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Efficient algorithms for max-norm and lexicographically optimized labelings

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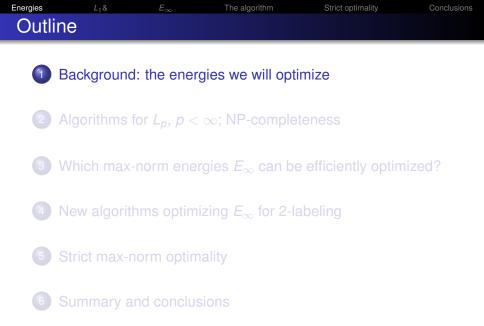
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Background: the energies we will optimize

- 2 Algorithms for L_p , $p < \infty$; NP-completeness
- 3 Which max-norm energies E_{∞} can be efficiently optimized?
- 4 New algorithms optimizing E_{∞} for 2-labeling
- 5 Strict max-norm optimality
- 6 Summary and conclusions

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Energies L_1 E_{∞} The algorithmStrict optimalityOptimization in image processing

- Many fundamental problems in image processing and computer vision, such as image filtering, segmentation, registration, and stereo vision, can naturally be formulated as optimization problems.
- Often, these optimization problems can be described as *labeling* problems, in which we wish to assign to each image element (pixel) an element from some finite set of labels.
- We identify each image with a vertex weighted graph *G* = (*V*, *E*, *f*), with vertices *V* being image voxels, edges *E* being pairs {*s*, *t*} of adjacent voxels, and *f*(*s*) image intensity at *s*. Its labeling is a map *l*: *V* → {0,...,*m*-1}, with *m* ≥ 2.

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L_p energies: the case of L_1

Energies

With any image *n*-labeling ℓ we associate local cost map $\phi_{\ell} \colon V \cup \mathcal{E} \to [0, \infty]$ consisting of

• unary terms $\phi_{\ell}(s) = \phi_{s}(\ell(s))$, depending on $s \in V$, its label $\ell(s)$, and image intensity;

The algorithm

- pairwise terms φ_ℓ(s, t) = φ_{s,t}(ℓ(s), ℓ(t)), depending on {s, t} ∈ E and their labeling. They reflect desirability of smoothness/regularity of labeling.
 All φ_{s,t}(0,0), φ_{s,t}(0,1), φ_{s,t}(1,0), φ_{s,t}(1,1) can be distinct!
- L1 (graph cut) energy is defined as

$$E_1(\ell) := \|\phi_\ell\|_1 = \sum_{\boldsymbol{s} \in V} \phi_{\boldsymbol{s}}(\ell(\boldsymbol{s})) + \sum_{\{\boldsymbol{s},t\} \in \mathcal{E}} \phi_{\boldsymbol{s}t}(\ell(\boldsymbol{s}),\ell(t)),$$

often represented as (with x_i denoting label of vertex i)

$$E(\mathbf{x}) = \sum_{i \in \mathcal{V}} \phi_i(x_i) + \sum_{i,j \in \mathcal{E}} \phi_{ij}(x_i, x_j).$$

Strict optimality

Energies $L_{1^{\&}}$ E_{∞} The algorithm Strict optimality L_p energies: the cases of $p \in (1,\infty]$

For
$$p \in [1,\infty)$$
:

$$E_{\rho}(\ell) := \|\phi_{\ell}\|_{\rho} = \left(\sum_{\boldsymbol{s} \in V} (\phi_{\boldsymbol{s}}(\ell(\boldsymbol{s})))^{\rho} + \sum_{\{\boldsymbol{s},t\} \in \mathcal{E}} (\phi_{\boldsymbol{s}t}(\ell(\boldsymbol{s}),\ell(t)))^{\rho}\right)^{1/\rho}$$

For $p = \infty$ (of main interest here)

$$\mathsf{E}_{\infty}(\ell) := \|\phi_{\ell}\|_{\infty} = \max\left\{\max_{s \in V} \phi_{s}(\ell(s)), \max_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))\right\}$$

Standard analysis fact: $E_p(\ell) \nearrow_{p \to \infty} E_{\infty}(\ell)$.

- The value *p* can be seen as a parameter controlling the balance between minimizing the overall cost $E_p(\ell)$ versus minimizing the magnitude of the individual terms $\phi_s(\ell(s))$ and $\phi_{st}(\ell(s), \ell(t))$.
- For *p* = 1, the optimal labeling may contain (few) arbitrarily large individual terms as long as the sum of the terms is small.
- As *p* increases, a larger penalty is assigned to solutions containing large individual terms. This forces local errors to be distributed more evenly across the image domain.



- 2 Algorithms for L_p , $p < \infty$; NP-completeness
- ${}_{\textcircled{3}}$ Which max-norm energies ${\it E}_{\infty}$ can be efficiently optimized?
- [4] New algorithms optimizing E_∞ for 2-labeling
- 5 Strict max-norm optimality
- 6 Summary and conclusions

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Energies L_1 E_{∞} The algorithm Strict optimality Conclusions p = 1: Graph Cut segmentation via min-cut/max-flow

$$E_1(\ell) := \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))$$

- $\phi_s(\ell(s)) = 0$ in all cases (except seeds);
- $\phi_{st}(\ell(s), \ell(t)) = 0$ when $\ell(s) = \ell(t);$
- Cost of cut: $\phi_{st}(\ell(s), \ell(t)) > 0$ (depending of f(s), f(t)) when $\ell(s) \neq \ell(t)$.

Min-cut/max-flow (efficiency between $O(n^2 \ln n)$ and $O(n^3)$) algorithm returns optimized labeling **for 2-labeling**.

Here and below $n := |V \cup \mathcal{E}|$.

Optimization is NP-hard for \geq 3-labeling.

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Energies $L_{1,k}$ E_{∞} The algorithm Strict optimality **2-labeling for general** $E_1(\ell)$ -optimization

$$E_1(\ell) := \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))$$

 E_1 (for 2-labeling) is submodular provided, for every $\{s, t\} \in \mathcal{E}$,

 $\phi_{st}(0,0) + \phi_{st}(1,1) \le \phi_{st}(0,1) + \phi_{st}(1,0).$

Theorem (Kolmogorov & Zabih 2004)

- If E₁ is submodular, then min-cut/max-flow algorithm returns optimized labeling.
- If E_1 is **NOT** submodular, then minimizing E_1 is NP-hard.

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Energies $L_{1,8}$ E_{∞} The algorithm $E_{ ho}(\ell)$ with $1 \leq p < \infty$ is as $E_1(\ell)$

$$(E_{\rho}(\ell))^{\rho} := \sum_{s \in V} (\phi_s(\ell(s)))^{\rho} + \sum_{\{s,t\} \in \mathcal{E}} (\phi_{st}(\ell(s),\ell(t)))^{\rho}$$

 E_p is *p*-submodular provided, for every $\{s, t\} \in \mathcal{E}$,

 $\phi_{st}(0,0)^{\rho} + \phi_{st}(1,1)^{\rho} \le \phi_{st}(0,1)^{\rho} + \phi_{st}(1,0)^{\rho}.$

Corollary (Obvious, Malmberg & Strand, IWCIA 2018)

- If E_p is p-submodular, then min-cut/max-flow algorithm returns optimized labeling.
- If E_p is **NOT** *p*-submodular, then minimizing E_p is NP-hard.

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Strict optimality

Energies L_1 E_∞ The algorithm Strict optimality Conclusions $E_{
ho}(\ell)$ with $1 \le
ho < \infty$ VS $E_\infty(\ell)$

$$\phi_{st}(0,0)^{\rho} + \phi_{st}(1,1)^{\rho} \le \phi_{st}(0,1)^{\rho} + \phi_{st}(1,0)^{\rho}.$$

p-submodular for every $p < \infty$ implies ∞ -submodularity:

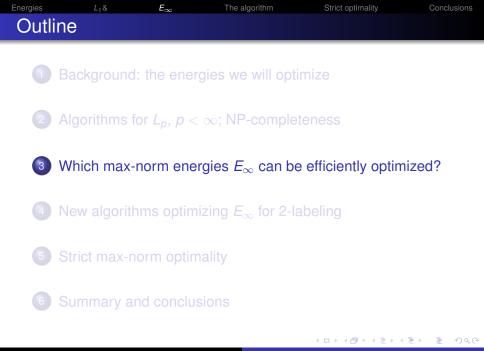
 $\max\{\phi_{st}(0,0),\phi_{st}(1,1)\} \le \max\{\phi_{st}(1,0),\phi_{st}(0,1)\}.$

Theorem (Malmberg & Strand, IWCIA 2018)

1- and ∞ -submodularity imply p-submodularity for all p. In such case min-cut/max-flow algorithm optimizes E_p for every $p < \infty$.

Actually, ϕ is ∞ -submodular iff there is an N so that ϕ is p-submodular for all $p \in (N, \infty)$.

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Energies L_1 E_{∞} The algorithm Strict optimality Conclusions FC segmentations are E_{∞} optimized segmentations

 $E_{\infty}(\ell) := \max \left\{ \max_{s \in V} \phi_{s}(\ell(s)), \max_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t)) \right\}$ We get FC segmentations (as minimization, not maximization)

- $\phi_s(\ell(s)) = 0$ in all cases (except seeds, when $= \infty$);
- $\phi_{st}(\ell(s), \ell(t)) = 0$ when $\ell(s) = \ell(t)$;
- Cost of cut: $\phi_{st}(\ell(s), \ell(t)) > 0$ (depending of f(s), f(t)) when $\ell(s) \neq \ell(t)$.

Dijkstra algorithm (efficiency between O(n) and $O(n \ln n)$) returns optimized labeling for *m*-labeling **for arbitrary large** *m*! Better than for $E_1(\ell)$ (i.e., GC) segmentations.

Q. For what other E_{∞} s are there efficient optimizing algorithms?

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Energies L_1 E_{∞} The algorithm Strict optimality Conclusions Efficient algorithm for 2-labeling ∞ -submodular E_{∞} ?

YES! ∞ -sub algotithm

Theorem (Malmberg, Ciesielski, Strand, DGCI 2019)

There is an algorithm, quasi-linear with respect to $n = |V \cup \mathcal{E}|$, returning minimal 2-labeling for any ∞ -submodular energy E_{∞} .

The algorithm, efficiency between O(n) and $O(n \ln n)$, is NOT Dijkstra-like! This is all that is in the DGCI 2019 paper.

Natural questions, towards post DGCI 2019 work:

Q1: Is ∞ -submodularity assumption essential in the thm?

Q2: Is there efficient algorithm for \geq 3-labelings?



Full answer to Q1: 2-sat algorithm

Theorem (Malmberg, Ciesielski, Strand; 2019 ???) There is an algorithm, $O(n^2)$ with respect to $n = |V \cup \mathcal{E}|$, returning minimal 2-labeling for any $E_{\infty}(\ell)$: $\max \{\max_{s \in V} \phi_s(\ell(s)), \max_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))\}.$

More about the algorithm latter.



Partial answer to Q2:

Theorem (Malmberg, Ciesielski, Strand; 2019 ???)

Optimization problem of the general form of E_∞ energy for more than 2 labels is NP-hard.

Remaining version of Q2:

Q: Under what conditions there exists an efficient (polynomial-time) algorithm for optimization of E_{∞} energy for 3 or more labels?

Can be done in FC/Dijkstra setting. Not (NP-hard) in general.

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 $\begin{array}{c|c} & E_{nergies} & L_{1} & E_{\infty} & \text{The algorithm} & \text{Strict optimality} & \text{Conclusions} \\ \hline & \text{Optimal} \geq 3\text{-labeling of } E_{\infty}(\ell) \text{ is NP-hard: proof} \end{array}$

 $E_{\infty}(\ell) := \max\left\{\max_{s \in V} \phi_s(\ell(s)), \max_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))\right\}$

For a graph $\mathcal{G} = (V, \mathcal{E})$ put:

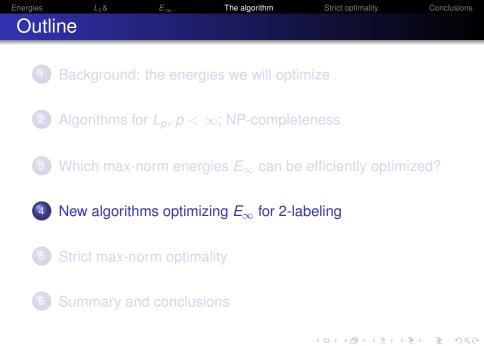
- $\phi_s(\ell(s)) = 0$ in all cases;
- $\phi_{st}(\ell(s), \ell(t)) = 1$ when $\ell(s) = \ell(t)$;
- $\phi_{st}(\ell(s), \ell(t) = 0 \text{ when } \ell(s) \neq \ell(t).$

Then, the minimal $E_{\infty}(\ell)$ is 0 if, and only if, ℓ is a coloring of \mathcal{G} .

But graph *m*-coloring problem for any $m \ge 3$ is NP-complete! It is not for m = 2.

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Energies L_1 E_{∞} The algorithm Strict optimality Conclusions Atoms of E_{∞} and their cost

 $E_{\infty}(\ell) := \max\left\{\max_{s \in V} \phi_s(\ell(s)), \max_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))\right\}$

Atoms $\mathcal{A}(\ell)$ of ℓ : input for ϕ .. and ϕ . (to calculate $E_{\infty}(\ell)$), i.e., $\mathcal{A}(\ell) := \{\{(s, \ell(s))\} : s \in V\} \cup \{\{(s, \ell(s)), (t, \ell(t))\} : \{s, t\} \in \mathcal{E}\}$ Atoms \mathcal{A} of E_{∞} : all such possible atoms, i.e.,

unary: two $\{(s, 0)\}$ and $\{(s, 1)\}$ for each $v \in V$ binary: four $\{(s, i), (t, j)\}$ $(i, j \in \{0, 1\})$ for each $\{s, t\} \in \mathcal{E}$.

- Cost of a unary atom $\{(s, i)\}$: $\phi_s(i)$
- Cost of a binary atom $\{(s, i), (t, j)\}$: $\phi_{st}(i, j)$

Set $\mathcal{A}' \subset \mathcal{A}$ of atoms is consistent when $\mathcal{A}(\ell) \subset \mathcal{A}'$ for some ℓ .

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- List all atoms in a list S in a decreasing cost so that if atoms A₀ and A₁ have the same cost and A₁ = {(s, i), (t, i)}, then A₁ proceeds A₀.
- While S is non-empty do
 - Remove the first atom A from S
 - If A is the last atom for its vertex/edge, insert it to list L
 - Consecutively remove from S all atoms that are locally inconsistent with current S ∪ L
- 3 Return labeling $\ell = \bigcup L$

The locally inconsistency loop is natural.

The trick is to show that the algorithm works property for all $\infty\mathchar`-submodular energies.$

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Towards 2-sat algorithm: 2-satisfiability

For atoms $A = \{(s, i)\}$ and $A' = \{(s, i), (t, j)\}$ define formulas

The algorithm

$$\psi_{\mathcal{A}}(\boldsymbol{s}) := "\boldsymbol{s}
eq \boldsymbol{i}" = egin{cases}
eg \boldsymbol{s} & ext{if } \boldsymbol{i} = 1, \ \boldsymbol{s} & ext{if } \boldsymbol{i} = 0 \end{cases}$$

$$\psi_{A}(s,t) := "(s \neq i) \lor (t \neq j)" = \psi_{\{(s,i)\}}(s) \lor \psi_{\{(t,j)\}}(t).$$

For a set $\mathcal{A}' = \{A_1, A_2, \dots, A_k\}$ of atoms the formula $\psi_{\mathcal{A}'} := \psi_{A_1} \wedge \dots \wedge \psi_{A_k}$ is in 2-conjunctive normal form.

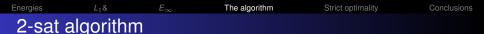
Theorem

Energies

A set $A_1 \subseteq A$ of atoms is consistent if, and only if, the 2-satisfiability problem for a formula $\psi_{A_1^c}$ has a positive solution.

So, consistency of $A_1 \subseteq A$ can be decided by in linear time. (Aspvall et al. algorithm.)

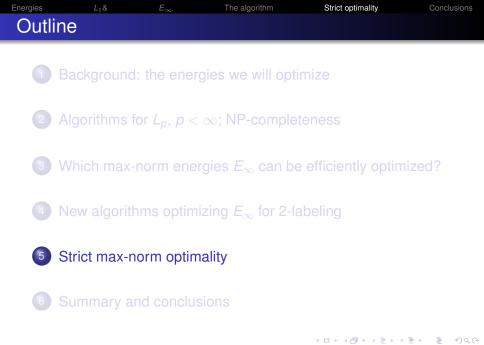
Strict optimality



- List all atoms in a list S in a decreasing cost
- While S is non-empty do
 - Remove the first atom A from S
 - If $S \cup L$ is not consistent, insert A to L
- 3 Return labeling $\ell = \bigcup L$

The $S \cup L$ is not consistent clause is decided by Aspvall et al. algorithm.

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Energies L_1 E_{∞} The algorithmStrict optimalityStrict optimality via lexicographical order

Max-norm identifies ℓ_1 and ℓ_2 when $E_{\infty}(\ell_1) = E_{\infty}(\ell_2)$.

Lexicographical order \leq is a sharper distinguishing tool.

For labeling ℓ , let $\vec{\ell} = \langle \ell_1, \dots, \ell_n \rangle = \langle \Phi(A_1), \dots, \Phi(A_n) \rangle$ non-increasing for an enumeration $\mathcal{A}(\ell) = \{A_1, \dots, A_n\}$.

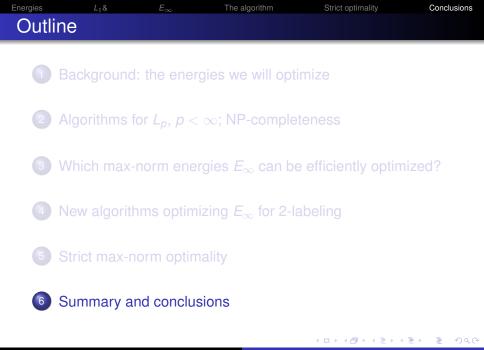
 $\ell \prec \ell'$ iff $\ell_i < \ell'_i$, where $i := \min\{k \colon \ell_k < \ell'_k\}$.

 ℓ is strictly optimal when it is maximal w.r.t. \leq .

Strictly optimal implies max-norm optimal, but not converse.

Q: Can we efficiently find also strict optimizers?

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Conclusions

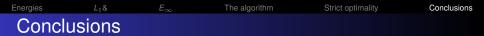
Summary (including new results)

| 2 labels | \geq 3 labels |
|---|---|
| NP-hard problem | NP-hard problem |
| $\max - \text{flow/min-cut} \\ O(n^2 \ln n) \le \cdot \le O(n^3)$ | NP-hard problem |
| 2-sat algorithm $O(n^2)$ | NP-hard problem |
| 2-sat algorithm; $O(n^2)$ | NP-hard problem |
| ∞ -sub algorithm $O(n) \le \cdot \le O(n \ln n)$ | NP-hard problem |
| Dijkstra algorithm $O(n) \le \cdot \le O(n \ln n)$ | Dijkstra algorithm $O(n) \le \cdot \le O(n \ln n)$ |
| | NP-hard problemmax-flow/min-cut $O(n^2 \ln n) \le \cdot \le O(n^3)$ 2-sat algorithm $O(n^2)$ 2-sat algorithm; $O(n^2)$ ∞ -sub algorithm $O(n) \le \cdot \le O(n \ln n)$ |

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- Optimization problems, specifically pixel labeling problems, are frequently occurring in image processing applications.
- We are specifically interested in problems where the objective function is given by the max-norm of the local errors.
- For many such problems, globally optimal solutions can be found very efficiently, in quasi linear or quadratic time.
- Some max-norm for \geq 3-labeling are NP-hard.

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Thank you for your attention!

K. Chris Ciesielski Optimization of Max-Norm Objective Functions 20 of 20

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