Path-value functions for which Dijkstra's algorithm returns optimal mapping

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Dijkstra Algorithm, DA: Why should you care?

DA discovered: V. Jarník 1930, R. Prim 1957, E. Dijkstra 1959 to find minimum spanning tree for a weighted undirected graph.

- It is one of the fastest algorithms used in image precessing, including image segmentation:

  (essentially) linear time with respect to image size

- It is the power engine behind
  - Fuzzy Connectedness, FC, segmentation software

- Can be used to find Watershed transform

- Usable in boundary tracking, morphological reconstructions, fast binary morphology, shape description, clustering, and classification

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- Q was investigated in the paper

  [FSL] Falcão, Stolfi, and Lotufo, *IFT*, TPAMI, 2004

- They found "sufficient" conditions for **DA** to be usable

- I started search for *necessary and sufficient* conditions

- Indeed, I found such conditions

- In the process, I found also that

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What's ahead: Talk's outline

1. The algorithm
2. Characterization Theorem for **DA**
3. **DA***: a slight modification of **DA**
4. What is in [FSL] paper
5. Final Remarks
6. Summary

Outline

Definitions and notation needed for DA

- $G = \langle V, E \rangle$ – finite directed graph

  (Applications and our examples use simple grids.)

- *Path (in G)*: $p = \langle v_0, \ldots, v_\ell \rangle$, $\langle v_j, v_{j+1} \rangle \in E$ for $j < \ell$;

  from $S \subset V$ to $v \in V$ when $v_0 \in S$ and $v_\ell = v$;

  $p\hat{\ }w = \langle v_0, \ldots, v_\ell, w \rangle$; Π_G – all paths in G.

- Path cost function: a map ψ from Π_G to $\langle [-\infty, \infty], \preceq \rangle$,

  $\preceq$ is either $\leq$ or $\geq$.

- DA for ψ tries to find, for every $v \in V$, the ψ-minimizer:

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  $\psi_{\mathrm{sum}}(\langle v_0, \ldots, v_\ell \rangle) = \sum_{1 \leq j \leq \ell} \omega_E(v_{j-1}, v_j)$ for $\ell > 0$

  $\psi_{\mathrm{sum}}(\langle v_0 \rangle) = 0$ if $v_0 \in S$, $\psi_{\mathrm{sum}}(\langle v_0 \rangle) = \infty$ if $v_0 \notin S$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\leq$)

More examples of path cost functions ψ

- Watershed transform: given *altitude* map $\omega_V\colon V \to [0,\infty)$,

  $\psi_{\text{peak}}(\langle v_0,\ldots,v_\ell\rangle) = \max_{1\le j\le\ell}\{h(v_0),\omega_V(v_j)\}$ for $\ell > 0$

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- Barrier Distance transform: given map $\omega_V\colon V \to [0,\infty)$,

  $\psi_{\text{dif}}(\langle v_0,\ldots,v_\ell\rangle) = \max_{0\le j\le\ell}\omega_V(v_j) - \min_{0\le j\le\ell}\omega_V(v_j)$ for $\ell > 0$

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- The last value: given a map $\omega_V \colon V \to [0, \infty)$,

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Dijkstra Algorithm, DA, aiming to find ψ-optimal map

Data: $G = \langle V, E \rangle$ and ψ from Π_G to $\langle [-\infty, \infty], \preceq \rangle$
Result: an array $\sigma[\,]$, aiming for being ψ-optimal map
Additional: an array $\pi[\,]$ of paths, such that, at any time, for any $v \in V$, $\pi[v]$ is a path to v with $\sigma[v] = \psi(\pi[v])$

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foreach v ∈ V do π[v] ← ⟨v⟩; σ[v] ← ψ(π[v])      /* init.          */
H ← V
while H ≠ ∅ do                                    /* the main loop  */
    remove an element w of arg ⪯-min_{u∈H} σ[u] from H
    foreach x such that ⟨w, x⟩ ∈ E do
        σ' ← ψ(π[w]^x)
        if σ[x] ≻ σ' then σ[x] ← σ'; π[x] ← π[w]^x
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Outline

Special paths

For fixed $\psi \colon \Pi_G \to \mathbb{R}$, a path $p = \langle v_0, \ldots, v_\ell \rangle \in \Pi_G$ to v:

- is *ψ-optimal* if it is $\preceq$-minimal, that is, provided $\psi(p) \preceq \psi(q)$ for any other path $q \in \Pi_G$ to v;
- is *hereditarily ψ-optimal* provided every initial segment $\langle v_0, \ldots, v_k \rangle$, $k \leq \ell$, of p is ψ-optimal;
- is *monotone* provided $\psi(\langle v_0, \ldots, v_i \rangle) \preceq \psi(\langle v_0, \ldots, v_j \rangle)$ whenever $0 \leq i \leq j \leq \ell$;
- is *hereditarily ψ-optimal monotone, HOM,* provided it is both hereditarily ψ-optimal and monotone;
- *has the replacement property* provided $\psi(\langle v_0, \ldots, v_i \rangle) = \psi(q\hat{}v_i)$ for every $i \in \{1, \ldots, \ell\}$ and every HOM path $q \in \Pi_G$ to v_{i-1}.

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Examples: for FC cost $\psi_{\min}$ with $S = \{s\}$

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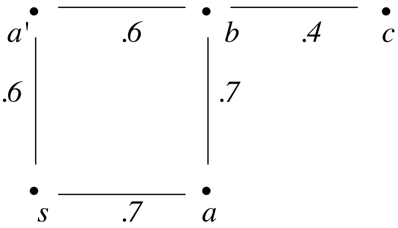

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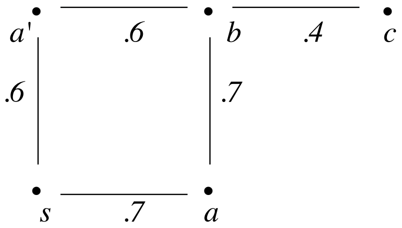

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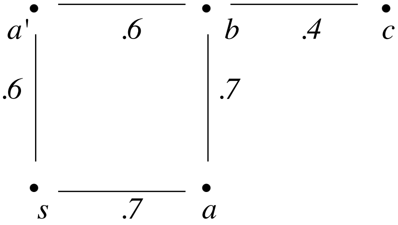

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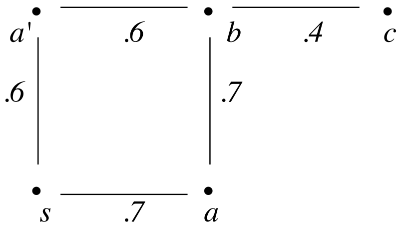

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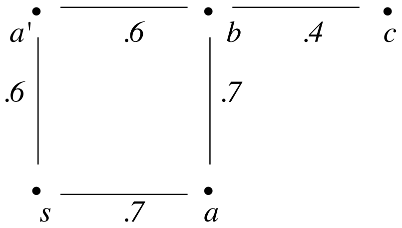
- $\langle s, a', b, c \rangle$ is $\psi_{\min}$-optimal but not hereditarily

Examples: for FC cost $\psi_{\min}$ with $S = \{s\}$

$\psi_{\min}(\langle v_0, \ldots, v_\ell \rangle) = \min_{1 \le j \le \ell} \psi(v_{j-1}, v_j)$ for $\ell > 0$



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Facts related to special paths

For costs $\psi_{\min}$, ψ_{sum}, and ψ_{peak} there is a map f s.t.

(I) $\psi(p\hat{\ }v) = f(\psi(p), a, v)$ for any path p to a and edge $\langle a, v\rangle$.

Any ψ-optimal path has replacement property if ψ satisfies (I).

$\psi_{\min}$, ψ_{sum}, and ψ_{peak} have strong replacement property:

(R*) $\psi(\langle v_0, \ldots, v_\ell\rangle) \preceq \psi(q\hat{\ }v_\ell)$ all paths $\langle v_0, \ldots, v_\ell\rangle$ and q to $v_{\ell-1}$ with $\psi(\langle v_0, \ldots, v_{\ell-1}\rangle) \preceq \psi(q)$.

For $\psi_{\min}$, ψ_{sum}, ψ_{peak}, and ψ_{dif}: (M) any path is monotone

(M) and (R*) imply that every v admits HOM path

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The theorem for **DA**

Theorem

Let $\psi\colon \Pi_G \to [-\infty,\infty]$ be a path cost function. If

(E) *for every $v \in V$ there exists an HOM path to v with the replacement property,*

then $\sigma[\,]$ returned by **DA is guaranteed** *to be ψ-optimal;*

$\pi[\,]$ returned by **DA***: $\pi[v] = \langle v_0, \ldots, v_\ell\rangle$ is HO path to v; $\pi[v_i] = \langle v_0, \ldots, v_i\rangle$ for every $i \in \{0, \ldots, \ell\}$.*

Conversely, if

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Corollary: Characterization Theorem

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If $\psi\colon \Pi_G \to \mathbb{R}$ satisfies (M) and

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then $\sigma[\,]$ returned by **DA** *is the ψ-optimal map if, and only if, for every $v \in V$ there exists a hereditarily ψ-optimal path to v.*

PROOF. (E) follows from (M) and (R).

The rest follows from Theorem. □

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Practical consequences

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ψ_{sum}, ψ_{min}, and ψ_{peak} satisfy (E).

DA *works correctly for these functions.*

PROOF. (R*) implies:

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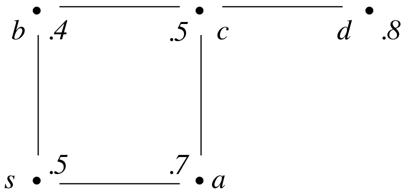

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DA *need not return optimal map for Barrier Distance* ψ_{dif}.

PROOF. No hereditarily ψ_{dif}-optimal path from $S = \{s\}$ to d.

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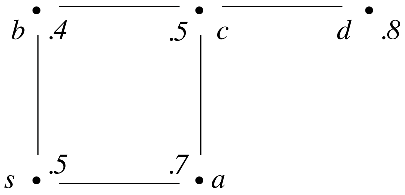

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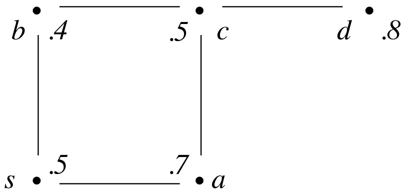

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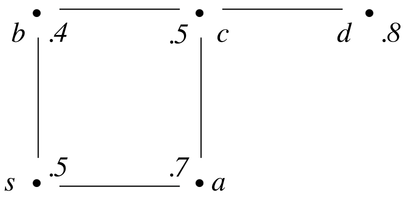

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Outline

Problems with **DA** for general path costs

Consider graph $s \longleftrightarrow a$

Put $\psi(\langle s\rangle) = .2$, $\psi(p) = 0$ for any other path from s, and

$\psi(p) = 0$ for p from a. For minimization, we get

There is no HOM path for any $v \in V$, since $\langle v\rangle$ is suboptimal.

ψ satisfies (R), in void, since there are no HO paths.

DA returns a non-trivial circular path: **DA** terminates with
$\pi[a] = \langle s, a\rangle$ and the cycle $\pi[s] = \langle s, a, s\rangle$.

This contradicts Lemma 2 from [FSL]

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DA*, which cannot return cycles for any ψ

Algorithm 1: DA*, aiming to find the ψ-optimal map

Data: $G = \langle V, E\rangle$ and ψ from Π_G to $\langle [-\infty,\infty], \preceq\rangle$

Result: an array $\sigma[\,]$, aiming for being ψ-optimal map

Additional: an array $\pi[\,]$ of paths, such that, at any time, for any $v \in V$, $\pi[v]$ is a path to v with $\sigma[v] = \psi(\pi[v])$

foreach v $\in V$ **do** $\pi[\mathrm{v}] \leftarrow \langle \mathrm{v}\rangle$; $\sigma[\mathrm{v}] \leftarrow \psi(\pi[\mathrm{v}])$ /* init. */

H $\leftarrow V$

while H $\neq \emptyset$ **do** /* the main loop */

  remove an element w of arg $\preceq$-$\min_{u\in \mathrm{H}} \sigma[u]$ **from** H

  foreach x *such that* $\langle \mathrm{w},\mathrm{x}\rangle \in E$ *and* x $\in$ H **do**

    $\sigma' \leftarrow \psi(\pi[\mathrm{w}]\hat{\ }\mathrm{x})$

    if $\sigma[\mathrm{x}] \succ \sigma'$ **then** $\sigma[\mathrm{x}] \leftarrow \sigma'$; $\pi[\mathrm{x}] \leftarrow \pi[\mathrm{w}]\hat{\ }\mathrm{x}$

Main Theorem for DA*: no cycles

Theorem

Let $\psi \colon \Pi_G \to [-\infty, \infty]$ be a path cost function.

- *If $\pi[\,]$ is returned by* **DA***, *then, for every $v \in V$, $\pi[v] = \langle v_0\ \ldots, v_\ell\rangle$ is a path to v with no repetitions such that $\pi[v_i] = \langle v_0\ \ldots, v_i\rangle$ for every $i \in \{0, \ldots, \ell\}$.*
- *If (E) holds, then $\sigma[\,]$ returned by* **DA*** **is guaranteed** *to be the ψ-optimal map. Moreover, the returned map $\pi[\,]$ consists of hereditary ψ-optimal paths.*
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Outline

Smooth functions from [FSL]

A path cost map ψ is a smooth function provided

for any v there exists ψ-optimal p to v s.t. either $p = \langle v \rangle$, or

$p = q\hat{\ }v$, where q is a path to w, $\langle w, v \rangle$ is an edge, and

C1. $\psi(q) \preceq \psi(p)$,

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It is claimed (incorrectly) in [FSL] that for smooth ψ **DA** must return ψ-optimal map $\sigma[\,]$.

There is no proof of this in [FSL]. Instead, authors claim (without proof) that C1-C3 imply stronger properties C1*-C3* and proceed to prove that they imply **DA**'s good behavior.

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C1*&C2* means that p is an HOM path

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C1-C3 does not imply C1*-C3*

Example

Graph: $\{0,\ldots,5\}\times\{0,\ldots,5\}$ with 4-adjacency.

Seed: $s=\langle 0,0\rangle$. Problem: minimization, i.e., $\preceq$ is $\leq$.

If s appears in $p=\langle v_0,\ldots,v_\ell\rangle$ only as v_0:
$\psi(p)=\ell$ when $\ell\leq 3$; $\psi(p)=0$ otherwise.

$\psi(p)=100$ for all other paths p.

- $\psi(v)=0$ for every v
- C1-C3 are satisfied (by any path of length ≥ 5)
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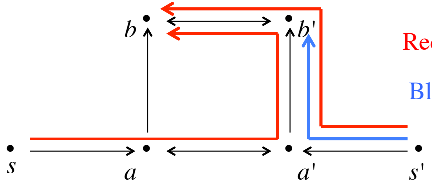
C1*-C3* do not imply good behavior of DA or DA*



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C1*-C3* satisfied: by $\langle s, a, a', b', b \rangle$ and $\langle s', a', a, b, b' \rangle$
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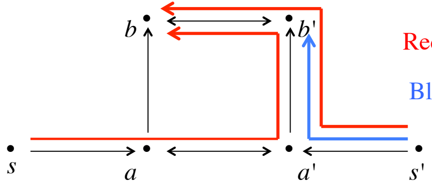

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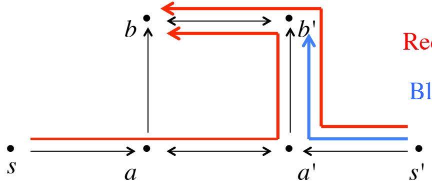

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May terminate with suboptimal σ: Starting with $\langle s, a \rangle$ and $\langle s', a' \rangle$

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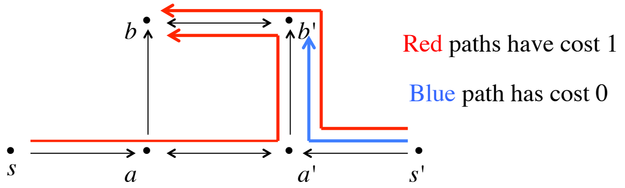
C1*-C3* do not imply good behavior of DA or DA*



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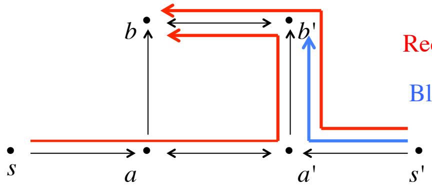

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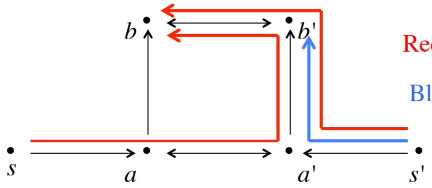

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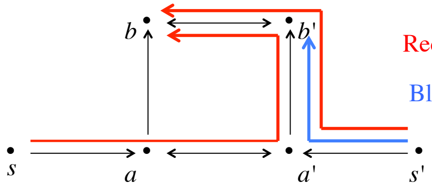

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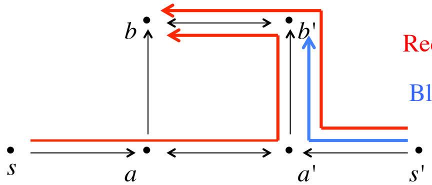

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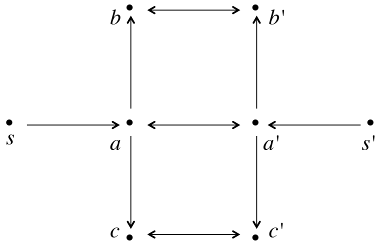

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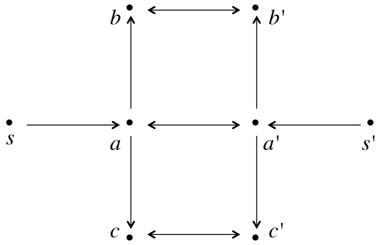

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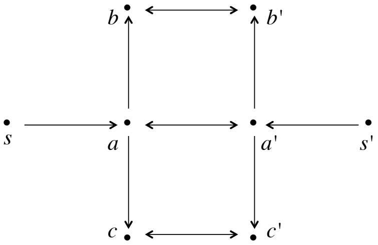

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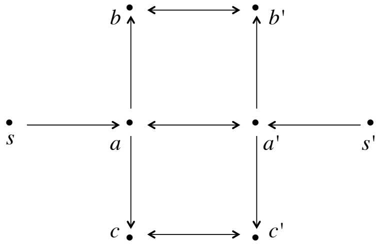

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Outline

Final tune-ups

If ψ, like $\psi_{\min}$, ψ_{sum}, and ψ_{peak}, satisfies

(I) $\psi(p\hat{\ }v) = f(\psi(p), a, b)$ for any path p to a and edge $\langle a, b\rangle$,

then, in **DA** and **DA***, there is no need to store paths in $\pi[\,]$. The similar trick can be used for ψ_{dif}.

If ψ satisfies (M), "x $\in$ H" in line 5 of **DA*** is redundant.

For such ψ it makes sense to replace, both in **DA** and **DA***, the condition in line 5 with "x such that $\langle$w, x$\rangle \in E$ and x $\in$ H," to avoid unnecessary compution of $\psi(\pi[\text{w}]\hat{\ }\text{x})$.

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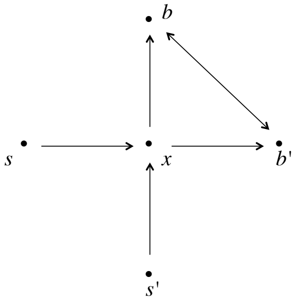

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b and b' admits no optimal path with the replacement property.

DA and **DA*** return optimal maps:

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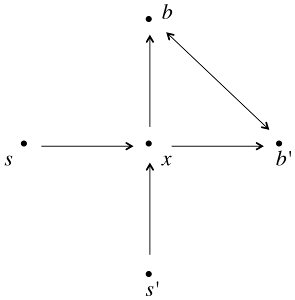

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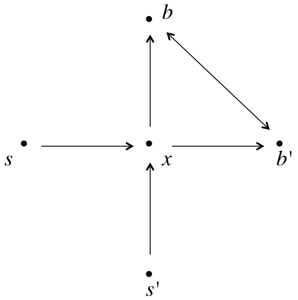

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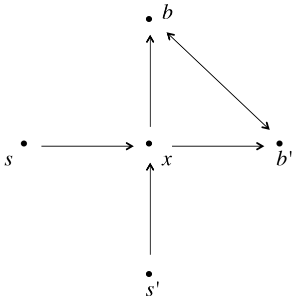

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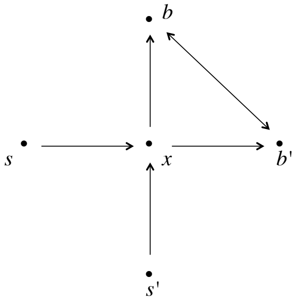

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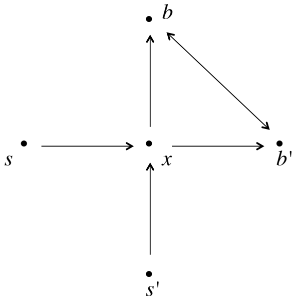

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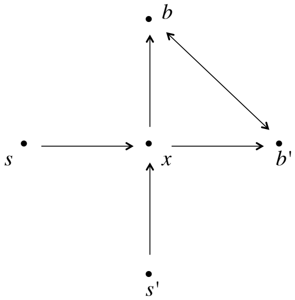

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