Tutorial on Martin's Axiom, by example

Krzysztof Chris Ciesielski

Department of Mathematics, West Virginia University
MIPG, Department of Radiology, University of Pennsylvania

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Set Theory for the Working Mathematician, Cambridge Univ. Press 1997.

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Outline

1. The theorem we will prove
2. Induction: a way to understand Martin’s Axiom
3. Martin’s Axiom and the proof of our theorem

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The theorem and its motivation

Theorem (What we will prove)

Assume MA. If $X \in [\mathbb{R}]^{<\mathfrak{c}}$ then every $Y \subset X$ is a G_δ subset of X, that is, there exists a G_δ set $G \subset \mathbb{R}$ such that $G \cap X = Y$.

Corollary (Explaining why Theorem is important)

If MA holds, then $2^\kappa = 2^\omega$ for every infinite cardinal $\kappa < \mathfrak{c}$.

Proof.

For $X \in [\mathbb{R}]^\kappa$

$$2^\kappa = |\mathcal{P}(X)| = |\{B \cap X \colon B \in Borel\}| \leq |Borel| = 2^\omega.$$

□

Reduction to a combinatorial statement

Thm: Under MA, every $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$ is a G_δ subset of X.

$\mathcal{B} := \{B_n \colon n < \omega\}$ a countable base for $\mathbb{R}$. It is enough to prove:

Proposition (Reduction)

Under MA, for every $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$ there is $\hat{A} \subset \omega$ such that for every $x \in X$

$$x \in Y \quad \Leftrightarrow \quad x \in B_n \text{ for infinitely many } n \text{ from } \hat{A}.$$

Proof of reduction.

Fix $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$ and let $\hat{A} \subset \omega$ be as in Proposition.
For every $k < \omega$ the set $G_k := \bigcup \{B_n \colon n \in \hat{A} \ \& \ n > k\}$ is open.
$G := \bigcap_{k<\omega} G_k$ is as needed, as for every $x \in X$

$$x \in Y \quad \Leftrightarrow \quad x \in G_k \text{ for all } k < \omega.$$

Outline

Inductive construction of $\hat{A}$ for countable Y

Fix countable $Y \subset \mathbb{R}$ and $X \subset Y$. Need $\hat{A} \subset \omega$ such that

$$\forall x \in X \left[x \in Y \quad \Leftrightarrow \quad x \in B_n \text{ for infinitely many } n \text{ from } \hat{A}. \right]$$

Proof.

Let $X \setminus Y = \{z_n \colon n < \omega\}$ and $Y \times \omega = \{\langle y_n, k_n \rangle \colon n < \omega\}$.

Construct increasing $\langle A_n \in [\omega]^{<\omega} \rangle_{n<\omega}$ aiming for $\hat{A} = \bigcup_{n<\omega} A_n$.

Ensuring infinity (**diagonalization**): For every $n < \omega$ insert to $A_n \supset A_{n-1}$ an m with $m > k_n$ and $y_n \in B_m$.

Preservation of finiteness: and such that $z_i \notin B_m$ for all $m \in A_n \setminus A_{n-1}$ and $i \leq n$.

Then $\hat{A} = \bigcup_{n<\omega} A_n$ is as needed. □

From induction to a partial order, PO, set

Fixed countable $Y \subset \mathbb{R}$ and $X \subset Y$. Need $\hat{A} \subset \omega$ such that

$$\forall x \in X\big[x \in Y \quad \Leftrightarrow \quad x \in B_n \text{ for infinitely many } n \text{ from } \hat{A}.\big]$$

Define PO set $\langle \mathbb{P}, \leq \rangle$ by $\mathbb{P} := [\omega]^{<\omega} \times [X \setminus Y]^{<\omega}$

and for $\langle A_1, C_1 \rangle, \langle A_0, C_0 \rangle \in \mathbb{P}$ we let

$\langle A_1, C_1 \rangle \leq \langle A_0, C_0 \rangle$ provided $A_1 \supset A_0$, $C_1 \supset C_0$, and

$$c \notin B_m \text{ for all } m \in A_1 \setminus A_0 \text{ and } c \in C_0.$$

We will construct $\langle A_0, C_0 \rangle \geq \langle A_1, C_1 \rangle \geq \langle A_2, C_2 \rangle \cdots$ aiming for $\hat{A} = \bigcup_{n<\omega} A_n$. Sequence needs contain an element from each:

$$D_y^k = \{\langle A, C \rangle \in \mathbb{P} \colon \exists m \in A\, (m \geq k \,\&\, y \in B_m)\} \text{ with } y \in Y,\, k < \omega;$$

$$E_z = \{\langle A, C \rangle \in \mathbb{P} \colon z \in C\} \text{ with } z \in X \setminus Y.$$

The construction is possible (**density**) and $\hat{A}$ is as needed.

Dense sets and generic filters

$D \subset \mathbb{P}$ is dense (in a PO set $\langle \mathbb{P}, \leq \rangle$) provided
for every $p \in \mathbb{P}$ there exists a $q \in D$ such that $q \leq p$.

We just proved that sets D_y^k and E_z are dense in our PO set.

$\mathcal{F} \subset \mathbb{P}$ is a filter provided

1. $q \in \mathcal{F}$ whenever $q \geq p \in \mathcal{F}$
2. for every $p, q \in \mathcal{F}$ there is $r \in \mathcal{F}$ with $r \leq p$ and $r \leq q$

For the constructed sequence $\langle \langle A_i, C_i \rangle : i < \omega \rangle$ we have a filter

$$\mathcal{F} := \{\langle A, C \rangle : \langle A, C \rangle \leq \langle A_i, C_i \rangle \text{ for some } i\}.$$

For a family $\mathcal{D}$ of sets, a filter $\mathcal{F}$ is $\mathcal{D}$-generic provided
$\mathcal{F} \cap D \neq \emptyset$ for every $D \in \mathcal{D}$ **dense** in the PO set.

Our $\mathcal{F}$ is $\mathcal{D}$-generic for $\mathcal{D} = \{D_y^k : y \in Y, k < \omega\} \cup \{E_z : z \in X \setminus Y\}$.

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Statement of Martin's Axiom MA

- $p, q \in \mathbb{P}$ are compatible provided there exists an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$;
- $A \subset \mathbb{P}$ is an antichain if no distinct $p, q \in \mathbb{P}$ are compatible.
- $\langle \mathbb{P}, \leq \rangle$ is ccc provided $\mathbb{P}$ contains no uncountable antichain.

MA: *For every ccc PO set $\langle \mathbb{P}, \leq \rangle$ and every family $\mathcal{D}$ of cardinality less than $\mathfrak{c}$ there exists a $\mathcal{D}$-generic filter $\mathcal{F}$ in $\mathbb{P}$.*

- For countable families $\mathcal{D}$ the MA statement is true in ZFC. No ccc is needed.This is Rasiowa-Sikorski lemma.
- MA is consistent with ZFC and the negation of CH, the continuum hypothesis.

Proof of our Theorem, part 1

Fix $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$; $\mathbb{P} := [\omega]^{<\omega} \times [X \setminus Y]^{<\omega}$ s.t.
$\langle A_1, C_1 \rangle \leq \langle A_0, C_0 \rangle$ provided $A_1 \supset A_0$, $C_1 \supset C_0$, and

$$c \notin B_m \text{ for all } m \in A_1 \setminus A_0 \text{ and } c \in C_0.$$

It is ccc. (Will prove this next.) The family
$\mathcal{D} = \{D_y^k \colon y \in Y, k < \omega\} \cup \{E_z \colon z \in X \setminus Y\}$ has cardinality $< \mathfrak{c}$.

So, by MA, there exists a $\mathcal{D}$-generic filter $\mathcal{F}$. We claim that

$$\hat{A} = \bigcup \{A \colon \langle A, C \rangle \in \mathcal{F}\}$$

is as needed.

Proof of our Theorem: $\mathbb{P}$ is ccc

$Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$ and PO set $\mathbb{P} := [\omega]^{<\omega} \times [X \setminus Y]^{<\omega}$ s.t. $\langle A_1, C_1 \rangle \leq \langle A_0, C_0 \rangle$ provided $A_1 \supset A_0$, $C_1 \supset C_0$, and

$$c \notin B_m \text{ for all } m \in A_1 \setminus A_0 \text{ and } c \in C_0.$$

Fix uncountable subset $\mathcal{A} := \{\langle A_\xi, C_\xi \rangle : \xi < \omega_1\}$ of $\mathbb{P}$.

Since $[\omega]^{<\omega}$ is countable, there are $A \in [\omega]^{<\omega}$ and $\zeta < \xi < \omega_1$ such that $A_\zeta = A_\xi = A$.

Then $\langle A_\zeta, C_\zeta \rangle = \langle A, C_\zeta \rangle$ and $\langle A_\xi, C_\xi \rangle = \langle A, C_\xi \rangle$ are compatible, since $\langle A, C_\zeta \cup C_\xi \rangle \in \mathbb{P}$ extends them both.

So, $\mathcal{A}$ is not an antichain.

Proof of our Theorem: “finitely many” part

$z \in Z = Y \setminus X \implies z \in B_n$ only for finitely many $n \in \hat{A}$.

$\mathbb{P} := [\omega]^{<\omega} \times [X \setminus Y]^{<\omega}$; $\langle A_1, C_1 \rangle \leq \langle A_0, C_0 \rangle$ provided
$A_1 \supset A_0$, $C_1 \supset C_0$, and $c \notin B_m$ for all $m \in A_1 \setminus A_0$ and $c \in C_0$.

$E_z = \{\langle A, C \rangle \in \mathbb{P} \colon z \in C\}$ with $z \in X \setminus Y$.

As $\mathcal{F}$ is $\mathcal{D}$-generic, there is $\langle A_0, C_0 \rangle \in \mathcal{F} \cap E_z$.
It is enough to prove that $z \notin B_m$ for every $m \in \hat{A} \setminus A_0$.
Take $m \in \hat{A} \setminus A_0$.
By the definition of $\hat{A}$ there is $\langle A, C \rangle \in \mathcal{F}$ such that $m \in A$.
$\exists \langle A_1, C_1 \rangle \in \mathcal{F}$ extending $\langle A, C \rangle$ and $\langle A_0, C_0 \rangle$ (as $\mathcal{F}$ is filter).

So $\langle A_1, C_1 \rangle \leq \langle A_0, C_0 \rangle$, $m \in A \subset A_1$, $m \notin A_0$, and $z \in C_0$.
Hence indeed $z \notin B_m$.

Proof of our Theorem: “infinitely many” part

$x \in Y \implies x \in B_n$ for infinitely many $n \in \hat{A}$.

$\mathbb{P} := [\omega]^{<\omega} \times [X \setminus Y]^{<\omega}$; $\langle A_1, C_1 \rangle \leq \langle A_0, C_0 \rangle$ provided $A_1 \supset A_0$, $C_1 \supset C_0$, and $c \notin B_m$ for all $m \in A_1 \setminus A_0$ and $c \in C_0$.

$D_y^k = \{\langle A, C \rangle \in \mathbb{P} \colon \exists m \in A\ (m \geq k\ \&\ y \in B_m)\}$.

As $\mathcal{F}$ is $\mathcal{D}$-generic, for every $k < \omega$ there is $\langle A, C \rangle \in F \cap D_x^k$.

So, there is an $m \in A \subset \hat{A}$ with $m > k$ such that $x \in B_m$.

Hence $x \in B_m$ for infinitely many m from $\hat{A}$.

End of the proof!

That is all!

Thank you for your attention!