# Efficient algorithms for max-norm and lexicographically optimized labelings

#### Krzysztof Chris Ciesielski

Department of Mathematics, West Virginia University and MIPG, Department of Radiology, University of Pennsylvania

Joint work with Filip Malmberg and Robin Strand

University of Campinas, Brazil, May 30, 2019



#### Outline

- Background: the energies we will optimize
- 2 Algorithms for  $L_p$ ,  $p < \infty$ ; NP-completeness
- 3 Which max-norm energies  $E_{\infty}$  can be efficiently optimized?
- 4 New algorithms optimizing  $E_{\infty}$  for 2-labeling
- 5 Strict max-norm optimality
- 6 Summary and conclusions



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- Many fundamental problems in image processing and computer vision, such as image filtering, segmentation, registration, and stereo vision, can naturally be formulated as optimization problems.
- Often, these optimization problems can be described as *labeling* problems, in which we wish to assign to each image element (pixel) an element from some finite set of labels.
- We identify each image with a vertex weighted graph  $\mathcal{G} = (V, \mathcal{E}, f)$ , with vertices V being image voxels, edges  $\mathcal{E}$  being pairs  $\{s, t\}$  of adjacent voxels, and f(s) image intensity at s. Its labeling is a map  $\ell \colon V \to \{0, \dots, m-1\}$ , with m > 2

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With any image n-labeling  $\ell$  we associate local cost map  $\phi_{\ell} \colon V \cup \mathcal{E} \to [0, \infty]$  consisting of

- unary terms  $\phi_{\ell}(s) = \phi_{s}(\ell(s))$ , depending on  $s \in V$ , its label  $\ell(s)$ , and image intensity;
- pairwise terms  $\phi_{\ell}(s,t) = \phi_{s,t}(\ell(s),\ell(t))$ , depending on  $\{s,t\} \in \mathcal{E}$  and their labeling. They reflect desirability of smoothness/regularity of labeling.

All  $\phi_{s,t}(0,0), \phi_{s,t}(0,1), \phi_{s,t}(1,0), \phi_{s,t}(1,1)$  can be distinct

L<sub>1</sub> (graph cut) energy is defined a

$$E_1(\ell) := \|\phi_\ell\|_1 = \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t)),$$

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For  $p = \infty$  (of main interest here)

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- The value p can be seen as a parameter controlling the balance between minimizing the overall cost  $E_p(\ell)$  versus minimizing the magnitude of the individual terms  $\phi_s(\ell(s))$  and  $\phi_{st}(\ell(s),\ell(t))$ .
- For p = 1, the optimal labeling may contain (few) arbitrarily large individual terms as long as the sum of the terms is small.
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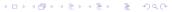


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Min-cut/max-flow (efficiency between  $O(n^2 \ln n)$  and  $O(n^3)$ ) algorithm returns optimized labeling **for 2-labeling**.

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 $E_1$  (for 2-labeling) is submodular provided, for every  $\{s,t\} \in \mathcal{E}$ ,

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Energies Strict optimality Conclusions

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p-submodular for every  $p < \infty$  implies  $\infty$ -submodularity:

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#### Theorem (Malmberg & Strand, IWCIA 2018)

1- and  $\infty$ -submodularity imply p-submodularity for all p. In such case min-cut/max-flow algorithm optimizes  $E_p$  for every  $p < \infty$ .

Actually,  $\phi$  is  $\infty$ -submodular iff there is an N so that  $\phi$  is p-submodular for all  $p \in (N, \infty)$ .

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Energies The algorithm Strict optimality Conclusions

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We get FC segmentations (as minimization, not maximization)

- $\phi_s(\ell(s)) = 0$  in all cases (except seeds, when  $= \infty$ );
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### Efficient algorithm for 2-labeling $\infty$ -submodular $E_{\infty}$ ?

#### YES! ∞-sub algorithm

#### Theorem (Malmberg, Ciesielski, Strand, DGCI 2019)

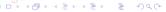
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The algorithm, efficiency between O(n) and  $O(n \ln n)$ , is NOT Dijkstra-like!

This is all that is in the DGCL 2019 paper.

Natural questions, towards post DGCI 2019 work:

Q1: Is ∞-submodularity assumption essential in the thm?



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Optimization problem of the general form of  $E_{\infty}$  energy for more than 2 labels is NP-hard.

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Atoms  $A(\ell)$  of  $\ell$ : input for  $\phi$ .. and  $\phi$ . (to calculate  $E_{\infty}(\ell)$ ), i.e.,

$$\mathcal{A}(\ell) := ig\{ \{(oldsymbol{s}, \ell(oldsymbol{s}))\} \colon oldsymbol{s} \in V ig\} \cup ig\{ \{(oldsymbol{s}, \ell(oldsymbol{s})), (t, \ell(t))\} \colon \{oldsymbol{s}, t\} \in \mathcal{E} ig]$$

Atoms A of  $E_{\infty}$ : all such possible atoms, i.e.,

```
binary: two \{(s,0)\} and \{(s,1)\} for each v \in V
```

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For atoms  $A = \{(s, i)\}$  and  $A' = \{(s, i), (t, j)\}$  define formulas

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A set  $A_1 \subseteq A$  of atoms is consistent if, and only if, the 2-satisfiability problem for a formula  $\psi_{A_1^c}$  has a positive solution.

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# Strict optimality via lexicographical order

Max-norm identifies  $\ell_1$  and  $\ell_2$  when  $E_{\infty}(\ell_1) = E_{\infty}(\ell_2)$ .

Lexicographical order  $\prec$  is a sharper distinguishing tool.

For labeling 
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$$\ell \prec \ell'$$
 iff  $\ell_i < \ell'_i$ , where  $i := \min\{k \colon \ell_k < \ell'_k\}$ .

 $\ell$  is strictly optimal when it is maximal w.r.t.  $\leq$ .

Strictly optimal implies max-norm optimal, but not converse.

Q: Can we efficiently find also strict optimizers?



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∞-submodular strict optimization		NP-hard problem
	2-sat algorithm $O(n^2)$	NP-hard problem
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general case	2-sat algorithm; $O(n^2)$	NP-hard problem
		NP-hard problem
		NP-hard problem  Dijkstra algorithm



	2 labels	≥ 3 labels
general case strict optimization	NP-hard problem	NP-hard problem
∞-submodular strict optimization	max-flow/min-cut $O(n^2 \ln n) \le \cdot \le O(n^3)$	NP-hard problem
unique weights strict optimization	2-sat algorithm $O(n^2)$	NP-hard problem
general case	2-sat algorithm; $O(n^2)$	NP-hard problem
∞-submodular	$\infty$ -sub algorithm $O(n) \le \cdot \le O(n \ln n)$	NP-hard problem
$\infty$ -submodular $\phi_{s}(i) = \phi_{st}(i,i) = 0;$		NP-hard problem  Dijkstra algorithm $O(n) < \cdot < O(n \ln n)$



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$\infty$ -submodular	$\infty$ -sub algorithm $O(n) \le \cdot \le O(n \ln n)$	NP-hard problem
$\phi_{s}(i) = \phi_{st}(i,i) = 0;$	Dijkstra algorithm	Dijkstra algorithm
$\phi_{st}(i,j) = \phi_{st}(j,i) \ge 0$	$O(n) \leq \cdot \leq O(n \ln n)$	$O(n) \leq \cdot \leq O(n \ln n)$



- Optimization problems, specifically pixel labeling problems, are frequently occurring in image processing applications.
- We are specifically interested in problems where the objective function is given by the max-norm of the local errors.
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# Thank you for your attention!

