A century of Sierpiński-Zygmund functions

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Based on survey, with the same title, written with
Juan Benigno Seoane-Sepúlveda, to appear in *Rev. R. Acad. Cien. Serie A. Mat.*
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Outline

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1. How did Sierpiński-Zygmund maps come about?
2. Generalizations of Blumberg's theorem
3. SZ maps with extra properties
4. Algebraic structures within SZ

Q: Must an arbitrary $f\colon \mathbb{R} \to \mathbb{R}$ have some continuity?

NO: $\chi_{\mathbb{Q}}\colon \mathbb{R} \to 2$, *Dirichlet function*, is continuous at no point.

Q: What about continuity of $f \restriction D$ for some $D \subset \mathbb{R}$?

A: $f \restriction D$ is continuous at any isolated point of D.

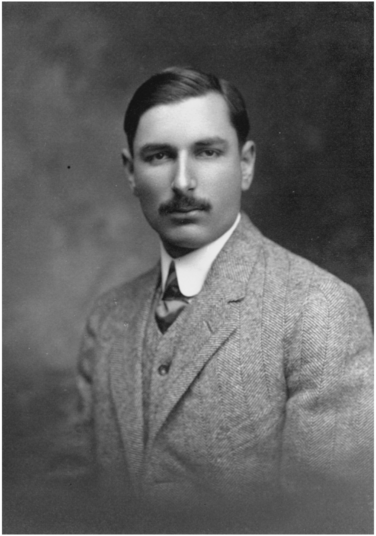
True Q: What about continuity of $f \restriction D$ for $D \subset \mathbb{R}$ with no isolated points?

Theorem (H. Blumberg, 1922, Trans AMS)

For an arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a dense subset D of $\mathbb{R}$ such that $f \restriction D$ is continuous.

For a short proof see K.C. Ciesielski, M.E. Martinez-Gomez, J.B. Seoane-Sepulveda, Amer. Math. Monthly 126(6) (2019)

Henry Blumberg (1886–1950)



Born in Russia, immigrated to the USA in 1891. Ph.D. in 1912 from University of Göttingen under Edmund Landau. Eight Ph.D. students between 1925 and 1950 while working at Ohio State University; including Casper Goffman (1913–2006). Interestingly, Baruch Blumberg, co-recipient of the 1976 Nobel Prize in Physiology or Medicine, was a nephew of Henry Blumberg.

Contribution of W. Sierpiński and A. Zygmund

Blumberg (1922): For any $f\colon \mathbb{R} \to \mathbb{R}$ there is dense $D \subset \mathbb{R}$ with continuous $f \restriction D$.

Fact: D in Blumberg theorem is countable.

Natural Q: Can set D in Blumberg's theorem be uncountable?

Theorem (Sierpiński and Zygmund 1923 in Fund. Math.)

There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction S$ is discontinuous for every $S \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.

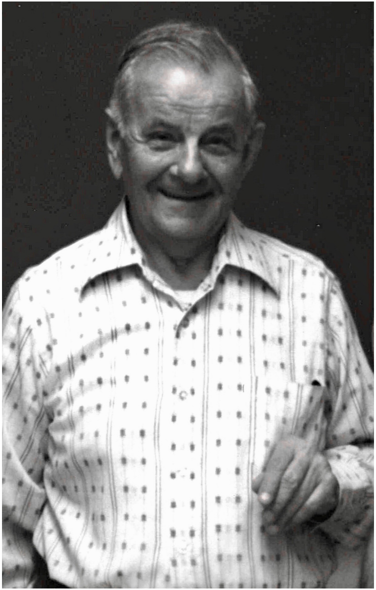
Such maps, denoted SZ, are called *SZ-functions*.

Under the Continuum Hypothesis, CH, this settles the matter.

Wacław Franciszek Sierpiński (1882–1969)

Polish mathematician famous for contributions to topology, set theory (proving that ZF set theory together with the GCH imply the Axiom of Choice), and number theory. Published over 700 papers and 50 books. Co-founded Fundamenta Mathematicae. He had 9 Ph.D. students. Currently, he counts $>$5000 mathematical descendants, including K.C. Ciesielski.

Antoni Zygmund (1900–1992)



Polish mathematician, considered as one of the greatest analysts of the 20th century. Ph.D. in 1923 from Warsaw University. In 1940, during the World War II, he emigrated to the USA. From 1947 until his passing he was a professor at the University of Chicago. In 1986 he received the National Medal of Science. Directed over 40 Ph.D. theses, including one of Paul Cohen (1937–2007), Fields medallist in 1966.

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Restrictions to uncountable sets under ¬CH

Theorem (Sierpiński and Zygmund 1923)

$CH \implies \exists f\colon \mathbb{R} \to \mathbb{R}\ \forall D \in [\mathbb{R}]^{\omega_1}\ \ f \restriction S$ is discontinuous.

Theorem (Gruenhage, see Recław 1993; also Shelah 1995)

In a model of ZFC obtained by adding at least ω_2 Cohen reals:

- *$\neg CH$ and $\exists f\colon \mathbb{R} \to \mathbb{R}\ \forall D \in [\mathbb{R}]^{\omega_1}\ \ f \restriction S$ is discontinuous.*

Theorem (S. Baldwin 1990, generalizing Shinoda 1973)

Under the Martin's Axiom MA,

($*$) For every $f\colon \mathbb{R} \to \mathbb{R}$ and infinite cardinal $\kappa < \mathfrak{c}$ there exists a κ-dense set $D \subset \mathbb{R}$ for which $f \restriction D$ is continuous.

So, MA+¬CH implies that set D can be ω_1-dense.

New short proof of this can be found in the survey.

Can set D in Blumberg's thm be of second category?

Theorem (Shelah 1995)

There exists a model of ZFC$+\neg$CH in which

- *For every $f\colon \mathbb{R} \to \mathbb{R}$ there is a second category set $D \subset \mathbb{R}$ with $f \restriction D$ continuous.*

Can above D be second category in any (a, b) with $a < b$? YES

Proposition (easy, from the survey)

The above property • implies that

- For every $f\colon \mathbb{R} \to \mathbb{R}$ there exists a category dense set D in $\mathbb{R}$ for which $f \restriction D$ continuous.

Can set D be nowhere Lebesgue null?

Theorem (Rosłanowski & Shelah 2006)

There exists a model of ZFC$+\neg$CH in which

- *For every map $f\colon \mathbb{R} \to \mathbb{R}$ there is a continuous $g\colon \mathbb{R} \to \mathbb{R}$ s.t. $f = g$ on a set D of positive Lebesgue outer measure.*

Can above D in Blumberg's theorem be of positive Lebesgue outer measure in any (a, b) with $a < b$? **NO!**

Theorem (J. Brown 1977)

There is an $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction D$ is discontinuous for every set $D \subset \mathbb{R}$ which is nowhere measure zero, that is, such that $D \cap I$ has positive outer measure for every non-trivial interval I.

Easy construction comes from 1997 paper of K.C. Ciesielski.

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The class SZ(Borel)

Let SZ(Borel) be the class of all $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction S$ is not Borel for every $S \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.

It is easy to see that $\emptyset \neq$ SZ(Borel) $\subseteq$ SZ.

Q: Are these classes equal? A: This is not decidable in ZFC.

Theorem (Bartoszewicz, Bienias, Głąb, Natkaniec 2016)

- *If dec(Borel, Cont) = $\mathfrak{c} = \kappa^+$, then SZ(Borel) $\subsetneq$ SZ;*
- *if dec(Borel, Cont) < $\mathfrak{c} = \kappa^+$, then SZ(Borel) $=$ SZ.*

More on this in 2019 draft of myself and T. Natkaniec.

SZ maps that are continuous in a generalized sense

SZ map can be neither measurable nor have Baire property.

Q: Can an SZ map be Darboux?
(i.e. very discontinuous but continuous-like)

($f\colon \mathbb{R} \to \mathbb{R}$ is Darboux if it has the Intermediate Value Property.)

Theorem (Balcerzak, Ciesielski, Natkaniec 1997)

- *If cov(Meager) = $\mathfrak{c}$, then there is Darboux SZ-map.*
- *In the iterated perfect set (Sacks) model there is no Darboux SZ-map.*

KCC and Pawlikowski 2003: CPA, axiom that holds in the Sacks model, implies that $f[\mathbb{R}]$ contains no perfect set when f is SZ.

Q: Is SZ $\cap$Darboux $\neq \emptyset$ equivalent to cov(Meager) = $\mathfrak{c}$?

Q asked a week ago by a referee.

By last thm: cov(Meager) = $\mathfrak{c}$ implies that SZ $\cap$Darboux $\neq \emptyset$.

A: No by the following

Theorem (Proved yesturday)

cov(Null) = $\mathfrak{c}$ implies that SZ $\cap$Darboux $\neq \emptyset$.

So, SZ $\cap$Darboux $\neq \emptyset$ holds in a model of ZFC
where cov(Null) = $\mathfrak{c}$ $>$cov(Meager).

Variations on Darboux SZ-maps

cov(Meager) = $\mathfrak{c}$ implies that

there is almost continuous SZ-map $f\colon \mathbb{R} \to \mathbb{R}$.
(almost continuous $\Longrightarrow$ connected graph $\Longrightarrow$ Darboux)

Banaszewski-Natkaniec 1997: such f can be also additive
(i.e., with $f(x+y) = f(x) + f(y)$).

1990s: There is ZFC example of an SZ map f with CIVP, i.e. s.t

for all $p < q$ with $f(p) \neq f(q)$ and perfect set K between $f(p)$ and $f(q)$, there is a perfect $C \subset (p, q)$ with $f[C] \subset K$.

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Lineability and algebrability of SZ

For a cardinal number κ we say that an $F \subset \mathbb{R}^{\mathbb{R}}$ is:

- κ-lineable if $F \cup \{0\}$ contains a vector subspace of $\mathbb{R}^{\mathbb{R}}$, over the field $\mathbb{R}$, of dimension κ;
- κ-algebrable if there is an algebra $A \subset F \cup \{0\}$ for which κ is the smallest cardinality of any $B \subset A$ generating A.

Theorem (Gámez-Merino, Seoane-Sepúlveda 2012)

For any cardinal number κ the following are equivalent:

1. *SZ is κ-algebrable.*
2. *SZ is κ-lineable.*
3. *There is a $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality κ.*

Moreover, there is a model of ZFC where these fail for $\kappa = 2^{\mathfrak{c}}$.

Lineability of SZ Darboux-like maps

This is still mainly work in progress.

- SZ$\cap$CIVP is $\mathfrak{c}^+$-lineable; its $2^{\mathfrak{c}}$-lineability is not provable;
- SZ$\cap$Darboux is $\mathfrak{c}^+$-lineable under cov(Meager) $= \mathfrak{c}$.

Relatively little more is known in this front.

Though some Ph.D. students work in these areas right now.

Inverses of SZ bijections

Q: Is there an SZ bijection f with f^{-1} also SZ? In ZFC?

No, as there are no SZ surjections (so bijections) in ZFC. But

Theorem (Ciesielski, Natkaniec 1997)

It is consistent, follows from cov(Meager) = $\mathfrak{c}$, that

(i) *there exists an SZ bijection $f \in \mathbb{R}^{\mathbb{R}}$ such that $f^{-1} = f$;*

(ii) *there exists an SZ bijection $f \in \mathbb{R}^{\mathbb{R}}$ such that $f^{-1} \notin SZ$.*

Q: What about SZ injections in ZFC? Yes, they (clearly) exist.

So, what about their inverses? In what sense they can be SZ?

Partial SZ maps and their inverses

Def. An f from an $X \in [\mathbb{R}]^{\mathfrak{c}}$ to $\mathbb{R}$ is SZ provided $f \restriction S$ is discontinuous for every $S \in [X]^{\mathfrak{c}}$.

Q: Is there, in ZFC, a partial SZ injection with SZ inverse? **NO:**

Theorem (Ciesielski, Natkaniec 1997)

The following properties are equivalent.

(i) *There is NO partial SZ injection so that f^{-1} is SZ.*

(ii) *There is family $\mathcal{H}$ of continuous maps from $X \in [\mathbb{R}]^{\mathfrak{c}}$ into $\mathbb{R}$ such that $\mathcal{H}$ has cardinality $< \mathfrak{c}$ and that $\mathbb{R}^2$ is covered by the graphs of $h \in \mathcal{H}$ and their inverses.*

Since (ii) is consistent with ZFC—it follows from CPA—so is (i).

Any ZFC result on the inverses of SZ injections?

Theorem (Ciesielski, Natkaniec 1997)

There is, in ZFC, an SZ injection f with f^{-1} continuous (so not SZ).

A construction, simple modification of original one of Sierpiński and Zygmund, is based on the lemma:

Lemma

For every continuous g from an $S \subset \mathbb{R}$ into $\mathbb{R}$, there is a G_δ-set $G \supset S$ and a continuous extension $\bar{g}\colon G \to \mathbb{R}$ of g.
In particular, g admits Borel extension $\hat{g}$.

Proof: $G := \{x \in \mathrm{cl}(S)\colon \mathrm{osc}_g(x) = 0\}$, $\quad \bar{g} := \mathrm{cl}(g) \cap (G \times \mathbb{R})$

Construction of SZ injection f with continuous f^{-1}

Let $h\colon \mathbb{R} \to \mathbb{R}$ be continuous surjection such that $h^{-1}(y)$ has cardinality $\mathfrak{c}$ for every $y \in \mathbb{R}$.

Let $\{x_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration, with no repetition, of $\mathbb{R}$ and let $\{\hat{g}_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of Borel functions. For every $\xi < \mathfrak{c}$ choose

$$f(x_\xi) \in h^{-1}(x_\xi) \setminus (\{\hat{g}_\zeta(x_\xi) \colon \zeta < \xi\} \cup \{f(x_\zeta) \colon \zeta < \xi\}).$$

Then $f \in \mathrm{SZ}$ and
f^{-1} is continuous, as its graph is contained in the graph of h.

That is all!

Thank you for your attention!