# Efficient algorithms for max-norm and lexicographically optimized labelings

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University of Saõ Paulo, Brazil, May 27, 2019

# Outline

1. Background: the energies we will optimize
2. Algorithms for $L_p$, $p < \infty$; NP-completeness
3. Which max-norm energies $E_\infty$ can be efficiently optimized?
4. New algorithms optimizing $E_\infty$ for 2-labeling
5. Strict max-norm optimality
6. Summary and conclusions

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# Optimization in image processing

- Many fundamental problems in image processing and computer vision, such as image filtering, segmentation, registration, and stereo vision, can naturally be formulated as optimization problems.
- Often, these optimization problems can be described as *labeling* problems, in which we wish to assign to each image element (pixel) an element from some finite set of labels.
- We identify each image with a vertex weighted graph $\mathcal{G} = (V, \mathcal{E}, f)$, with vertices $V$ being image voxels, edges $\mathcal{E}$ being pairs $\{s, t\}$ of adjacent voxels, and $f(s)$ image intensity at $s$. Its labeling is a map $\ell \colon V \to \{0, \ldots, m-1\}$, with $m \geq 2$.

# $L_p$ energies: the case of $L_1$

With any image $n$-labeling $\ell$ we associate local cost map $\phi_\ell \colon V \cup \mathcal{E} \to [0, \infty]$ consisting of

- unary terms $\phi_\ell(s) = \phi_s(\ell(s))$, depending on $s \in V$, its label $\ell(s)$, and image intensity;
- pairwise terms $\phi_\ell(s,t) = \phi_{s,t}(\ell(s), \ell(t))$, depending on $\{s,t\} \in \mathcal{E}$ and their labeling. They reflect desirability of smoothness/regularity of labeling.
  All $\phi_{s,t}(0,0)$, $\phi_{s,t}(0,1)$, $\phi_{s,t}(1,0)$, $\phi_{s,t}(1,1)$ can be distinct!

$L_1$ (graph cut) energy is defined as

$$E_1(\ell) := \|\phi_\ell\|_1 = \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t)),$$

often represented as (with $x_i$ denoting label of vertex $i$)

$$E(\mathbf{x}) = \sum_{i \in \mathcal{V}} \phi_i(x_i) + \sum_{i,j \in \mathcal{E}} \phi_{ij}(x_i, x_j).$$

# $L_p$ energies: the cases of $p \in (1, \infty]$

For $p \in [1, \infty)$:

$$E_p(\ell) := \|\phi_\ell\|_p = \left( \sum_{s \in V} (\phi_s(\ell(s)))^p + \sum_{\{s,t\} \in \mathcal{E}} (\phi_{st}(\ell(s), \ell(t)))^p \right)^{1/p}$$

For $p = \infty$ (of main interest here)

$$E_\infty(\ell) := \|\phi_\ell\|_\infty = \max \left\{ \max_{s \in V} \phi_s(\ell(s)), \max_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t)) \right\}$$

**Standard analysis fact:** $E_p(\ell) \nearrow_{p \to \infty} E_\infty(\ell)$.

# What is the effect of $p$?

- The value $p$ can be seen as a parameter controlling the balance between minimizing the overall cost $E_p(\ell)$ versus minimizing the magnitude of the individual terms $\phi_s(\ell(s))$ and $\phi_{st}(\ell(s), \ell(t))$.

- For $p = 1$, the optimal labeling may contain (few) arbitrarily large individual terms as long as the sum of the terms is small.

- As $p$ increases, a larger penalty is assigned to solutions containing large individual terms. This forces local errors to be distributed more evenly across the image domain.

# Outline

# $p = 1$: Graph Cut segmentation via min-cut/max-flow

$$E_1(\ell) := \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))$$

- $\phi_s(\ell(s)) = 0$ in all cases (except seeds);
- $\phi_{st}(\ell(s), \ell(t)) = 0$ when $\ell(s) = \ell(t)$;
- Cost of cut: $\phi_{st}(\ell(s), \ell(t)) > 0$ (depending of $f(s)$, $f(t)$) when $\ell(s) \neq \ell(t)$.

Min-cut/max-flow (efficiency between $O(n^2 \ln n)$ and $O(n^3)$) algorithm returns optimized labeling **for 2-labeling.**

Here and below $n := |V \cup \mathcal{E}|$.

Optimization is NP-hard **for** $\geq$ **3-labeling**.

# 2-labeling for general $E_1(\ell)$-optimization

$$E_1(\ell) := \sum_{s \in V} \phi_s(\ell(s)) + \sum_{\{s,t\} \in \mathcal{E}} \phi_{st}(\ell(s), \ell(t))$$

$E_1$ (for 2-labeling) is submodular provided, for every $\{s,t\} \in \mathcal{E}$,

$$\phi_{st}(0,0) + \phi_{st}(1,1) \leq \phi_{st}(0,1) + \phi_{st}(1,0).$$

**Theorem (Kolmogorov & Zabih 2004)**

- *If $E_1$ is submodular, then min-cut/max-flow algorithm returns optimized labeling.*
- *If $E_1$ is **NOT** submodular, then minimizing $E_1$ is NP-hard.*

# $E_p(\ell)$ with $1 \le p < \infty$ is as $E_1(\ell)$

$$\left(E_p(\ell)\right)^p := \sum_{s \in V} \left(\phi_s(\ell(s))\right)^p + \sum_{\{s,t\} \in \mathcal{E}} \left(\phi_{st}(\ell(s), \ell(t))\right)^p$$

$E_p$ is $p$-submodular provided, for every $\{s,t\} \in \mathcal{E}$,

$$\phi_{st}(0,0)^p + \phi_{st}(1,1)^p \le \phi_{st}(0,1)^p + \phi_{st}(1,0)^p.$$

**Corollary (Obvious, Malmberg & Strand, IWCIA 2018)**

- *If $E_p$ is p-submodular, then min-cut/max-flow algorithm returns optimized labeling.*
- *If $E_p$ is* **NOT** *p-submodular, then minimizing $E_p$ is NP-hard.*

# $E_p(\ell)$ with $1 \leq p < \infty$ vs $E_\infty(\ell)$

$$\phi_{st}(0,0)^p + \phi_{st}(1,1)^p \leq \phi_{st}(0,1)^p + \phi_{st}(1,0)^p.$$

$p$-submodular for every $p < \infty$ implies $\infty$-submodularity:

$$\max\{\phi_{st}(0,0), \phi_{st}(1,1)\} \leq \max\{\phi_{st}(1,0), \phi_{st}(0,1)\}.$$

**Theorem (Malmberg & Strand, IWCIA 2018)**

*1- and $\infty$-submodularity imply p-submodularity for all p. In such case min-cut/max-flow algorithm optimizes $E_p$ for every $p < \infty$.*

Actually, $\phi$ is $\infty$-submodular iff there is an $N$ so that $\phi$ is $p$-submodular for all $p \in (N, \infty)$.

# Outline

# FC segmentations are $E_\infty$ optimized segmentations

$$E_\infty(\ell) := \max\left\{\max_{s\in V}\phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}}\phi_{st}(\ell(s),\ell(t))\right\}$$

We get FC segmentations (as minimization, not maximization)

- $\phi_s(\ell(s)) = 0$ in all cases (except seeds, when $= \infty$);
- $\phi_{st}(\ell(s),\ell(t)) = 0$ when $\ell(s) = \ell(t)$;
- Cost of cut: $\phi_{st}(\ell(s),\ell(t)) > 0$ (depending of $f(s)$, $f(t)$) when $\ell(s) \neq \ell(t)$.

**Dijkstra algorithm** (efficiency between $O(n)$ and $O(n \ln n)$) returns optimized labeling for $m$-labeling **for arbitrary large** $m$**!** Better than for $E_1(\ell)$ (i.e., GC) segmentations.

**Q.** For what other $E_\infty$s are there efficient optimizing algorithms?

# Efficient algorithm for 2-labeling $\infty$-submodular $E_\infty$?

**YES! $\infty$-sub algotithm**

**Theorem (Malmberg, Ciesielski, Strand, DGCI 2019)**

*There is an algorithm, quasi-linear with respect to $n = |V \cup \mathcal{E}|$, returning minimal 2-labeling for any $\infty$-submodular energy $E_\infty$.*

The algorithm, efficiency between $O(n)$ and $O(n \ln n)$,
is NOT Dijkstra-like!
This is all that is in the DGCI 2019 paper.

Natural questions, towards post DGCI 2019 work:

Q1: Is $\infty$-submodularity assumption essential in the thm?

Q2: Is there efficient algorithm for $\geq$ 3-labelings?

# Optimal 2-labeling for $E_\infty(\ell)$ with no $\infty$-submodularity

**Full answer to Q1: 2-sat algorithm**

Theorem (Malmberg, Ciesielski, Strand; 2019 ??? )

**There is an algorithm,**
$O(n^2)$ **with respect to** $n = |V \cup \mathcal{E}|$**,**
**returning minimal** 2**-labeling for any** $E_\infty(\ell)$**:**

$$\max\left\{\max_{s\in V}\phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}}\phi_{st}(\ell(s),\ell(t))\right\}.$$

More about the algorithm latter.

# Q2: What about optimal $\geq$ 3-labeling for $E_\infty(\ell)$?

Partial answer to Q2:

**Theorem (Malmberg, Ciesielski, Strand; 2019 ??? )**

**Optimization problem of the general form of $E_\infty$ energy for more than 2 labels is NP-hard.**

Remaining version of Q2:

Q: Under what conditions there exists an efficient (polynomial-time) algorithm for optimization of $E_\infty$ energy for 3 or more labels?

Can be done in FC/Dijkstra setting. Not (NP-hard) in general.

# Optimal $\geq$ 3-labeling of $E_\infty(\ell)$ is NP-hard: proof

$$E_\infty(\ell) := \max\left\{\max_{s\in V}\phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}}\phi_{st}(\ell(s),\ell(t))\right\}$$

For a graph $\mathcal{G} = (V, \mathcal{E})$ put:

- $\phi_s(\ell(s)) = 0$ in all cases;
- $\phi_{st}(\ell(s), \ell(t)) = 1$ when $\ell(s) = \ell(t)$;
- $\phi_{st}(\ell(s), \ell(t) = 0$ when $\ell(s) \neq \ell(t)$.

Then, the minimal $E_\infty(\ell)$ is 0 if, and only if, $\ell$ is a coloring of $\mathcal{G}$.

But graph $m$-coloring problem for any $m \geq 3$ is NP-complete!
It is not for $m = 2$.

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# Atoms of $E_\infty$ and their cost

$$E_\infty(\ell) := \max\left\{\max_{s\in V}\phi_s(\ell(s)), \max_{\{s,t\}\in\mathcal{E}}\phi_{st}(\ell(s),\ell(t))\right\}$$

Atoms $\mathcal{A}(\ell)$ of $\ell$: input for $\phi_{..}$ and $\phi_.$ (to calculate $E_\infty(\ell)$), i.e.,

$$\mathcal{A}(\ell) := \{\{(s,\ell(s))\}\colon s\in V\}\cup\{\{(s,\ell(s)),(t,\ell(t))\}\colon \{s,t\}\in\mathcal{E}\}$$

Atoms $\mathcal{A}$ of $E_\infty$: all such possible atoms, i.e.,

unary: two $\{(s,0)\}$ and $\{(s,1)\}$ for each $v\in V$

binary: four $\{(s,i),(t,j)\}$ $(i,j\in\{0,1\})$ for each $\{s,t\}\in\mathcal{E}$.

- Cost of a unary atom $\{(s,i)\}$: $\phi_s(i)$
- Cost of a binary atom $\{(s,i),(t,j)\}$: $\phi_{st}(i,j)$

Set $\mathcal{A}'\subset\mathcal{A}$ of atoms is consistent when $\mathcal{A}(\ell)\subset\mathcal{A}'$ for some $\ell$.

# $\infty$-sub algorithm

1. List all atoms in a list $S$ in a decreasing cost so that if atoms $A_0$ and $A_1$ have the same cost and $A_1 = \{(s,i),(t,i)\}$, then $A_1$ proceeds $A_0$.
2. While $S$ is non-empty do
   - Remove the first atom $A$ from $S$
   - If $A$ is the last atom for its vertex/edge, insert it to list $L$
   - Consecutively remove from $S$ all atoms that are locally inconsistent with current $S \cup L$
3. Return labeling $\ell = \bigcup L$

The locally inconsistency loop is natural.

The trick is to show that the algorithm works property for all $\infty$-submodular energies.

# Towards 2-sat algorithm: 2-satisfiability

For atoms $A = \{(s,i)\}$ and $A' = \{(s,i),(t,j)\}$ define formulas

$$\psi_A(s) := \text{“}s \neq i\text{”} = \begin{cases} \neg s & \text{if } i = 1, \\ s & \text{if } i = 0 \end{cases}$$

$$\psi_A(s,t) := \text{“}(s \neq i) \vee (t \neq j)\text{”} = \psi_{\{(s,i)\}}(s) \vee \psi_{\{(t,j)\}}(t).$$

For a set $\mathcal{A}' = \{A_1, A_2, \ldots, A_k\}$ of atoms the formula $\psi_{\mathcal{A}'} := \psi_{A_1} \wedge \cdots \wedge \psi_{A_k}$. is in *2-conjunctive normal form*.

## Theorem

*A set $\mathcal{A}_1 \subseteq \mathcal{A}$ of atoms is consistent if, and only if, the 2-satisfiability problem for a formula $\psi_{\mathcal{A}_1^c}$ has a positive solution.*

So, consistency of $\mathcal{A}_1 \subseteq \mathcal{A}$ can be decided by in linear time. (Aspvall et al. algorithm.)

# 2-sat algorithm

1. List all atoms in a list $S$ in a decreasing cost
2. While $S$ is non-empty do
   - Remove the first atom $A$ from $S$
   - If $S \cup L$ is not consistent, insert $A$ to $L$
3. Return labeling $\ell = \bigcup L$

The $S \cup L$ is not consistent clause is decided by Aspvall et al. algorithm.

# Outline

# Strict optimality via lexicographical order

Max-norm identifies $\ell_1$ and $\ell_2$ when $E_\infty(\ell_1) = E_\infty(\ell_2)$.

Lexicographical order $\preceq$ is a sharper distinguishing tool.

For labeling $\ell$, let $\vec{\ell} = \langle \ell_1, \ldots, \ell_n \rangle = \langle \Phi(A_1), \ldots, \Phi(A_n) \rangle$ non-increasing for an enumeration $\mathcal{A}(\ell) = \{A_1, \ldots, A_n\}$.

$\ell \prec \ell'$ iff $\ell_i < \ell'_i$, where $i := \min\{k \colon \ell_k < \ell'_k\}$.

$\ell$ is strictly optimal when it is maximal w.r.t. $\preceq$.

Strictly optimal implies max-norm optimal, but not converse.

Q: Can we efficiently find also strict optimizers?

# Outline

# Summary (including new results)

| | 2 labels | $\geq$ 3 labels |
|---|---|---|
| general case<br>strict optimization | **NP-hard problem** | NP-hard problem |
| $\infty$-submodular<br>strict optimization | max-flow/min-cut<br>$O(n^2 \ln n) \leq \cdot \leq O(n^3)$ | NP-hard problem |
| unique weights<br>strict optimization | **2-sat algorithm**<br>$O(n^2)$ | NP-hard problem |
| general case | 2-sat algorithm; $O(n^2)$ | NP-hard problem |
| $\infty$-submodular | $\infty$-sub algorithm<br>$O(n) \leq \cdot \leq O(n \ln n)$ | NP-hard problem |
| $\phi_s(i) = \phi_{st}(i,i) = 0$;<br>$\phi_{st}(i,j) = \phi_{st}(j,i) \geq 0$ | Dijkstra algorithm<br>$O(n) \leq \cdot \leq O(n \ln n)$ | Dijkstra algorithm<br>$O(n) \leq \cdot \leq O(n \ln n)$ |

# Conclusions

- Optimization problems, specifically pixel labeling problems, are frequently occurring in image processing applications.
- We are specifically interested in problems where the objective function is given by the max-norm of the local errors.
- For many such problems, globally optimal solutions can be found very efficiently, in quasi linear or quadratic time.
- Some max-norm for $\geq$ 3-labeling are NP-hard.

Thank you for your attention!