# Path-value functions for which Dijkstra's algorithm returns optimal mapping

Krzysztof Chris Ciesielski

Department of Mathematics, West Virginia University
and
MIPG, Department of Radiology, University of Pennsylvania

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# Dijkstra Algorithm, DA: Why should you care?

DA discovered: V. Jarník 1930, R. Prim 1957, E. Dijkstra 1959 to find minimum spanning tree for a weighted undirected graph.

- It is one of the fastest algorithms used in image precessing, including image segmentation:

  (essentially) linear time with respect to image size

- It is the power engine behind
  - Fuzzy Connectedness, FC, segmentation software

- Can be used to find Watershed transform

- Usable in boundary tracking, morphological reconstructions, fast binary morphology, shape description, clustering, and classification

# Q: In what other situations DA can be used?

- Q was investigated in the paper

  [FSL] Falcão, Stolfi, and Lotufo, *IFT*, TPAMI, 2004

- They found "sufficient" conditions for **DA** to be usable

- I started search for *necessary and sufficient* conditions

- Indeed, I found such conditions

- In the process, I found also that

  "sufficient" conditions in [FSL] are not sufficient!

  (Practical conclusions from [FSL] seem to be intact.)

# What's ahead: Talk's outline

1. The algorithm
2. Characterization Theorem for **DA**
3. **DA***: a slight modification of **DA**
4. What is in [FSL] paper
5. Final Remarks
6. Summary

# Outline

1. The algorithm
2. Characterization Theorem for **DA**
3. **DA***: a slight modification of **DA**
4. What is in [FSL] paper
5. Final Remarks
6. Summary

# Definitions and notation needed for DA

- $G = \langle V, E \rangle$ – finite directed graph

  (Applications and our examples use simple grids.)

- *Path (in G)*: $p = \langle v_0, \ldots, v_\ell \rangle$, $\langle v_j, v_{j+1} \rangle \in E$ for $j < \ell$;

  from $S \subset V$ to $v \in V$ when $v_0 \in S$ and $v_\ell = v$;

  $p\hat{\ }w = \langle v_0, \ldots, v_\ell, w \rangle$; $\Pi_G$ – all paths in $G$.

- Path cost function: a map $\psi$ from $\Pi_G$ to $\langle [-\infty, \infty], \preceq \rangle$,

  $\preceq$ is either $\leq$ or $\geq$.

- DA for $\psi$ tries to find, for every $v \in V$, the $\psi$-minimizer:

$$\psi(v) = \preceq\text{-min}\{\psi(p) \colon p \text{ is a path to } v\}$$

# Examples of path cost functions $\psi$

$G = \langle V, E \rangle$ and non-empty $S \subset V$ are fixed

- Fuzzy connectedness: given *affinity* map $\psi\colon E \to [0,1]$,

  seeks for maximizers (i.e., $\preceq$-minimizers with $\preceq$ being $\geq$):

  $\psi_{\min}(\langle v_0, \ldots, v_\ell \rangle) = \min_{1 \leq j \leq \ell} \psi(v_{j-1}, v_j)$ for $\ell > 0$

  $\psi_{\min}(\langle v_0 \rangle) = 1$ if $v_0 \in S$, $\psi_{\min}(\langle v_0 \rangle) = 0$ if $v_0 \notin S$

- Shortest path (classic DA): given *distance* $\omega_E\colon E \to [0, \infty)$,

  $\psi_{\text{sum}}(\langle v_0, \ldots, v_\ell \rangle) = \sum_{1 \leq j \leq \ell} \omega_E(v_{j-1}, v_j)$ for $\ell > 0$

  $\psi_{\text{sum}}(\langle v_0 \rangle) = 0$ if $v_0 \in S$, $\psi_{\text{sum}}(\langle v_0 \rangle) = \infty$ if $v_0 \notin S$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\leq$)

# More examples of path cost functions $\psi$

- Watershed transform: given *altitude* map $\omega_V \colon V \to [0,\infty)$,

  $\psi_{\rm peak}(\langle v_0,\ldots,v_\ell\rangle) = \max_{1\le j\le\ell}\{h(v_0),\omega_V(v_j)\}$ for $\ell>0$

  $\psi_{\rm peak}(\langle v_0\rangle) = h(v_0)$ for some $h$, $h(v_0)\ge\omega_V(v_0)$ for $v_0\in V$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\le$)

- Barrier Distance transform: given map $\omega_V \colon V \to [0,\infty)$,

  $\psi_{\rm dif}(\langle v_0,\ldots,v_\ell\rangle) = \max_{0\le j\le\ell}\omega_V(v_j) - \min_{0\le j\le\ell}\omega_V(v_j)$ for $\ell>0$

  $\psi_{\rm dif}(\langle v_0\rangle) = 0$ if $v_0\in S$, $\psi_{\rm dif}(\langle v_0\rangle)=\infty$ if $v_0\notin S$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\le$)

# Yet another example of a path cost function $\psi$

- The last value: given a map $\omega_V\colon V \to [0,\infty)$,

  $\psi_{\text{last}}(\langle v_0,\ldots,v_\ell\rangle) = \omega_V(v_\ell)$ for $\ell > 0$

  $\psi_{\text{last}}(\langle v_0\rangle) = \omega_V(v_0)$ if $v_0 \in S$, $\psi_{\text{last}}(\langle v_0\rangle) = \infty$ if $v_0 \notin S$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\le$)

Its applications are concerned with a particular case of the riverbed boundary tracking and can be used to support connectivity constraints in region-based image segmentation.

# Dijkstra Algorithm, DA, aiming to find $\psi$-optimal map

**Data**: $G = \langle V, E\rangle$ and $\psi$ from $\Pi_G$ to $\langle[-\infty,\infty],\preceq\rangle$
**Result**: an array $\sigma[\,]$, aiming for being $\psi$-optimal map
**Additional:** an array $\pi[\,]$ of paths, such that, at any time, for any $v \in V$, $\pi[v]$ is a path to $v$ with $\sigma[v] = \psi(\pi[v])$

**foreach** $\mathsf{v} \in V$ **do** $\pi[\mathsf{v}] \leftarrow \langle \mathsf{v}\rangle$; $\sigma[\mathsf{v}] \leftarrow \psi(\pi[\mathsf{v}])$ /* init. */
$\mathsf{H} \leftarrow V$
**while** $\mathsf{H} \neq \emptyset$ **do** /* the main loop */
  **remove** an element w of arg $\preceq$-$\min_{u\in\mathsf{H}} \sigma[u]$ **from** H
  **foreach** x *such that* $\langle \mathsf{w}, \mathsf{x}\rangle \in E$ **do**
    $\sigma' \leftarrow \psi(\pi[\mathsf{w}]\hat{\ }\mathsf{x})$
    **if** $\sigma[\mathsf{x}] \succ \sigma'$ **then** $\sigma[\mathsf{x}] \leftarrow \sigma'$; $\pi[\mathsf{x}] \leftarrow \pi[\mathsf{w}]\hat{\ }\mathsf{x}$

# Outline

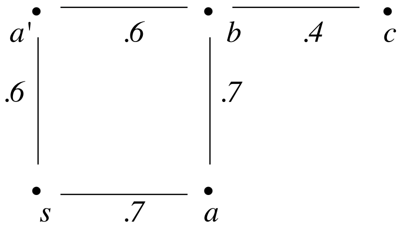
# Special paths

For fixed $\psi\colon \Pi_G \to \mathbb{R}$, a path $p = \langle v_0, \ldots, v_\ell \rangle \in \Pi_G$ to $v$:

- is *$\psi$-optimal* if it is $\preceq$-minimal, that is, provided $\psi(p) \preceq \psi(q)$ for any other path $q \in \Pi_G$ to $v$;
- is *hereditarily $\psi$-optimal* provided every initial segment $\langle v_0, \ldots, v_k \rangle$, $k \leq \ell$, of $p$ is $\psi$-optimal;
- is *monotone* provided $\psi(\langle v_0, \ldots, v_i \rangle) \preceq \psi(\langle v_0, \ldots, v_j \rangle)$ whenever $0 \leq i \leq j \leq \ell$;
- is *hereditarily $\psi$-optimal monotone, HOM,* provided it is both hereditarily $\psi$-optimal and monotone;
- *has the replacement property* provided $\psi(\langle v_0, \ldots, v_i \rangle) = \psi(q\hat{\ } v_i)$ for every $i \in \{1, \ldots, \ell\}$ and every HOM path $q \in \Pi_G$ to $v_{i-1}$.

# Examples: for FC cost $\psi_{\min}$ with $S = \{s\}$

$\psi_{\min}(\langle v_0, \ldots, v_\ell \rangle) = \min_{1 \le j \le \ell} \psi(v_{j-1}, v_j)$ for $\ell > 0$




- $\langle s, a, b \rangle$ is hereditarily $\psi_{\min}$-optimal
- $\langle s, a', b \rangle$ is not $\psi_{\min}$-optimal
- $\langle s, a, b, c \rangle$ is hereditarily $\psi_{\min}$-optimal
- $\langle s, a', b, c \rangle$ is $\psi_{\min}$-optimal but not hereditarily

# Facts related to special paths

For costs $\psi_{\min}$, $\psi_{sum}$, and $\psi_{peak}$ there is a map $f$ s.t.

(I) $\psi(p\hat{\ }v) = f(\psi(p), a, v)$ for any path $p$ to $a$ and edge $\langle a, v\rangle$.

Any $\psi$-optimal path has replacement property if $\psi$ satisfies (I).

$\psi_{\min}$, $\psi_{sum}$, and $\psi_{peak}$ have strong replacement property:

(R*) $\psi(\langle v_0, \ldots, v_\ell\rangle) \preceq \psi(q\hat{\ }v_\ell)$ all paths $\langle v_0, \ldots, v_\ell\rangle$ and $q$ to $v_{\ell-1}$ with $\psi(\langle v_0, \ldots, v_{\ell-1}\rangle) \preceq \psi(q)$.

For $\psi_{\min}$, $\psi_{sum}$, $\psi_{peak}$, and $\psi_{dif}$: (M) any path is monotone

(M) and (R*) imply that every $v$ admits HOM path

So, for $\psi_{\min}$, $\psi_{sum}$, and $\psi_{peak}$, every $v$ admits HOM path

# The theorem for **DA**

## Theorem

*Let* $\psi\colon \Pi_G \to [-\infty, \infty]$ *be a path cost function. If*

(E) *for every* $v \in V$ *there exists an HOM path to* $v$ *with the replacement property,*

*then* $\sigma[\,]$ *returned by* **DA is guaranteed** *to be* $\psi$*-optimal;*

$\pi[\,]$ *returned by* **DA***:* $\pi[v] = \langle v_0, \ldots, v_\ell \rangle$ *is HO path to* $v$*;*
$\pi[v_i] = \langle v_0, \ldots, v_i \rangle$ *for every* $i \in \{0, \ldots, \ell\}$.

*Conversely, if*

(M) $\psi(q) \preceq \psi(p)$ *for every path* $p$ *and its initial segment* $q$*,*

*then* $\sigma[\,]$ *returned by* **DA cannot be** $\psi$*-optimal,*

*unless for every* $v$ *there is a hereditarily* $\psi$*-optimal path to* $v$.

$\psi_{\mathrm{last}}$ satisfies (E) but is not monotone!

# Corollary: Characterization Theorem

**Corollary**

*If $\psi\colon \Pi_G \to \mathbb{R}$ satisfies (M) and*

(R) $\psi(p) = \psi(q\hat{\ }v)$ *for every HOM* $p = \langle v_0, \ldots, v_\ell \rangle$ *& q to* $v_{\ell-1}$,

*then* $\sigma[\,]$ *returned by* **DA** *is the* $\psi$*-optimal map if, and only if, for every* $v \in V$ *there exists a hereditarily* $\psi$*-optimal path to* $v$.

PROOF. (E) follows from (M) and (R).

The rest follows from Theorem. □

# Practical consequences

## Corollary

*$\psi_{\text{sum}}$, $\psi_{\text{min}}$, and $\psi_{peak}$ satisfy (E).*

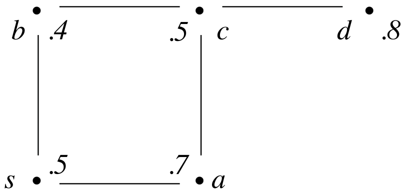
**DA** *works correctly for these functions.*

PROOF. (R*) implies:

- $\psi(\langle v_0, \ldots, v_\ell \rangle) = \psi(q \hat{} v_\ell)$ for all optimal paths $\langle v_0, \ldots, v_\ell \rangle$ and $q$ to $v_{\ell-1}$ with $\psi(\langle v_0, \ldots, v_{\ell-1} \rangle) \preceq \psi(q)$.

So, (E) holds. □

# Another consequence




**Corollary**

**DA** *need not return optimal map for Barrier Distance* $\psi_{dif}$.

PROOF. No hereditarily $\psi_{dif}$-optimal path from $S = \{s\}$ to $d$.

As $\psi_{dif}$ satisfies (M), the result follows from the Theorem. □

# Outline

# Problems with **DA** for general path costs

Consider graph $s \longleftrightarrow a$

Put $\psi(\langle s \rangle) = .2$, $\psi(p) = 0$ for any other path from $s$, and

$\psi(p) = 0$ for $p$ from $a$. For minimization, we get

There is no HOM path for any $v \in V$, since $\langle v \rangle$ is suboptimal.

$\psi$ satisfies (R), in void, since there are no HO paths.

**DA** returns a non-trivial circular path: **DA** terminates with

$\pi[a] = \langle s, a \rangle$ and the cycle $\pi[s] = \langle s, a, s \rangle$.

This contradicts Lemma 2 from [FSL]

**DA** returns optimal $\sigma[\,]$

# DA*, which cannot return cycles for any $\psi$

**Algorithm 1: DA***, aiming to find the $\psi$-optimal map

**Data**: $G = \langle V, E \rangle$ and $\psi$ from $\Pi_G$ to $\langle [-\infty, \infty], \preceq \rangle$

**Result**: an array $\sigma[\,]$, aiming for being $\psi$-optimal map

**Additional:** an array $\pi[\,]$ of paths, such that, at any time, for any $v \in V$, $\pi[v]$ is a path to $v$ with $\sigma[v] = \psi(\pi[v])$

**foreach** $\mathrm{v} \in V$ **do** $\pi[\mathrm{v}] \leftarrow \langle \mathrm{v} \rangle$; $\sigma[\mathrm{v}] \leftarrow \psi(\pi[\mathrm{v}])$ /* **init.** */

$\mathrm{H} \leftarrow V$

**while** $\mathrm{H} \neq \emptyset$ **do** /* **the main loop** */

    **remove** an element w of $\arg \preceq\text{-}\min_{u \in \mathrm{H}} \sigma[u]$ **from** H

    **foreach** x *such that* $\langle \mathrm{w}, \mathrm{x} \rangle \in E$ *and* $\mathrm{x} \in \mathrm{H}$ **do**

        $\sigma' \leftarrow \psi(\pi[\mathrm{w}]\hat{\ }\mathrm{x})$

        **if** $\sigma[\mathrm{x}] \succ \sigma'$ **then** $\sigma[\mathrm{x}] \leftarrow \sigma'$; $\pi[\mathrm{x}] \leftarrow \pi[\mathrm{w}]\hat{\ }\mathrm{x}$

# Main Theorem for DA*: no cycles

## Theorem

*Let $\psi\colon \Pi_G \to [-\infty, \infty]$ be a path cost function.*

- *If $\pi[\,]$ is returned by* **DA***, *then, for every $v \in V$, $\pi[v] = \langle v_0 \ldots, v_\ell \rangle$ is a path to $v$ with no repetitions such that $\pi[v_i] = \langle v_0 \ldots, v_i \rangle$ for every $i \in \{0, \ldots, \ell\}$.*
- *If (E) holds, then $\sigma[\,]$ returned by* **DA* is guaranteed** *to be the $\psi$-optimal map. Moreover, the returned map $\pi[\,]$ consists of hereditary $\psi$-optimal paths.*
- *Conversely, $\sigma[\,]$ returned by* **DA* cannot be** *$\psi$-optimal, unless for every $v \in V$ there exists a HOM path to $v$.*

# Outline

# Smooth functions from [FSL]

A path cost map $\psi$ is a smooth function provided

for any $v$ there exists $\psi$-optimal $p$ to $v$ s.t. either $p = \langle v \rangle$, or

$p = q\hat{\ }v$, where $q$ is a path to $w$, $\langle w, v \rangle$ is an edge, and

C1. $\psi(q) \preceq \psi(p)$,

C2. $q$ is $\psi$-optimal,

C3. for any $\psi$-optimal path $r$ to $w$, $\psi(r\hat{\ }v) = \psi(p)$.

It is claimed (incorrectly) in [FSL] that for smooth $\psi$ **DA** must return $\psi$-optimal map $\sigma[\,]$.

There is no proof of this in [FSL]. Instead, authors claim (without proof) that C1-C3 imply stronger properties C1*-C3* and proceed to prove that they imply **DA**'s good behavior.

# Properties C1*-C3*: hereditary versions of C1-C3

For any $v$ there exists a $\psi$-optimal path $p = \langle v_0, \ldots, v_\ell \rangle$ to $v$ s.t. for any $k \in \{0, \ldots, \ell - 1\}$

C1*. $\psi(\langle v_0, \ldots, v_k \rangle) \preceq \psi(p)$,

C2*. $\langle v_0, \ldots, v_k \rangle$ is $\psi$-optimal,

C3*. for any $\psi$-optimal path $q$ to $v_k$, $\psi(q\hat{\ }\langle v_{k+1}, \ldots, v_\ell \rangle) = \psi(p)$.

C1*&C2* means that $p$ is an HOM path

C3* is close to our (R), demanding that

$$\psi(q\hat{\ }v_{k+1}) = \psi(\langle v_0, \ldots, v_{k+1} \rangle)$$

**Q**. Why did I bother, when [FSL] contains proof that C1*-C3* are sufficient?

**A.** The proof in [FSL], using C1*-C3*, is incorrect!

# C1-C3 does not imply C1*-C3*

## Example

Graph: $\{0,\dots,5\} \times \{0,\dots,5\}$ with 4-adjacency.

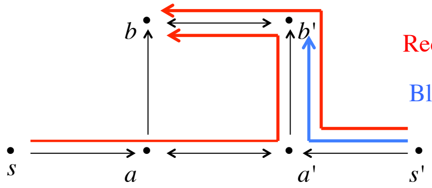
Seed: $s = \langle 0, 0 \rangle$. Problem: minimization, i.e., $\preceq$ is $\leq$.

If $s$ appears in $p = \langle v_0, \dots, v_\ell \rangle$ only as $v_0$:
$\psi(p) = \ell$ when $\ell \leq 3$; $\psi(p) = 0$ otherwise.

$\psi(p) = 100$ for all other paths $p$.

- $\psi(v) = 0$ for every $v$
- C1-C3 are satisfied (by any path of length $\geq 5$)
- C1*-C2* are not satisfied (only $s$ admits HOM path)
- for any $v$ adjacent to $s$, **DA** returns a suboptimal value $1$.

# C1*-C3* do not imply good behavior of DA or DA*




$S = \{s, s'\}$; maximization problem (i.e., $\preceq$ is $\geq$)
$\psi(p) = 1$ for any $p$ from $S$ of the form ($\psi(p) = 0$ otherwise):

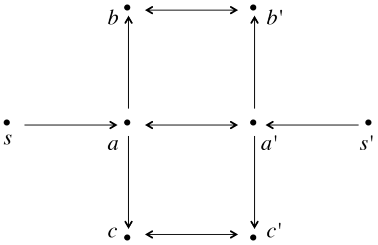
- to a $v \in \{s, s', a, a'\}$ or having repeated vertices;
- $\langle \ldots, a', b', b \rangle$, $\langle s, a, a', b' \rangle$, $\langle \ldots, a, b, b' \rangle$, or $\langle s', a', a, b \rangle$.

C1*-C3* satisfied: by $\langle s, a, a', b', b \rangle$ and $\langle s', a', a, b, b' \rangle$

May terminate with suboptimal $\sigma$: Starting with $\langle s, a \rangle$ and $\langle s', a' \rangle$

May terminate with optimal $\sigma$: Starting with $\langle s, a, a' \rangle$

# Stronger example: $\sigma$ cannot be optimal




$\psi(p) = 1$ for any $p$ from $\{s, s'\}$ of the form ($\psi(p) = 0$ otherwise):

- to a $v \in \{s, s', a, a'\}$ or having repeated vertices;
- $\langle \ldots, a', b', b \rangle$, $\langle s, a, a', b' \rangle$, $\langle \ldots, a, b, b' \rangle$, or $\langle s', a', a, b \rangle$.
- $\langle \ldots, a', c', c \rangle$, $\langle s', a', c' \rangle$, $\langle \ldots, a, c, c' \rangle$, or $\langle s, a, c \rangle$.

# Outline

# Final tune-ups

If $\psi$, like $\psi_{\min}$, $\psi_{sum}$, and $\psi_{peak}$, satisfies

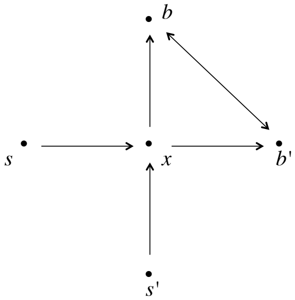
(I) $\psi(p\hat{\ }v) = f(\psi(p), a, b)$ for any path $p$ to $a$ and edge $\langle a, b\rangle$,

then, in **DA** and **DA***, there is no need to store paths in $\pi[\,]$.
The similar trick can be used for $\psi_{dif}$.

If $\psi$ satisfies (M), "x $\in$ H" in line 5 of **DA*** is redundant.

For such $\psi$ it makes sense to replace, both in **DA** and **DA***, the condition in line 5 with "x such that $\langle \mathrm{w}, \mathrm{x}\rangle \in E$ and x $\in$ H," to avoid unnecessary compution of $\psi(\pi[\mathrm{w}]\hat{\ }\mathrm{x})$.

# Is the replacement requirement necessary?




$S = \{s, s'\}$; maximization problem (i.e., $\preceq$ is $\geq$)
$\psi(p) = 1$ for any $p$ from $S$ of the form ($\psi(p) = 0$ otherwise):

- $\langle s, x, b, b' \rangle$, $\langle s', x, b', b \rangle$, and their initial segments.

$b$ and $b'$ admits no optimal path with the replacement property.

**DA** and **DA*** return optimal maps:

with $\pi[b] = \langle s', x, b', b \rangle$ or $\pi[b'] = \langle s, x, b, b' \rangle$.

# Outline

# Summary

- For some classes of path cost functions $\psi$, we found a necessary and sufficient conditions on $\psi$, for Dijkstra algorithm to return correct optimizer.
- We identified the errors in the [FSL] paper and shown how these errors can be patched.
- We showed how our characterization theorem can be used for some practically used path cost functions.
- The application of these characterization theorem to other path cost functions is currently investigated.

# Thank you for your attention!