac{\psi_i(y) - \psi_i(x)}{y - x} \leq 6\psi_i'(x)$.

(ii) and (iii) easily follow from (i). □

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x - t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D} \big((-d + Z) \cap (d - Z) \big)$ works.

Clearly f is differentiable with $f'(x) = h'(x - t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t + d) = h'(d) - h'(t + d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d - t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. $\quad\square$

Outline

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F \colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

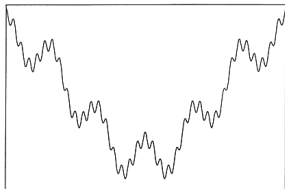Bernard Bolzano (1781-1848)   Karl Weierstrass (1815–1897)

Weierstrass' Monster: $W(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x)$

Teiji Takagi (1875–1960)

Bartel van der Waerden
(1903–1996)

$F(x) = \sum_{n=0}^{\infty} 4^n min\{|x - \frac{k}{8^n}|: k \in \mathbb{Z}\}$

Weierstrass' Monster of

Takagi from 1903, and

van der Waerden, from 1930

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f : \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \upharpoonright Q$ is differentiable.

Remark

There are continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f \upharpoonright Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2) : [0, 1] \to [0, 1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. $\quad\square$

New proof of differentiable restriction theorem

Goal: If $f \colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \upharpoonright Q$ is diff. for some perfect Q.

Theorem (With new (2017/18) simple proof, by KC)

For every continuous **increasing** $f \colon [a, b] \to \mathbb{R}$ *there is perfect P such that* $f \upharpoonright P$ *is Lipschitz.*

Proof based on the following results, due to Riesz:

Lemma (Rising sun lemma 1932, proof is an easy exercise)

If $g \colon [a, b] \to \mathbb{R}$ *is cont, then* $g(c) \leq g(d)$ *for every component* (c, d) *of* $U = \{x \in [a, b] \colon g(x) < g(y)$ *for some* $y \in (x, b]\}$.

Fact (Proved by induction)

Let $a < b$ *and* $\mathcal{J}$ *be a family of open intervals with* $\bigcup \mathcal{J} \subset (a, b)$.

(i) *If* $[\alpha, \beta] \subset \bigcup \mathcal{J}$, *then* $\sum_{I \in \mathcal{J}} \ell(I) > \beta - \alpha$.

(ii) *If* $I \in \mathcal{J}$ *are pairwise disjoint, then* $\sum_{I \in \mathcal{J}} \ell(I) \leq b - a$.

Riesz' Rising sun lemma

If $g\colon [a, b] \to \mathbb{R}$ is cont, then $g(c) \leq g(d)$ for every component (c, d) of $U = \{x \in [a, b]\colon g(x) < g(y) \text{ for some } y \in (x, b]\}$.

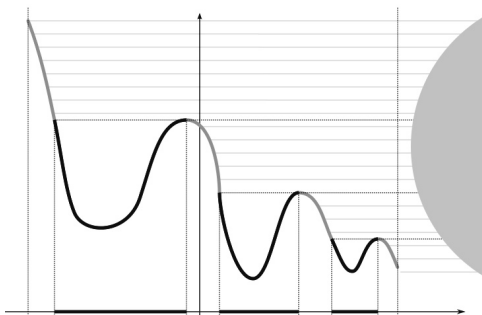Frigyes Riesz (1880-1956)

Illustration of the Rising Sun Lemma

The points in the set $U \cap (a, b)$ are those lying in the shadow.

Proof of Lipschitz restriction theorem

Goal: If $f: \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Have: If $g: [a, b] \to \mathbb{R}$ is cont, then $g(c) \leq g(d)$ for every comp. (c, d) of $\{x \in [a, b): g(x) < g(y)$ for some $y \in (x, b]\}$.

Sketch of proof. Fix $L > \frac{f(b)-f(a)}{b-a}$, put $g(t) = f(t) - Lt$, and

$$U = \{x \in [a, b): g(y) > g(x) \text{ for some } y \in (x, b]\}.$$

f is Lipschitz on $P = [\bar{a}, b] \setminus U$ with constant L, where

$\bar{a} = \sup\{x: [a, x) \subset U)\}$. Fix $X = \{x_n: n \in \mathbb{N}\}$. Need $P \setminus X \neq \emptyset$.

If $\mathcal{J} =$ open components of U, then $\ell(f[J]) \geq L\ell(J)$ for $J \in \mathcal{J}$.

By Fact (ii), $\sum_{J \in \mathcal{J}} \ell(f[J]) \leq f(b) - f(\bar{a})$. So,

$\sum_{J \in \mathcal{J}} \ell(J) \leq \frac{1}{L} \sum_{J \in \mathcal{J}} \ell(f[J]) \leq \frac{f(b)-f(\bar{a})}{L} < b - \bar{a}$, and by Fact (i),

$P \neq \emptyset$. To get $P \setminus X \neq \emptyset$ increase slightly $\mathcal{J}$.

End of proof of differentiable restriction theorem

Goal: If $f \colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Have: If $f \colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Proof of differentiable restriction theorem.

f is Lipschitz on some perfect P: proved above for somewhere monotone f; otherwise f is constant on some perfect set.

For function $f \restriction P$ use Morayne theorem to find perfect $Q \subset P$ such that the quotient map for $f \restriction Q$ is uniformly continuous. Then Q is as needed. $\qquad\square$

Outline

Differentiable monster (# 2)

Are differentiable $f \colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

> **Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)**
>
> There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself.   Also

> **Theorem (Edelstein 1962, almost contradicting above thm)**
>
> *If $f \colon X \to X$ is LC and X is compact, then f has a periodic point,*

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.
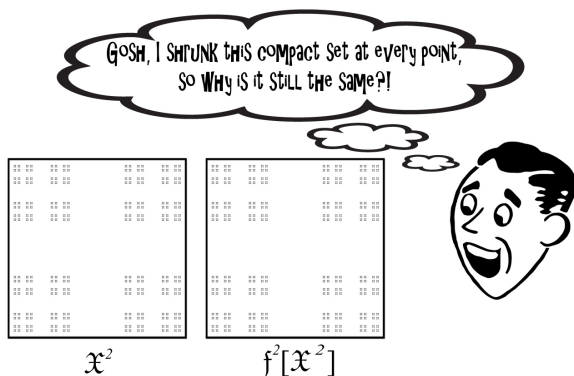
Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h \colon 2^\omega \to \mathbb{R}$ is embedding and
$\sigma \colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0, 0, 0, \ldots \rangle & \text{if } s = \langle 1, 1, 1, \ldots \rangle, \\ \langle 0, 0, \ldots, 0, 1, s_{k+1}, , \ldots \rangle & \text{if } s = \langle 1, 1, \ldots, 1, 0, s_{k+1}, \ldots \rangle. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1) N(s \upharpoonright n)}$,

where $N(s \upharpoonright 0) = 1$ and, for $n > 0$,

$$\begin{aligned} N(s \upharpoonright n) &= \sum_{i < n-1} s_i 2^i + (1 - s_{n-1}) 2^{n-1} + 2^n \\ &= (1(1 - s_{n-1}) s_{n-2} \ldots s_0)_2. \end{aligned}$$

E.g. $N(101101) = (1001101)_2$

Proof of $f' \equiv 0$ for $f = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\upharpoonright n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\upharpoonright n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\upharpoonright n)}.$$

Also (a): $\forall s \in 2^\omega \, \exists k < \omega \; N(\sigma(s) \upharpoonright n) = N(s \upharpoonright n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $f' \equiv 0$.

To see $f'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t.
$0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|f(h(s)) - f(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\upharpoonright n)}}{3^{-(n+1)N(s\upharpoonright n)}} = 3 \cdot 3^{-(n+1)}.$$

So $f'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. $\qquad\square$

Dynamical system ƒ

Every orbit $\{x, \mathfrak{f}(x), \mathfrak{f}^2(x), \ldots\}$ of $\mathfrak{f}$ is dense in $\mathfrak{X}$.

So, $\mathfrak{f}$ is a minimal dynamical system. Must it be?

Theorem (KC & JJ 2016: YES, essentially)

If $f: X \to X$ is onto, PC, and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \upharpoonright P$ is a minimal dynamical system,

where *f* is *pointwise contractive, PC*, if for every $x \in X$ there is open $U \ni x$ and $L \in [0, 1)$ s.t. $|f(x) - f(y)| \leq L|x - y|$ for all $y \in U$.

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Notation

For $J = (a, a + h)$ let $I_J = [a + h/3, a + 2h/3]$, middle third of J.

For closed $Q \subset \mathbb{R}$ and $f \colon Q \to \mathbb{R}$ let

$\hat{Q} = Q \cup \bigcup \{ I_J \colon J \text{ is a bounded connected component of } \mathbb{R} \setminus Q \}$,

$\bar{f} \colon \mathbb{R} \to \mathbb{R}$ — "the" linear interpolation of f, $\hat{f} = \bar{f} \upharpoonright \hat{Q}$.
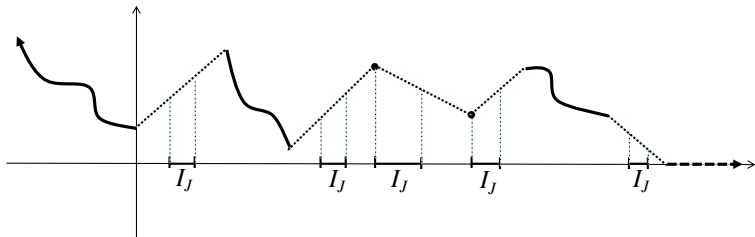

Figure: The linear interpolation $\bar{f}$ of f, represented by thick curves.

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f : Q \to \mathbb{R}$ has differentiable extension $F : \mathbb{R} \to \mathbb{R}$.

Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.
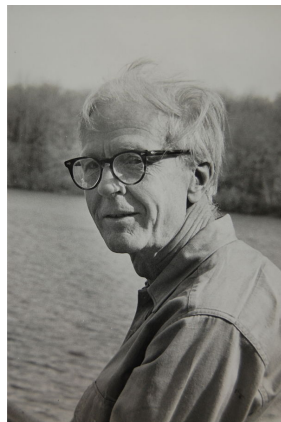
Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Vojtěch Jarník and Hassler Whitney
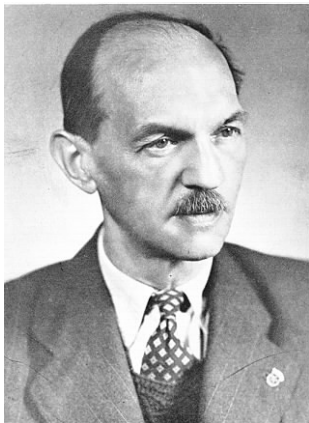
Vojtěch Jarník (1897–1970)

Hassler Whitney (1907-1989)

Jarník and Whitney differentiable extension theorems

> **Theorem (Jarník and Whitney thms, version of MC&KC 2017)**
>
> *If $Q \subset \mathbb{R}$ is closed, than any differentiable $f \colon Q \to \mathbb{R}$ has differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \upharpoonright \hat{Q}$ is continuously differentiable.*

> **Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)**
>
> *For every continuous $f \colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g \colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.*

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \upharpoonright Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f \colon Q \to \mathbb{R}$ has differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f \colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

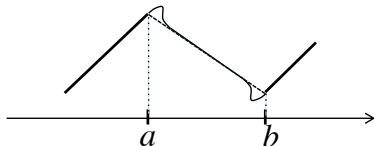**The right extension**: Small modification of $\bar{f}$: $F = \bar{f} + g$:

Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Differentiable extensions of f, Monster # 2

By Jarník's theorem, our $f \colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F \colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

> **Theorem (KC & JJ 2016: No)**
>
> *If $f \colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.*

Can such F can be bad? Yes, very bad!

> **Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)**
>
> *For every closed set $P \subseteq \mathbb{R}$ and differentiable $f \colon P \to \mathbb{R}$, there exists a differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.*

Differentiable monster (#3)

> **Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)**
>
> There exists everywhere differentiable nowhere monotone
> function $F \colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1)
> such that $F \upharpoonright \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3,   as #1+ #2 = #3

> **Proof.**
>
> Use previous theorem to $\mathfrak{f}$.   □

Outline

Summary of new (2017+) presented results

Example (New simple construction of a classic example)

There exists a differentiable nowhere monotone map $f \colon \mathbb{R} \to \mathbb{R}$.

Example (Greatly simplified construction of 2016 example)

There exists a differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect $\mathfrak{X} \subset \mathbb{R}$ with $\mathfrak{f}' \equiv 0$.

Theorem (C^1 interpolation thm, no Lebesgue measure needed)

For every continuous $f \colon \mathbb{R} \to \mathbb{R}$:

- *there is perfect $P \subset \mathbb{R}$ s.t. $f \upharpoonright P$ is Lipschitz;*
- *there is C^1 map $g \colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.*

Theorem (Simple proof of Whitney and Jarník extension thms)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f \colon Q \to \mathbb{R}$ has differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff a simple (new) condition for f holds.

BAMS survey contains a lot of other results

But this is all for today

Thank you for your attention!