Differentiability versus continuity: Restriction and extension theorems and monstrous examples

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Based on BAMS survey written with Juan B. Seoane–Sepúlveda

Colloquium at Universidad Complutense de Madrid, Spain, November 8, 2018.

Preamble: new (2017) results that are behind this talk

Example (New simple construction of a classic example)

There exists a differentiable nowhere monotone map $f \colon \mathbb{R} \to \mathbb{R}$.

Example (Greatly simplified construction of 2016 example)

There exists a differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect $\mathfrak{X} \subset \mathbb{R}$ with $\mathfrak{f}' \equiv 0$.

Theorem (C^1 interpolation thm, no Lebesgue measure needed)

For every continuous $f \colon \mathbb{R} \to \mathbb{R}$:

- *there is perfect $P \subset \mathbb{R}$ s.t. $f \restriction P$ is Lipschitz;*
- *there is C^1 map $g \colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.*

Theorem (Simple proof of Whitney and Jarník extension thms)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f \colon Q \to \mathbb{R}$ has differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff a simple (new) condition for f holds.

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Outline

1. Continuity from differentiability: classical results
2. Continuity from differentiability: newer results
3. Differentiability from continuity: differentiable restrictions
4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$
5. Differentiable extensions: Jarník and Whitney theorems

Outline

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

True question: *To what extend $f = F'$ must be continuous?*

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About $F(x) = x^2 \sin\left(x^{-1}\right)$

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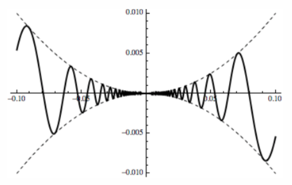
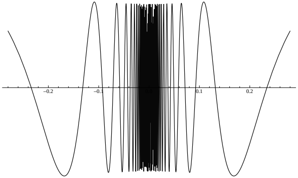


Graph of F



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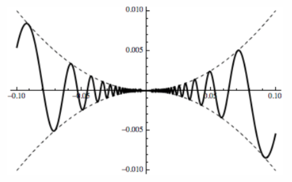

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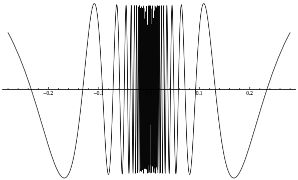

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Jean-Gaston Darboux
(1842-1917)

Theorem (Darboux 1875)

Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called Darboux functions.

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The derivative of any differentiable $F\colon \mathbb{R} \to \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a dense G_δ-set.

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Proof of previous theorem and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R}\to\mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^{\infty} V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0} \{x\in\mathbb{R}\colon |g(s)-f(g)| < 1/n \text{ for all } s,t\in(x-\delta,x+\delta)\}.$$

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Let $G\subset\mathbb{R}$.
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Fixed point property

Theorem (Relatively new)

If $f = f_n \circ \cdots \circ f_1$, where each $f_i \colon [0,1] \to [0,1]$ is a derivative, then f has a fixed point.

For $n = 1$: easy exercise, as $h(x) = f(x) - x$ is Darboux.

For $n = 2$: proved independently in 2001
by Csörnyei, O'Neil & Preiss and by Elekes, Keleti & Prokaj.

For arbitrary n: Szuca 2003.

Open Problem

Must f as in the theorem have connected graph?

Yes for $n = 1$. Positive answer would imply the theorem.

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Baire classification of composition of the derivatives.

Let $f = f_n \circ \cdots \circ f_1$, where each f_i is a derivative.

Then f is Darboux.

Any Darboux Baire class one map has connected graph.

A natural question: must f be of Baire class 1? NO

Theorem (*Andy* Bruckner and K. Ciesielski 2018)

There exist derivatives $\varphi, \gamma\colon [-1,1] \to [-1,1]$ such that their composition $\psi := \varphi \circ \gamma$ is not of Baire class one.

We use $\gamma(x) := \cos(x^{-1})$ and φ Pompeiu's map, see below.

Problem (could be easy) Find derivatives f_i such that $f = f_n \circ \cdots \circ f_1$ is of Baire class not lower than n.

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There exist derivatives $\varphi, \gamma \colon [-1,1] \to [-1,1]$ such that their composition $\psi := \varphi \circ \gamma$ is not of Baire class one.

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Differentiable monster (# 1)

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
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Arnaud Denjoy and Dimitrie Pompeiu

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A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$, *with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

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(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity follows from $|g(x)| \le \sum_{i=1}^{\infty} r^i (|x| + i + 1)$.

Differentiability requires $g'(x) = \sum_{i=1}^{\infty} r^i \frac{1}{3} \frac{1}{(x-q_i)^{2/3}}$. Easy when series $= \infty$. Other case follows from $0 < \frac{\psi_i(y) - \psi_i(x)}{y - x} \le 6\psi_i'(x)$.

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New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D}((-d + Z) \cap (d - Z))$ works.

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Outline

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F\colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

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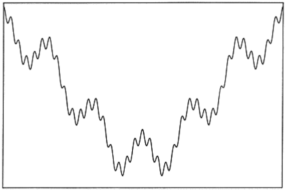
Weierstrass' Monster: $W(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x)$

Teiji Takagi (1875–1960)

Bartel van der Waerden (1903–1996)



$F(x) = \sum_{n=0}^{\infty} 4^n min\{|x - \frac{k}{8^n}|: k \in \mathbb{Z}\}$

Weierstrass' Monster of

Takagi from 1903, and

van der Waerden, from 1930

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

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Some differentiability after all!

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New proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Theorem (With new (2017/18) simple proof, by KC)

For every continuous increasing $f\colon [a,b] \to \mathbb{R}$ there is perfect P such that $f \restriction P$ is Lipschitz.

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Let $a < b$ and $\mathcal{J}$ be a family of open intervals with $\bigcup \mathcal{J} \subset (a,b)$.

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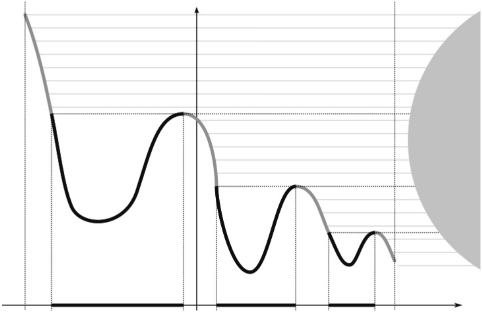

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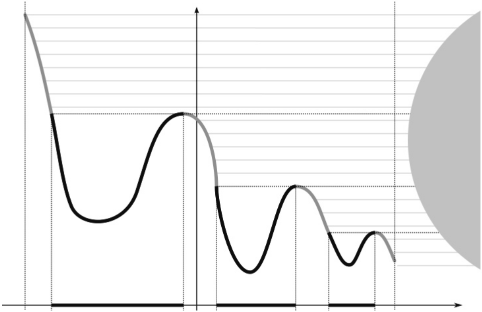

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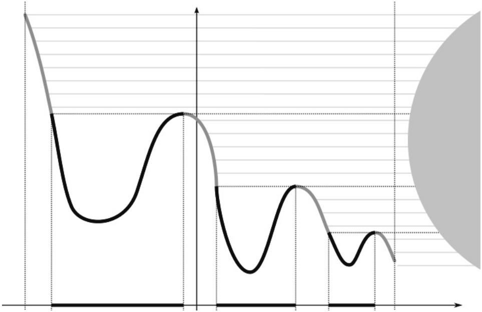

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For function $f \restriction P$ use Morayne theorem to find perfect $Q \subset P$ such that the quotient map for $f \restriction Q$ is uniformly continuous. Then Q is as needed. □

End of proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Have: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

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Outline

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
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Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

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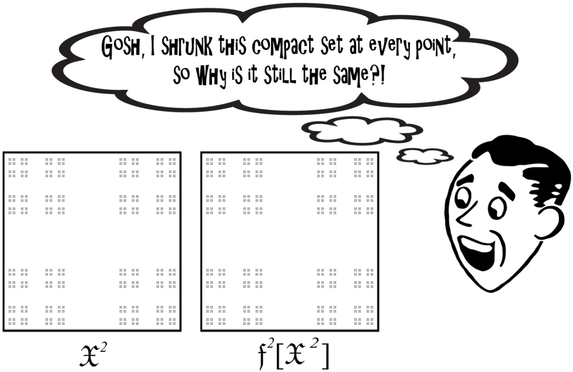

Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

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$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h \colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma \colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0, 0, 0, \ldots \rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0, 0, \ldots, 0, 1, s_{k+1}, s_{k+2}, \ldots \rangle & \text{if } s_k = 0 \ \& \ s_i = 1 \text{ for } i < k. \end{cases}$$

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To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Dynamical system $\mathfrak{f}$

Every orbit $\{x, \mathfrak{f}(x), \mathfrak{f}^2(x), \ldots\}$ of $\mathfrak{f}$ is dense in $\mathfrak{X}$.

So, $\mathfrak{f}$ is a minimal dynamical system. Must it be?

Theorem (KC & JJ 2016: YES, essentially)

If $f\colon X \to X$ is onto, PC, and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamical system,

where *f is pointwise contractive, PC*, if for every $x \in X$ there is open $U \ni x$ and $L \in [0, 1)$ s.t. $|f(x) - f(y)| \le L|x - y|$ for all $y \in U$.

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Outline

1. Continuity from differentiability: classical results
2. Continuity from differentiability: newer results
3. Differentiability from continuity: differentiable restrictions
4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$
5. Differentiable extensions: Jarník and Whitney theorems

Notation

For $J = (a, a+h)$ let $I_J = [a + h/3, a + 2h/3]$, middle third of J.

For closed $Q \subset \mathbb{R}$ and $f\colon Q \to \mathbb{R}$ let

$\hat{Q} = Q \cup \bigcup \{I_J\colon J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}$,

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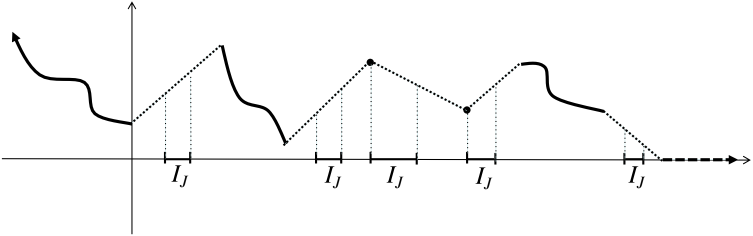

Figure: The linear interpolation $\bar{f}$ of f, represented by thick curves.

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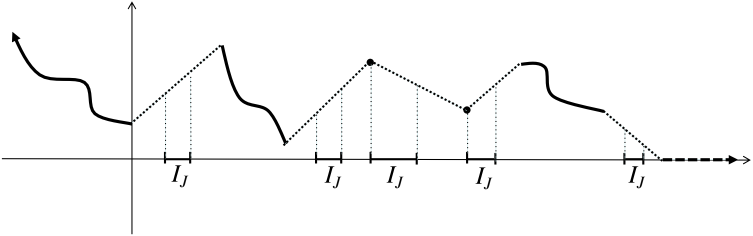

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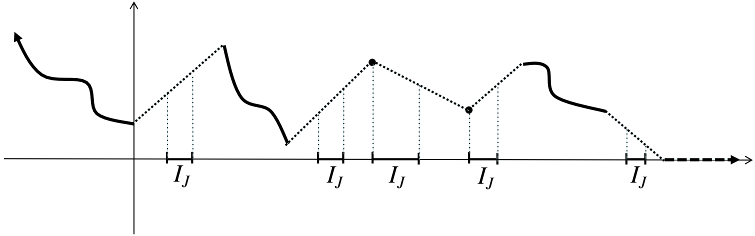

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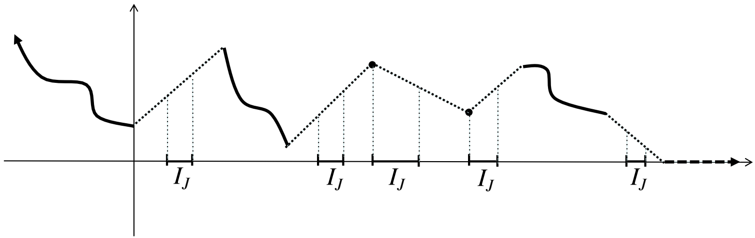

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Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

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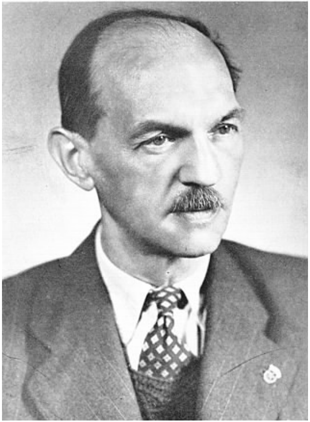
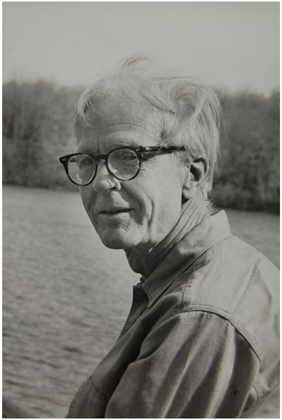
Vojtěch Jarník and Hassler Whitney



Vojtěch Jarník (1897–1970)



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Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

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The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

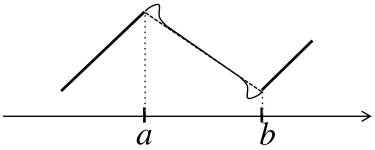

Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

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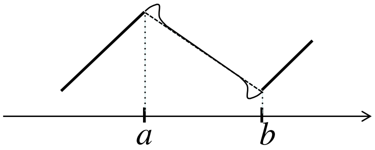

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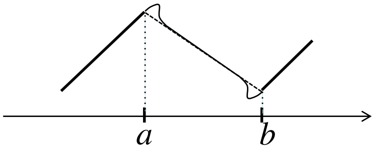

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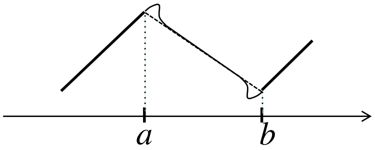

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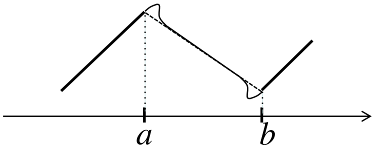

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By Jarník's theorem, our $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

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If $f\colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

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Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f\colon P \to \mathbb{R}$, there exists a differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

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Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1)
such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

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such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

Differentiable monster (#3)

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