Tutorial on Martin's Axiom, by example

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Based on Theorem 8.2.3 from my book
Set Theory for the Working Mathematician, Cambridge Univ. Press 1997.

Workshop: Topological and Algebraic Genericity II, Spain, November 13, 2018.

Outline

1. The theorem we will prove
2. Induction: a way to understand Martin's Axiom
3. Martin's Axiom and the proof of our theorem

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The theorem and its motivation

Theorem (What we will prove)

Assume MA. If $X \in [\mathbb{R}]^{<\mathfrak{c}}$ then every $Y \subset X$ is a G_δ subset of X, that is, there exists a G_δ set $G \subset \mathbb{R}$ such that $G \cap X = Y$.

Corollary (Explaining why Theorem is important)

If MA holds, then $2^\kappa = 2^\omega$ for every infinite cardinal $\kappa < \mathfrak{c}$.

Proof.

For $X \in [\mathbb{R}]^\kappa$

$$2^\kappa = |\mathcal{P}(X)| = |\{B \cap X : B \in Borel\}| \leq |Borel| = 2^\omega.$$

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Reduction to a combinatorial statement

Thm: Under MA, every $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$ is a G_δ subset of X.

$\mathcal{B} := \{B_n : n < \omega\}$ a countable base for $\mathbb{R}$. It is enough to prove:

Proposition (Reduction)

Under MA, for every $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$ there is $\hat{A} \subset \omega$ such that for every $x \in X$

$$x \in Y \quad \Leftrightarrow \quad x \in B_n \text{ for infinitely many } n \text{ from } \hat{A}.$$

Proof of reduction.

Fix $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$ and let $\hat{A} \subset \omega$ be as in Proposition.
For every $k < \omega$ the set $G_k := \bigcup\{B_n : n \in \hat{A} \,\&\, n > k\}$ is open.
$G := \bigcap_{k<\omega} G_k$ is as needed, as for every $x \in X$

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Proof.

Let $X \setminus Y = \{z_n : n < \omega\}$ and $Y \times \omega = \{\langle y_n, k_n \rangle : n < \omega\}$.

Construct increasing $\langle A_n \in [\omega]^{<\omega} \rangle_{n<\omega}$ aiming for $\hat{A} = \bigcup_{n<\omega} A_n$.

Ensuring infinity (**diagonalization**): For every $n < \omega$ insert to $A_n \supset A_{n-1}$ an m with $m > k_n$ and $y_n \in B_m$.

Preservation of finiteness: and such that $z_i \notin B_m$ for all $m \in A_n \setminus A_{n-1}$ and $i \leq n$.

Then $\hat{A} = \bigcup_{n<\omega} A_n$ is as needed. □

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$$D_y^k = \{\langle A, C \rangle \in \mathbb{P} \colon \exists m \in A \, (m \geq k \,\&\, y \in B_m)\} \text{ with } y \in Y, k < \omega;$$

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Dense sets and generic filters

$D \subset \mathbb{P}$ is dense (in a PO set $\langle \mathbb{P}, \leq \rangle$) provided
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We just proved that sets D^k_y and E_z are dense in our PO set.

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Outline

Statement of Martin's Axiom MA

- $p, q \in \mathbb{P}$ are compatible provided there exists an $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$;
- $A \subset \mathbb{P}$ is an antichain if no distinct $p, q \in \mathbb{P}$ are compatible.
- $\langle \mathbb{P}, \leq \rangle$ is ccc provided $\mathbb{P}$ contains no uncountable antichain.

MA: *For every ccc PO set $\langle \mathbb{P}, \leq \rangle$ and every family $\mathcal{D}$ of cardinality less than $\mathfrak{c}$ there exists a $\mathcal{D}$-generic filter $\mathcal{F}$ in $\mathbb{P}$.*

- For countable families $\mathcal{D}$ the MA statement is true in ZFC. No ccc is needed.This is Rasiowa-Sikorski lemma.
- MA is consistent with ZFC and the negation of CH, the continuum hypothesis.

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Proof of our Theorem, part 1

Fix $Y \subset X \in [\mathbb{R}]^{<\mathfrak{c}}$; $\mathbb{P} := [\omega]^{<\omega} \times [X \setminus Y]^{<\omega}$ s.t. $\langle A_1, C_1 \rangle \le \langle A_0, C_0 \rangle$ provided $A_1 \supset A_0$, $C_1 \supset C_0$, and

$$c \notin B_m \text{ for all } m \in A_1 \setminus A_0 \text{ and } c \in C_0.$$

It is ccc. (Will prove this next.) The family $\mathcal{D} = \{D_y^k \colon y \in Y, k < \omega\} \cup \{E_z \colon z \in X \setminus Y\}$ has cardinality $< \mathfrak{c}$.

So, by MA, there exists a $\mathcal{D}$-generic filter $\mathcal{F}$. We claim that

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Since $[\omega]^{<\omega}$ is countable, there are $A \in [\omega]^{<\omega}$ and $\zeta < \xi < \omega_1$ such that $A_\zeta = A_\xi = A$.

Then $\langle A_\zeta, C_\zeta \rangle = \langle A, C_\zeta \rangle$ and $\langle A_\xi, C_\xi \rangle = \langle A, C_\xi \rangle$ are compatible, since $\langle A, C_\zeta \cup C_\xi \rangle \in \mathbb{P}$ extends them both.

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Since $[\omega]^{<\omega}$ is countable, there are $A \in [\omega]^{<\omega}$ and $\zeta < \xi < \omega_1$ such that $A_\zeta = A_\xi = A$.

Then $\langle A_\zeta, C_\zeta \rangle = \langle A, C_\zeta \rangle$ and $\langle A_\xi, C_\xi \rangle = \langle A, C_\xi \rangle$ are compatible, since $\langle A, C_\zeta \cup C_\xi \rangle \in \mathbb{P}$ extends them both.

So, $\mathcal{A}$ is not an antichain.

Proof of our Theorem: "finitely many" part

$z \in Z = Y \setminus X \implies z \in B_n$ only for finitely many $n \in \hat{A}$.

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$D_y^k = \{\langle A, C \rangle \in \mathbb{P} \colon \exists m \in A\, (m \geq k \;\&\; y \in B_m)\}$.

As $\mathcal{F}$ is $\mathcal{D}$-generic, for every $k < \omega$ there is $\langle A, C \rangle \in F \cap D_x^k$.

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