Efficient algorithms for max-norm and lexicographically optimized labelings

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Outline

1. Background: the energies we will optimize
2. Algorithms for L_p, $p < \infty$; NP-completeness
3. Which max-norm energies E_∞ can be efficiently optimized?
4. New algorithms optimizing E_∞ for 2-labeling
5. Strict max-norm optimality
6. Summary and conclusions

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Optimization in image processing

- Many fundamental problems in image processing and computer vision, such as image filtering, segmentation, registration, and stereo vision, can naturally be formulated as optimization problems.
- Often, these optimization problems can be described as *labeling* problems, in which we wish to assign to each image element (pixel) an element from some finite set of labels.
- We identify each image with a vertex weighted graph $\mathcal{G} = (V, \mathcal{E}, f)$, with vertices V being image voxels, edges $\mathcal{E}$ being pairs $\{s, t\}$ of adjacent voxels, and $f(s)$ image intensity at s. Its labeling is a map $\ell \colon V \to \{0, \ldots, m-1\}$, with $m \geq 2$.

L_p energies: the case of L_1

With any image n-labeling ℓ we associate local cost map $\phi_\ell\colon V \cup \mathcal{E} \to [0,\infty]$ consisting of

- unary terms $\phi_\ell(s) = \phi_s(\ell(s))$, depending on $s \in V$, its label $\ell(s)$, and image intensity;
- pairwise terms $\phi_\ell(s,t) = \phi_{s,t}(\ell(s),\ell(t))$, depending on $\{s,t\} \in \mathcal{E}$ and their labeling. They reflect desirability of smoothness/regularity of labeling.
  All $\phi_{s,t}(0,0)$, $\phi_{s,t}(0,1)$, $\phi_{s,t}(1,0)$, $\phi_{s,t}(1,1)$ can be distinct!

L_1 (graph cut) energy is defined as

$$E_1(\ell) := \|\phi_\ell\|_1 = \sum_{s\in V} \phi_s(\ell(s)) + \sum_{\{s,t\}\in\mathcal{E}} \phi_{st}(\ell(s),\ell(t)),$$

often represented as (with x_i denoting label of vertex i)

$$E(\mathbf{x}) = \sum_{i\in\mathcal{V}} \phi_i(x_i) + \sum_{i,j\in\mathcal{E}} \phi_{ij}(x_i,x_j).$$

L_p energies: the cases of $p \in (1, \infty]$

For $p \in [1, \infty)$:

$$E_p(\ell) := \|\phi_\ell\|_p = \left(\sum_{s \in V} (\phi_s(\ell(s)))^p + \sum_{\{s,t\} \in \mathcal{E}} (\phi_{st}(\ell(s), \ell(t)))^p \right)^{1/p}$$

For $p = \infty$ (of main interest here)

$$E_\infty(\ell) := \|\phi_\ell\|_\infty = \max \left\{ \max_{s \in V} \phi_s(\ell(s)), \max_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t)) \right\}$$

Standard analysis fact: $E_p(\ell) \nearrow_{p \to \infty} E_\infty(\ell)$.

What is the effect of p?

- The value p can be seen as a parameter controlling the balance between minimizing the overall cost $E_p(\ell)$ versus minimizing the magnitude of the individual terms $\phi_s(\ell(s))$ and $\phi_{st}(\ell(s), \ell(t))$.
- For $p = 1$, the optimal labeling may contain (few) arbitrarily large individual terms as long as the sum of the terms is small.
- As p increases, a larger penalty is assigned to solutions containing large individual terms. This forces local errors to be distributed more evenly across the image domain.

Outline

$p = 1$: Graph Cut segmentation via min-cut/max-flow

$$E_1(\ell) := \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))$$

- $\phi_s(\ell(s)) = 0$ in all cases (except seeds);
- $\phi_{st}(\ell(s), \ell(t)) = 0$ when $\ell(s) = \ell(t)$;
- Cost of cut: $\phi_{st}(\ell(s), \ell(t)) > 0$ (depending of $f(s)$, $f(t)$) when $\ell(s) \neq \ell(t)$.

Min-cut/max-flow (efficiency between $O(n^2 \ln n)$ and $O(n^3)$) algorithm returns optimized labeling **for 2-labeling.**

Here and below $n := |V \cup \mathcal{E}|$.

Optimization is NP-hard **for** $\geq$ **3-labeling**.

2-labeling for general $E_1(\ell)$-optimization

$$E_1(\ell) := \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))$$

E_1 (for 2-labeling) is submodular provided, for every $\{s, t\} \in \mathcal{E}$,

$$\phi_{st}(0,0) + \phi_{st}(1,1) \leq \phi_{st}(0,1) + \phi_{st}(1,0).$$

Theorem (Kolmogorov & Zabih 2004)

- *If E_1 is submodular, then min-cut/max-flow algorithm returns optimized labeling.*
- *If E_1 is* **NOT** *submodular, then minimizing E_1 is NP-hard.*

$E_p(\ell)$ with $1 \leq p < \infty$ is as $E_1(\ell)$

$$\left(E_p(\ell)\right)^p := \sum_{s\in V}\left(\phi_s(\ell(s))\right)^p + \sum_{\{s,t\}\in\mathcal{E}}\left(\phi_{st}(\ell(s),\ell(t))\right)^p$$

E_p is p-submodular provided, for every $\{s,t\} \in \mathcal{E}$,

$$\phi_{st}(0,0)^p + \phi_{st}(1,1)^p \leq \phi_{st}(0,1)^p + \phi_{st}(1,0)^p.$$

Corollary (Obvious, Malmberg & Strand, IWCIA 2018)

- *If E_p is p-submodular, then min-cut/max-flow algorithm returns optimized labeling.*
- *If E_p is* **NOT** *p-submodular, then minimizing E_p is NP-hard.*

$E_p(\ell)$ with $1 \leq p < \infty$ vs $E_\infty(\ell)$

$$\phi_{st}(0,0)^p + \phi_{st}(1,1)^p \leq \phi_{st}(0,1)^p + \phi_{st}(1,0)^p.$$

p-submodular for every $p < \infty$ implies ∞-submodularity:

$$\max\{\phi_{st}(0,0), \phi_{st}(1,1)\} \leq \max\{\phi_{st}(1,0), \phi_{st}(0,1)\}.$$

Theorem (Malmberg & Strand, IWCIA 2018)

1- and ∞-submodularity imply p-submodularity for all p. In such case min-cut/max-flow algorithm optimizes E_p for every $p < \infty$.

Actually, ϕ is ∞-submodular iff there is an N so that ϕ is p-submodular for all $p \in (N, \infty)$.

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FC segmentations are E_∞ optimized segmentations

$$E_\infty(\ell) := \max\left\{\max_{s\in V}\phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}}\phi_{st}(\ell(s),\ell(t))\right\}$$

We get FC segmentations (as minimization, not maximization)

- $\phi_s(\ell(s)) = 0$ in all cases (except seeds, when $= \infty$);
- $\phi_{st}(\ell(s),\ell(t)) = 0$ when $\ell(s) = \ell(t)$;
- Cost of cut: $\phi_{st}(\ell(s),\ell(t)) > 0$ (depending of $f(s)$, $f(t)$) when $\ell(s) \neq \ell(t)$.

Dijkstra algorithm (efficiency between $O(n)$ and $O(n \ln n)$) returns optimized labeling for m-labeling **for arbitrary large** m**!**

Better than for $E_1(\ell)$ (i.e., GC) segmentations.

Q. For what other E_∞s are there efficient optimizing algorithms?

Efficient algorithm for 2-labeling ∞-submodular E_∞?

YES! ∞-sub algotithm

Theorem (Malmberg, Ciesielski, Strand, DGCI 2019)

There is an algorithm, quasi-linear with respect to $n = |V \cup \mathcal{E}|$, returning minimal 2-labeling for any ∞-submodular energy E_∞.

The algorithm, efficiency between $O(n)$ and $O(n \ln n)$,
is NOT Dijkstra-like!
This is all that is in the DGCI 2019 paper.

Natural questions, towards post DGCI 2019 work:

Q1: Is ∞-submodularity assumption essential in the thm?

Q2: Is there efficient algorithm for $\geq$ 3-labelings?

Optimal 2-labeling for $E_\infty(\ell)$ with no ∞-submodularity

Full answer to Q1: 2-sat algorithm

Theorem (Malmberg, Ciesielski, Strand; 2019 ???)

There is an algorithm,
$O(n^2)$ **with respect to** $n = |V \cup \mathcal{E}|$**,**
returning minimal 2-labeling for any $E_\infty(\ell)$**:**

$$\max\left\{\max_{s\in V}\phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}}\phi_{st}(\ell(s),\ell(t))\right\}.$$

More about the algorithm latter.

Q2: What about optimal $\geq$ 3-labeling for $E_\infty(\ell)$?

Partial answer to Q2:

Theorem (Malmberg, Ciesielski, Strand; 2019 ???)

Optimization problem of the general form of E_∞ energy for more than 2 labels is NP-hard.

Remaining version of Q2:

Q: Under what conditions there exists an efficient (polynomial-time) algorithm for optimization of E_∞ energy for 3 or more labels?

Can be done in FC/Dijkstra setting. Not (NP-hard) in general.

Optimal $\geq$ 3-labeling of $E_\infty(\ell)$ is NP-hard: proof

$$E_\infty(\ell) := \max\left\{\max_{s\in V} \phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}} \phi_{st}(\ell(s),\ell(t))\right\}$$

For a graph $\mathcal{G} = (V, \mathcal{E})$ put:

- $\phi_s(\ell(s)) = 0$ in all cases;
- $\phi_{st}(\ell(s), \ell(t)) = 1$ when $\ell(s) = \ell(t)$;
- $\phi_{st}(\ell(s), \ell(t) = 0$ when $\ell(s) \neq \ell(t)$.

Then, the minimal $E_\infty(\ell)$ is 0 if, and only if, ℓ is a coloring of $\mathcal{G}$.

But graph m-coloring problem for any $m \geq 3$ is NP-complete!
It is not for $m = 2$.

Outline

Atoms of E_∞ and their cost

$$E_\infty(\ell) := \max\left\{\max_{s\in V}\phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}}\phi_{st}(\ell(s),\ell(t))\right\}$$

Atoms $\mathcal{A}(\ell)$ of ℓ: input for $\phi_{..}$ and $\phi_.$ (to calculate $E_\infty(\ell)$), i.e.,

$$\mathcal{A}(\ell) := \{\{(s,\ell(s))\} : s \in V\} \cup \{\{(s,\ell(s)),(t,\ell(t))\} : \{s,t\} \in \mathcal{E}\}$$

Atoms $\mathcal{A}$ of E_∞: all such possible atoms, i.e.,

unary: two $\{(s,0)\}$ and $\{(s,1)\}$ for each $v \in V$

binary: four $\{(s,i),(t,j)\}$ $(i,j \in \{0,1\})$ for each $\{s,t\} \in \mathcal{E}$.

- Cost of a unary atom $\{(s,i)\}$: $\phi_s(i)$
- Cost of a binary atom $\{(s,i),(t,j)\}$: $\phi_{st}(i,j)$

Set $\mathcal{A}' \subset \mathcal{A}$ of atoms is consistent when $\mathcal{A}(\ell) \subset \mathcal{A}'$ for some ℓ.

∞-sub algorithm

1. List all atoms in a list S in a decreasing cost so that if atoms A_0 and A_1 have the same cost and $A_1 = \{(s,i),(t,i)\}$, then A_1 proceeds A_0.
2. While S is non-empty do
   - Remove the first atom A from S
   - If A is the last atom for its vertex/edge, insert it to list L
   - Consecutively remove from S all atoms that are locally inconsistent with current $S \cup L$
3. Return labeling $\ell = \bigcup L$

The locally inconsistency loop is natural.

The trick is to show that the algorithm works property for all ∞-submodular energies.

Towards 2-sat algorithm: 2-satisfiability

For atoms $A = \{(s, i)\}$ and $A' = \{(s, i), (t, j)\}$ define formulas

$$\psi_A(s) := \text{“}s \neq i\text{”} = \begin{cases} \neg s & \text{if } i = 1, \\ s & \text{if } i = 0 \end{cases}$$

$$\psi_A(s, t) := \text{“}(s \neq i) \vee (t \neq j)\text{”} = \psi_{\{(s,i)\}}(s) \vee \psi_{\{(t,j)\}}(t).$$

For a set $\mathcal{A}' = \{A_1, A_2, \ldots, A_k\}$ of atoms the formula $\psi_{\mathcal{A}'} := \psi_{A_1} \wedge \cdots \wedge \psi_{A_k}$. is in *2-conjunctive normal form*.

Theorem

A set $\mathcal{A}_1 \subseteq \mathcal{A}$ of atoms is consistent if, and only if, the 2-satisfiability problem for a formula $\psi_{\mathcal{A}_1^c}$ has a positive solution.

So, consistency of $\mathcal{A}_1 \subseteq \mathcal{A}$ can be decided by in linear time. (Aspvall et al. algorithm.)

2-sat algorithm

1. List all atoms in a list S in a decreasing cost
2. While S is non-empty do
   - Remove the first atom A from S
   - If $S \cup L$ is not consistent, insert A to L
3. Return labeling $\ell = \bigcup L$

The $S \cup L$ is not consistent clause is decided by Aspvall et al. algorithm.

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Strict optimality via lexicographical order

Max-norm identifies ℓ_1 and ℓ_2 when $E_\infty(\ell_1) = E_\infty(\ell_2)$.

Lexicographical order $\preceq$ is a sharper distinguishing tool.

For labeling ℓ, let $\vec{\ell} = \langle \ell_1, \ldots, \ell_n \rangle = \langle \Phi(A_1), \ldots, \Phi(A_n) \rangle$ non-increasing for an enumeration $\mathcal{A}(\ell) = \{A_1, \ldots, A_n\}$.

$\ell \prec \ell'$ iff $\ell_i < \ell'_i$, where $i := \min\{k \colon \ell_k < \ell'_k\}$.

ℓ is strictly optimal when it is maximal w.r.t. $\preceq$.

Strictly optimal implies max-norm optimal, but not converse.

Q: Can we efficiently find also strict optimizers?

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Summary (including new results)

	2 labels	≥ 3 labels
general case strict optimization	**NP-hard problem**	NP-hard problem
∞-submodular strict optimization	max-flow/min-cut $O(n^2 \ln n) \leq \cdot \leq O(n^3)$	NP-hard problem
unique weights strict optimization	**2-sat algorithm** $O(n^2)$	NP-hard problem
general case	2-sat algorithm; $O(n^2)$	NP-hard problem
∞-submodular	∞-sub algorithm $O(n) \leq \cdot \leq O(n \ln n)$	NP-hard problem
$\phi_s(i) = \phi_{st}(i,i) = 0;$ $\phi_{st}(i,j) = \phi_{st}(j,i) \geq 0$	Dijkstra algorithm $O(n) \leq \cdot \leq O(n \ln n)$	Dijkstra algorithm $O(n) \leq \cdot \leq O(n \ln n)$

Conclusions

- Optimization problems, specifically pixel labeling problems, are frequently occurring in image processing applications.
- We are specifically interested in problems where the objective function is given by the max-norm of the local errors.
- For many such problems, globally optimal solutions can be found very efficiently, in quasi linear or quadratic time.
- Some max-norm for $\geq$ 3-labeling are NP-hard.

Thank you for your attention!