Differentiability versus continuity: Restriction and extension theorems and monstrous examples

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Preamble: new (2017) results that are behind this talk

Example (New simple construction of a classic example)

There exists a differentiable nowhere monotone map $f\colon \mathbb{R} \to \mathbb{R}$.

Example (Greatly simplified construction of 2016 example)

There exists a differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect $\mathfrak{X} \subset \mathbb{R}$ with $\mathfrak{f}' \equiv 0$.

Theorem (C^1 interpolation thm, no Lebesgue measure needed)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$:

- *there is perfect $P \subset \mathbb{R}$ s.t. $f \restriction P$ is Lipschitz;*
- *there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.*

Theorem (Simple proof of Whitney and Jarník extension thms)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.This F is C^1 iff such extension exists iff a simple (new) condition for f holds.

No familiarity

with Lebesgue measure

is needed to follow any proof

behind this talk

Outline

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Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

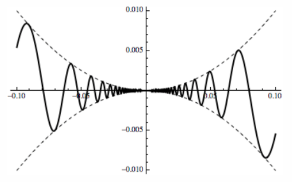
However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

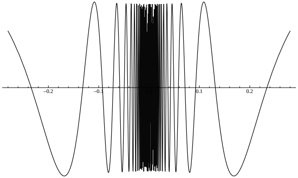
True question: *To what extend $f = F'$ must be continuous?*

About $F(x) = x^2 \sin\left(x^{-1}\right)$

This F appeared already in the 1881 paper of Vito Volterra (1860-1940)



Graph of F



Graph of F'

To what extend $f = F'$ must be continuous?

Jean-Gaston Darboux
(1842-1917)

Theorem (Darboux 1875)

Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called Darboux functions.

Baire result

René-Louis Baire
(1874-1932)

Theorem (1899 dissertation of Baire)

The derivative of any differentiable $F: \mathbb{R} \to \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a dense G_δ-set.

Proof of previous theorem and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R} \to \mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^{\infty} V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0} \{x \in \mathbb{R}\colon |g(s) - f(g)| < 1/n \text{ for all } s,t \in (x-\delta, x+\delta)\}.$$

If $g = \lim_{n\to\infty} g_n$, $g_n\colon \mathbb{R} \to \mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} U_N^n$, where each U_N^n is the interior of the closed set

$$\{x \in \mathbb{R}\colon |f_k(x) - f_m(x)| \le 1/n \text{ for all } m,k \ge N\}.$$

Theorem

Let $G \subset \mathbb{R}$.
There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

Outline

Fixed point property

Theorem (Relatively new)

If $f = f_n \circ \cdots \circ f_1$, where each $f_i \colon [0,1] \to [0,1]$ is a derivative, then f has a fixed point.

For $n = 1$: easy exercise, as $h(x) = f(x) - x$ is Darboux.

For $n = 2$: proved independently in 2001
by Csörnyei, O'Neil & Preiss and by Elekes, Keleti & Prokaj.

For arbitrary n: Szuca 2003.

Open Problem

Must f as in the theorem have connected graph?

Yes for $n = 1$. Positive answer would imply the theorem.

Baire classification of composition of the derivatives.

Let $f = f_n \circ \cdots \circ f_1$, where each f_i is a derivative.

Then f is Darboux.

Any Darboux Baire class one map has connected graph.

A natural question: must f be of Baire class 1? NO

Theorem (*Andy* Bruckner and K. Ciesielski 2018)

There exist derivatives $\varphi, \gamma \colon [-1,1] \to [-1,1]$ such that their composition $\psi := \varphi \circ \gamma$ is not of Baire class one.

We use $\gamma(x) := \cos(x^{-1})$ and φ Pompeiu's map, see below.

Problem (could be easy) Find derivatives f_i such that $f = f_n \circ \cdots \circ f_1$ is of Baire class not lower than n.

Differentiable monster (# 1)

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Arnaud Denjoy and Dimitrie Pompeiu

Arnaud Denjoy (1884–1974)

Dimitrie Pompeiu (1873–1954)

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i : i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

(ii) $h = g^{-1} : \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} : h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity follows from $|g(x)| \le \sum_{i=1}^{\infty} r^i (|x| + i + 1)$.

Differentiability requires $g'(x) = \sum_{i=1}^{\infty} r^i \frac{1}{3} \frac{1}{(x-q_i)^{2/3}}$. Easy when series $= \infty$. Other case follows from $0 < \frac{\psi_i(y)-\psi_i(x)}{y-x} \le 6\psi_i'(x)$.

(ii) and (iii) easily follow from (i).

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h \colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x - t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D} \left((-d + Z) \cap (d - Z) \right)$ works.

Clearly f is differentiable with $f'(x) = h'(x - t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t + d) = h'(d) - h'(t + d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d - t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

Outline

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F\colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

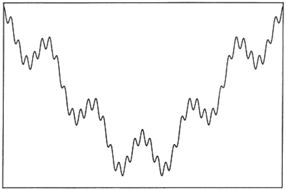
Bernard Bolzano (1781-1848)

Karl Weierstrass (1815–1897)

Weierstrass' Monster: $W(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x)$

Teiji Takagi (1875–1960)

Bartel van der Waerden (1903–1996)



$F(x) = \sum_{n=0}^{\infty} 4^n min\{|x - \frac{k}{8^n}| \colon k \in \mathbb{Z}\}$

Weierstrass' Monster of

Takagi from 1903, and

van der Waerden, from 1930

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

New proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Theorem (With new (2017/18) simple proof, by KC)

For every continuous increasing $f\colon [a,b] \to \mathbb{R}$ there is perfect P such that $f \restriction P$ is Lipschitz.

Proof based on the following results, due to Riesz:

Lemma (Rising sun lemma 1932, proof is an easy exercise)

If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \le g(d)$ for every component (c,d) of $U = \{x \in [a,b)\colon g(x) < g(y) \text{ for some } y \in (x,b]\}$.

Fact (Proved by induction)

Let $a < b$ and $\mathcal{J}$ be a family of open intervals with $\bigcup \mathcal{J} \subset (a,b)$.

(i) *If $[\alpha,\beta] \subset \bigcup \mathcal{J}$, then $\sum_{I\in\mathcal{J}} \ell(I) > \beta - \alpha$.*

(ii) *If $I \in \mathcal{J}$ are pairwise disjoint, then $\sum_{I\in\mathcal{J}} \ell(I) \le b - a$.*

Riesz' Rising sun lemma

If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \leq g(d)$ for every component (c,d) of $U = \{x \in [a,b)\colon g(x) < g(y) \text{ for some } y \in (x,b]\}$.

Frigyes Riesz (1880-1956)

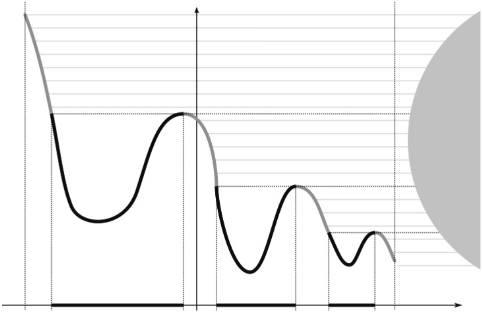


Illustration of the Rising Sun Lemma

The points in the set $U \cap (a,b)$ are those lying in the shadow.

Proof of Lipschitz restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Have: If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \le g(d)$ for every comp. (c,d) of $\{x \in [a,b)\colon g(x) < g(y) \text{ for some } y \in (x,b]\}$.

Sketch of proof. Fix $L > \frac{f(b)-f(a)}{b-a}$, put $g(t) = f(t) - Lt$, and

$$U = \{x \in [a,b)\colon g(y) > g(x) \text{ for some } y \in (x,b]\}.$$

f is Lipschitz on $P = [\bar{a}, b] \setminus U$ with constant L, where

$\bar{a} = \sup\{x\colon [a,x) \subset U)\}$. Fix $X = \{x_n\colon n \in \mathbb{N}\}$. Need $P \setminus X \neq \emptyset$.

If $\mathcal{J}$ = open components of U, then $\ell(f[J]) \ge L\ell(J)$ for $J \in \mathcal{J}$.

By Fact (ii), $\sum_{J\in\mathcal{J}} \ell(f[J]) \le f(b) - f(\bar{a})$. So,

$\sum_{J\in\mathcal{J}} \ell(J) \le \frac{1}{L} \sum_{J\in\mathcal{J}} \ell(f[J]) \le \frac{f(b)-f(\bar{a})}{L} < b - \bar{a}$, and by Fact (i),

$P \neq \emptyset$. To get $P \setminus X \neq \emptyset$ increase slightly $\mathcal{J}$.

End of proof of differentiable restriction theorem

Goal: If $f \colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Have: If $f \colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Proof of differentiable restriction theorem.

f is Lipschitz on some perfect P: proved above for somewhere monotone f; otherwise f is constant on some perfect set.

For function $f \restriction P$ use Morayne theorem to find perfect $Q \subset P$ such that the quotient map for $f \restriction Q$ is uniformly continuous. Then Q is as needed. □

Outline

1. Continuity from differentiability: classical results
2. Continuity from differentiability: newer results
3. Differentiability from continuity: differentiable restrictions
4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$
5. Differentiable extensions: Jarník and Whitney theorems
6. Bonus: Russian connection

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC,* provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

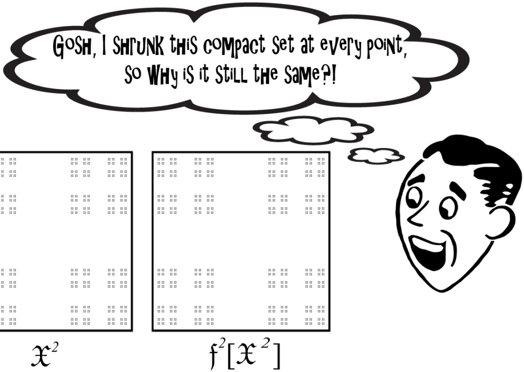


Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma\colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0,0,\ldots,0,1,s_{k+1},s_{k+2},\ldots\rangle & \text{if } s_k = 0 \ \& \ s_i = 1 \text{ for } i < k. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

where $N(s \restriction 0) = 1$ and, for $n > 0$,

$$\begin{aligned} N(s \restriction n) &= \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n \\ &= (1(1 - s_{n-1})s_{n-2}\ldots s_0)_2. \end{aligned}$$

E.g. $N(101101) = (1001101)_2$

Proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \le |h(s) - h(t)| \le 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \le \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Dynamical system $\mathfrak{f}$

Every orbit $\{x, \mathfrak{f}(x), \mathfrak{f}^2(x), \ldots\}$ of $\mathfrak{f}$ is dense in $\mathfrak{X}$.

So, $\mathfrak{f}$ is a minimal dynamical system. Must it be?

Theorem (KC & JJ 2016: YES, essentially)

If $f\colon X \to X$ is onto, PC, and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamical system,

where *f is pointwise contractive, PC*, if for every $x \in X$ there is open $U \ni x$ and $L \in [0,1)$ s.t. $|f(x) - f(y)| \le L|x - y|$ for all $y \in U$.

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Notation

For $J=(a,a+h)$ let $I_J=[a+h/3,a+2h/3]$, middle third of J.

For closed $Q\subset\mathbb{R}$ and $f\colon Q\to\mathbb{R}$ let

$\hat{Q}=Q\cup\bigcup\{I_J\colon J \text{ is a bounded connected component of } \mathbb{R}\setminus Q\}$,

$\bar{f}\colon\mathbb{R}\to\mathbb{R}$ — “the” linear interpolation of f, $\hat{f}=\bar{f}\restriction\hat{Q}$.

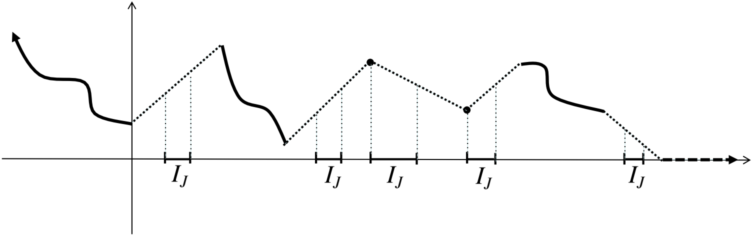


Figure: The linear interpolation $\bar{f}$ of f, represented by thick curves.

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

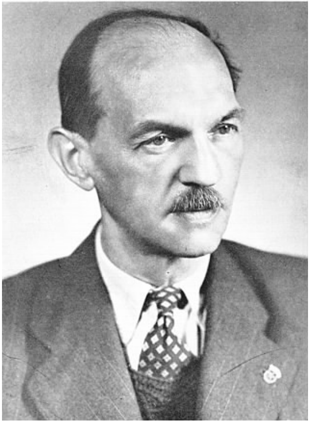
Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.

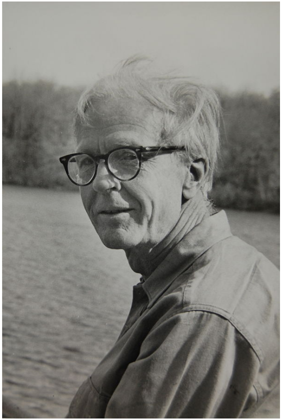
Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Vojtěch Jarník and Hassler Whitney



Vojtěch Jarník (1897–1970)



Hassler Whitney (1907-1989)

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \restriction \hat{Q}$ is continuously differentiable.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

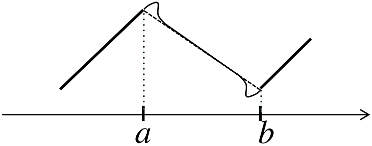


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f\colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f\colon P \to \mathbb{R}$, there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1)
such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

That is all!

Thank you for your attention!