A century of Sierpiński-Zygmund functions

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Based on survey, with the same title, written with
Juan Benigno Seoane-Sepúlveda, to appear in *Rev. R. Acad. Cien. Serie A. Mat.*
Text of this talk available at https://math.wvu.edu/˜kcies/presentations.html

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Outline

1. How did Sierpiński-Zygmund maps come about?
2. Generalizations of Blumberg's theorem
3. SZ maps with extra properties
4. Algebraic structures within SZ

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3. SZ maps with extra properties
4. Algebraic structures within SZ

Q: Must an arbitrary $f\colon \mathbb{R} \to \mathbb{R}$ have some continuity?

NO: $\chi_{\mathbb{Q}}\colon \mathbb{R} \to 2$, *Dirichlet function*, is continuous at no point.

Q: What about continuity of $f \restriction D$ for some $D \subset \mathbb{R}$?

A: $f \restriction D$ is continuous at any isolated point of D.

True Q: What about continuity of $f \restriction D$ for $D \subset \mathbb{R}$ with no isolated points?

Theorem (H. Blumberg, 1922, Trans AMS)

For an arbitrary function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a dense subset D of $\mathbb{R}$ such that $f \restriction D$ is continuous.

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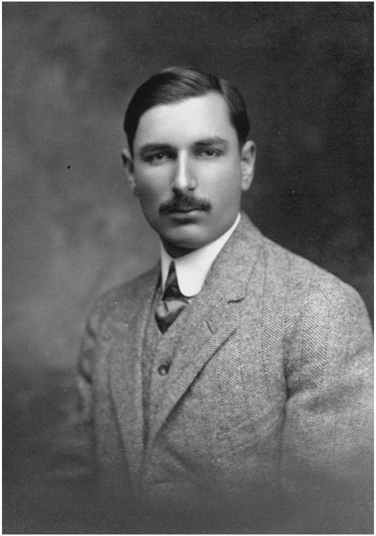

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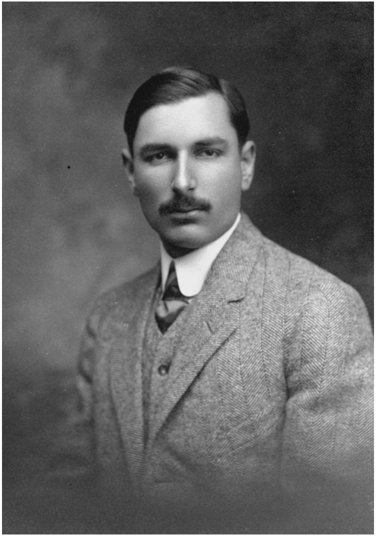

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Blumberg (1922): For any $f\colon \mathbb{R} \to \mathbb{R}$ there is dense $D \subset \mathbb{R}$ with continuous $f \restriction D$.

Fact: *D in Blumberg theorem is countable.*

Natural Q: Can set D in Blumberg's theorem be uncountable?

Theorem (Sierpiński and Zygmund 1923 in Fund. Math.)

There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction S$ is discontinuous for every $S \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.

Such maps, denoted SZ, are called *SZ-functions*.

Under the Continuum Hypothesis, CH, this settles the matter.

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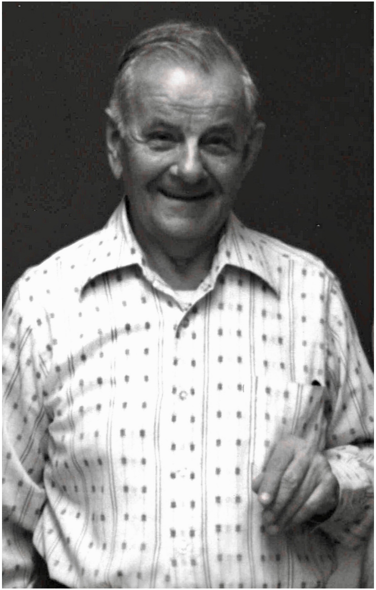
Wacław Franciszek Sierpiński (1882–1969)

Polish mathematician famous for contributions to topology, set theory (proving that ZF set theory together with the GCH imply the Axiom of Choice), and number theory. Published over 700 papers and 50 books. Co-founded Fundamenta Mathematicae. He had 9 Ph.D. students. Currently, he counts $>$5000 mathematical descendants, including K.C. Ciesielski.

Antoni Zygmund (1900–1992)



Polish mathematician, considered as one of the greatest analysts of the 20th century. Ph.D. in 1923 from Warsaw University. In 1940, during the World War II, he emigrated to the USA. From 1947 until his passing he was a professor at the University of Chicago. In 1986 he received the National Medal of Science. Directed over 40 Ph.D. theses, including one of Paul Cohen (1937–2007), Fields medallist in 1966.

Outline

Restrictions to uncountable sets under ¬CH

Theorem (Sierpiński and Zygmund 1923)

CH $\implies \exists f\colon \mathbb{R}\to\mathbb{R}\ \forall D\in[\mathbb{R}]^{\omega_1}\ \ f\restriction S$ *is discontinuous.*

Theorem (Gruenhage, see Recław 1993; also Shelah 1995)

In a model of ZFC obtained by adding at least ω_2 *Cohen reals:*

- $\neg$*CH and* $\exists f\colon \mathbb{R}\to\mathbb{R}\ \forall D\in[\mathbb{R}]^{\omega_1}\ \ f\restriction S$ *is discontinuous.*

Theorem (S. Baldwin 1990, generalizing Shinoda 1973)

Under the Martin's Axiom MA,

$(*)$ *For every* $f\colon\mathbb{R}\to\mathbb{R}$ *and infinite cardinal* $\kappa<\mathfrak{c}$ *there exists a* κ*-dense set* $D\subset\mathbb{R}$ *for which* $f\restriction D$ *is continuous.*

So, MA+$\neg$*CH implies that set D can be* ω_1*-dense.*

New short proof of this can be found in the survey.

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Can set D in Blumberg's thm be of second category?

Theorem (Shelah 1995)

There exists a model of ZFC+$\neg$CH in which

- *For every $f : \mathbb{R} \to \mathbb{R}$ there is a second category set $D \subset \mathbb{R}$ with $f \restriction D$ continuous.*

Can above D be second category in any (a, b) with $a < b$? YES

Proposition (easy, from the survey)

The above property • implies that

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Can set D be nowhere Lebesgue null?

Theorem (Rosłanowski & Shelah 2006)

There exists a model of ZFC+$\neg$CH in which

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The class SZ(Borel)

Let SZ(Borel) be the class of all $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction S$ is not Borel for every $S \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.

It is easy to see that $\emptyset \neq \text{SZ(Borel)} \subseteq \text{SZ}$.

Q: Are these classes equal? A: This is not decidable in ZFC.

Theorem (Bartoszewicz, Bienias, Głąb, Natkaniec 2016)

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More on this in 2019 draft of myself and T. Natkaniec.

The class SZ(Borel)

Let SZ(Borel) be the class of all $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction S$ is not Borel for every $S \subset \mathbb{R}$ of cardinality $\mathfrak{c}$.

It is easy to see that $\emptyset \neq$ SZ(Borel) $\subseteq$ SZ.

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SZ maps that are continuous in a generalized sense

SZ map can be neither measurable nor have Baire property.

Q: Can an SZ map be Darboux?
(i.e. very discontinuous but continuous-like)

($f\colon \mathbb{R} \to \mathbb{R}$ is Darboux if it has the Intermediate Value Property.)

Theorem (Balcerzak, Ciesielski, Natkaniec 1997)

- *If cov(Meager) = $\mathfrak{c}$, then there is Darboux SZ-map.*
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By last thm: cov(Meager) $= \mathfrak{c}$ implies that SZ $\cap$ Darboux $\neq \emptyset$.

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cov(Null) $= \mathfrak{c}$ implies that SZ $\cap$ Darboux $\neq \emptyset$.

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cov(Meager) = $\mathfrak{c}$ implies that

there is almost continuous SZ-map $f \colon \mathbb{R} \to \mathbb{R}$.
(almost continuous $\implies$ connected graph $\implies$ Darboux)

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(i.e., with $f(x+y) = f(x) + f(y)$).

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Outline

Lineability and algebrability of SZ

For a cardinal number κ we say that an $F \subset \mathbb{R}^{\mathbb{R}}$ is:

- κ-lineable if $F \cup \{0\}$ contains a vector subspace of $\mathbb{R}^{\mathbb{R}}$, over the field $\mathbb{R}$, of dimension κ;
- κ-algebrable if there is an algebra $A \subset F \cup \{0\}$ for which κ is the smallest cardinality of any $B \subset A$ generating A.

Theorem (Gámez-Merino, Seoane-Sepúlveda 2012)

For any cardinal number κ the following are equivalent:

1. *SZ is κ-algebrable.*
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3. *There is a $\mathfrak{c}$-almost disjoint family $\mathcal{F} \subset [\mathfrak{c}]^{\mathfrak{c}}$ of cardinality κ.*

Moreover, there is a model of ZFC where these fail for $\kappa = 2^{\mathfrak{c}}$.

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Def. An f from an $X \in [\mathbb{R}]^{\mathfrak{c}}$ to $\mathbb{R}$ is SZ provided $f \restriction S$ is discontinuous for every $S \in [X]^{\mathfrak{c}}$.

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Any ZFC result on the inverses of SZ injections?

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There is, in ZFC, an SZ injection f with f^{-1} continuous (so not SZ).

A construction, simple modification of original one of Sierpiński and Zygmund, is based on the lemma:

Lemma

For every continuous g from an $S \subset \mathbb{R}$ into $\mathbb{R}$, there is a G_δ-set $G \supset S$ and a continuous extension $\bar{g} \colon G \to \mathbb{R}$ of g.
In particular, g admits Borel extension $\hat{g}$.

Proof: $G := \{x \in \mathrm{cl}(S) \colon \mathrm{osc}_g(x) = 0\}$, $\quad \bar{g} := \mathrm{cl}(g) \cap (G \times \mathbb{R})$

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