

Hierarchical segmentation in a directed graph setting which optimizes a graph cut energy

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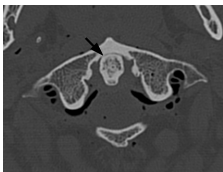
Outline

- 1 Image segmentation in graph cut setting
- 2 Dijkstra algorithm in general setting
- 3 Oriented IFT and graph cut optimization
- 4 HLOIFT: Hierarchical Layered OIFT algorithm
- 5 Experimental results for HLOIFT
- 6 Summary

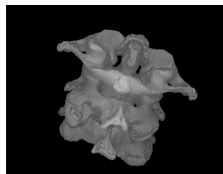
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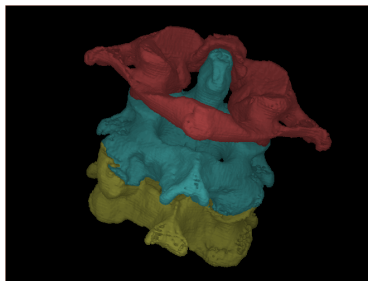
Image segmentation example 1: CT, cervical spine



A slice of an original 3D image



Surface rendition of segmented three vertebrae, together



Color surface rendition of the segmented three vertebra

Example 2: CT, thoracic-abdominal axial cross section

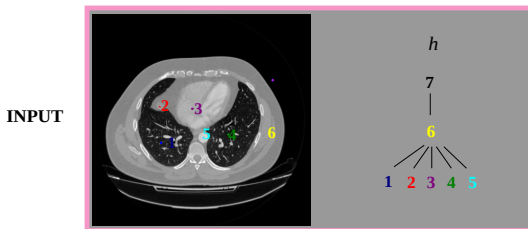


Figure: right lung (O_1), liver (O_2), heart (O_3), left lung (O_4), aorta (O_5) and the thoracic-abdominal region (O_6).

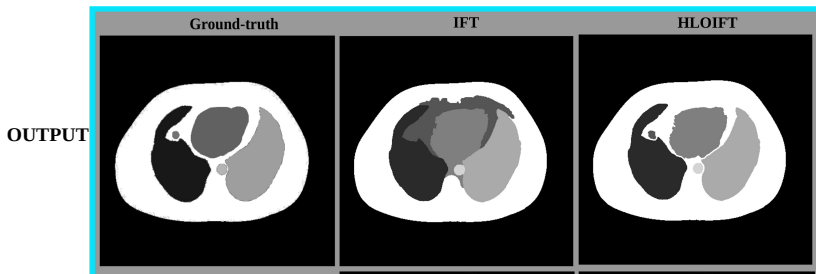
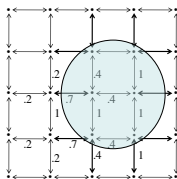
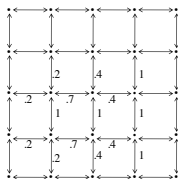


Image segmentation — formal setting

- An *image* is a map f from a set V (of spels) into \mathbb{R}^k
The value $f(c)$ represents **image intensity at c** , a k -dimensional vector each component of which indicates a measure of some aspect of the signal, like color.
- **Segmentation problem**: Given an image $f: V \rightarrow \mathbb{R}^k$, find a “**desired**” family $\{O_1, \dots, O_M\}$ of subsets of V .
- We will assume the objects are indicated by disjoint sets S_j of **seeds**, imposing that $S_j \subset O_j$.

Image, its graph, and graph cut



An image, with intensity map $f: V \rightarrow \mathbb{R}^k$

Its graph $G = \langle V, E \rangle$, with some edge weights

Object O and its graph cut edges $c(O)$ in bold

- **Vertices** $v \in V$ are image pixels. **Direct edges**: all $\langle c, d \rangle, \langle d, c \rangle \in E$, with $c, d \in V$ nearby (e.g. 4 adjacency).
- **Edge weights**: $w(\langle c, d \rangle) =$ some function of $f(c) - f(d)$.
- **Graph cut of O** : $c(O) = \{ \langle c, d \rangle \in E : c \in O \text{ \& } d \notin O \}$.

Only in one direction!

Graph cut measures: ℓ_p -norms, $1 \leq p \leq \infty$

Assuming $\langle c, d \rangle \in E \iff \langle d, c \rangle \in E$ and $w(\langle c, d \rangle) \geq 0$

ℓ_p -norm of $c(O)$ is defined as

$$\varepsilon_p(O) \stackrel{\text{def}}{=} \|w \upharpoonright c(O)\|_p = \begin{cases} \left(\sum_{e \in c(O)} w(e)^p \right)^{1/p} & \text{if } p < \infty \\ \max_{e \in c(O)} w(e) & \text{if } p = \infty. \end{cases}$$

Standard analysis fact: $\|w\|_p \rightarrow_{p \rightarrow \infty} \|w\|_\infty$ for any map w .

Known algorithms minimizing ℓ_p -norms of graph cut

$p = 1$: Minimization solved by classic **min-cut/max-flow algorithm**.

Graph Cut, GC, delineation algorithm minimizes ε_1 .

$p = \infty$: Minimization solved by (versions of) **Dijkstra algorithm**.

ε_∞ minimized objects are returned by the algorithms:

Power Watershed, PW [C. Couprie *et al*, 2011]

Relative Fuzzy Connectedness, RFC, **Iterative RFC, IRFC**,

Image Foresting Transform, IFT, [Ciesielski, Udupa,

Falcão, Miranda, 2012].

$p = 2$: **Random Walker, RW**, algorithm [Grady, 2006].

Fact: Inclusion-minimal ℓ_p -normed minimized delineations converge, as $p \rightarrow \infty$ to ℓ_∞ -normed minimized delineation.

This talk's Main Algorithm, **HLOIFT**, minimizes ℓ_∞ -norm of cut

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Paths and Optimal Path Forest OPF

- Fix directed graph $G = \langle V, E \rangle$ (with edge weight map w)
- *Path (in G):* $p = \langle v_0, \dots, v_\ell \rangle$ s.t. $\langle v_j, v_{j+1} \rangle \in E$ for $j < \ell$;
 p is from $S \subset V$ to $v \in V$ when $v_0 \in S$ and $v_\ell = v$;
 $p \hat{w} = \langle v_0, \dots, v_\ell, w \rangle$; Π_G – all paths in G .
- **Path cost** function: any map $\psi: \Pi_G \rightarrow \mathbb{R}$.
- A path p (from $S \subset V$) to v is **ψ -optimal** provided

$$\psi(p) = \max\{\psi(q) : q \text{ is a path (from } S) \text{ to } v\}.$$

- Jarník-Prim-Dijkstra algorithm **DA** for ψ and $S \subset V$ tries to find (S -rooted) forest, OPF, composed of ψ -optimal paths.
- **HLOIFT** is a **DA** for appropriate path cost map and graph.

Dijkstra Algorithm, **DA**, aiming to find ψ -optimal forest

Data: $G = \langle V, E \rangle$ and a path cost map $\psi: \Pi_G \rightarrow \mathbb{R}$

Result: an array $\pi[\cdot]$ of paths, aiming for being ψ -optimal

```

1 foreach  $v \in V$  do  $\pi[v] \leftarrow \langle v \rangle$ 
2  $Q \leftarrow V$ 
3 while  $Q \neq \emptyset$  do
4   remove an element  $w$  of  $\max_{u \in Q} \psi(\pi[u])$  from  $Q$ 
5   foreach  $x$  such that  $\langle w, x \rangle \in E$  do
6     if  $\psi(\pi[x]) < \psi(\pi[w] \wedge x)$  then  $\pi[x] \leftarrow \pi[w] \wedge x$ 

```

DA is very efficient: **quasi-linear** w.r.t. the size of the graph.

For what path cost ψ DA works properly?

Studied in JMIV paper [Ciesielski, Falcão, Miranda, Sept. 2018]

correcting errors of TPAMI paper [Falcão, Stolfi, Lotufo, 2004].

If w is an edge weight map for undirected graph $G = \langle V, E \rangle$, then DA works properly for:

- **FC/IFT**: $\psi_{\min}(\langle v_0, \dots, v_\ell \rangle) = \min_{1 \leq j \leq \ell} w(v_{j-1}, v_j)$ for $\ell > 0$
 $\psi_{\min}(\langle v_0 \rangle) = \infty$ if $v_0 \in S$, $\psi_{\min}(\langle v_0 \rangle) = -\infty$ if $v_0 \notin S$
- $\psi_{\text{sum}}(\langle v_0, \dots, v_\ell \rangle) = -\sum_{1 \leq j \leq \ell} w(v_{j-1}, v_j)$ for $\ell > 0$
 $\psi_{\text{sum}}(\langle v_0 \rangle) = \infty$ if $v_0 \in S$, $\psi_{\text{sum}}(\langle v_0 \rangle) = -\infty$ if $v_0 \notin S$
- **HLOIFT** uses **DA** with ψ_{\min} and **oriented w , a problem!**

DA with oriented variant of ψ_{\min}

In JMIV paper [Ciesielski, Herman, Kong, 2016]

we studied DA with i th object O_i having its oriented weights w_i and

$\psi_{\min}^*(\langle v_0, \dots, v_\ell \rangle) = \min_{1 \leq j \leq \ell} w_i(v_{j-1}, v_j)$ with v_0 a seed of O_i .

Theorem (Ciesielski, Herman, Kong, 2016)

For ψ_{\min}^* as above

- The output of DA is **completely robust under** (unaffected by) small (within CORE sets) **seed changes**.
- The output of DA has a nice characterization in terms of **path strength competition**.

However, for ψ_{\min}^* , the forest returned by DA need not be optimal. Also, in general, no minimality of a cut for ψ_{\min}^* .

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ψ_{\min}^* for which DA returns delineation with optimal cut

Let ψ_{\min}^* denotes ψ_{\min}^* in object/background setting such that

$$w_1(c, d) = w_0(d, c) \text{ for all } \langle c, d \rangle \in E.$$

Theorem (preliminary; & Leon, Ciesielski, Miranda, submitted)

If object O is an output of DA run with ψ_{\min}^* , then the graph cut

$$c(O) = \{\langle c, d \rangle \in E: c \in O \ \& \ d \notin O\}$$

minimizes the ℓ_∞ norm $\varepsilon_\infty(O) \stackrel{\text{def}}{=} \max_{\langle c, d \rangle \in c(O)} w_1(c, d)$ among all objects satisfying the constrains.

Assumption $w_1(c, d) = w_0(d, c)$ is needed to ensure that incorporating $\langle c, d \rangle$ in a path from either object or background influences the path strength the same way.

Oriented Image Foresting Transform algorithm **OIFT**

Is **OIFT** a DA run with ψ_{\min}^* ? **Close, but formally not.**

Assume that $w_1(c, d) = w_0(d, c)$ for all $\langle c, d \rangle \in E$ and let

$\psi_{\text{last}}(\langle v_0, \dots, v_\ell \rangle) = w_i(v_{\ell-1}, v_\ell)$ when $\ell > 0$ and v_0 a seed of O_i .

$\psi_{\text{last}}(\langle v_0 \rangle) = \infty$ when v_0 a seed and $\psi_{\text{last}}(\langle v_0 \rangle) = -\infty$ otherwise.

Definition

OIFT is a DA run with ψ_{last} as above.

Theorem (preliminary result: OIFT as DA with ψ_{\min}^*)

Any output of OIFT is an output of a particular implementation of DA with ψ_{\min}^ .*

Thus, a graph cut of any object returned by OIFT minimizes the ℓ_∞ norm among all objects satisfying the constrains.

Some properties of OIFT

- Can incorporate image brightness increase/decrease in weight function. If we like to favor transitions from bright to dark pixels when passing from object to the background, we can define, for some $\alpha \in (0, 1)$,

$$w_1(c, d) = \begin{cases} (1 - \alpha)e^{-\|f(c) - f(d)\|} & \text{if } \|f(c)\| > \|f(d)\| \\ (1 + \alpha)e^{-\|f(c) - f(d)\|} & \text{otherwise.} \end{cases}$$

- Can incorporate shape constraints like geodesic star convexity [Mansilla, Jackowski, Miranda, 2013], geodesic band constraints [Braz, Miranda, 2014], Hedgehog Shape Prior, and other to be explored.

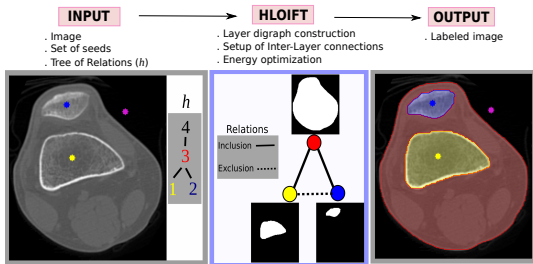
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HLOIFT: input and output

HLOIFT is, essentially, OIFT algorithm run on a **modified graph**.

Input: Image, a tree representing inclusion/exclusion relations between the objects we seek, seeds representing the objects; $\rho \geq 0$ giving minimal distance between boundaries of objects.



Forming HLOIFT's graph

Let $f: V \rightarrow \mathbb{R}^k$ be an (n -dimensional) image containing objects $O_1, \dots, O_m, O_{m+1} = V$. A hierarchy tree is indicated by a **parent map** h , with $h(i) = j$ meaning that O_j is a parent of O_i .

For every $i \in \mathcal{L} = \{1, \dots, m\}$ let $\langle V, E_i, w_i \rangle$ be an edge weighted graph associated with image f and object O_i . The edges and weights can include other constraints, like shape.

HLOIFT weighted digraph is defined as $\langle \mathcal{L} \times V, E, w \rangle$, where its restriction to i th object layer, $\langle \{i\} \times V, E^i, w^i \rangle$, is an isomorphic copy of $\langle V, E_i, w_i \rangle$.

We still need to define inter-layer edges and their weights on the **HLOIFT graph** $\mathcal{N} = \mathcal{L} \times V$.

Let $p: \mathcal{N} \rightarrow V$ be a projection, $p(i, c) = c$.

Labeling of objects

HLOIFT, being essentially OIFT run on \mathcal{N} , returns a single object $O \subset \mathcal{N}$.

It encodes the objects and the background as

$$O_i = \{t \in V : (i, t) \in O\} = p[O \cap (\{i\} \times V)] \text{ \& } O_0 = V \setminus \bigcup_{i \in \mathcal{L}} O_i.$$

This indicates how to define inter-layer edges and their weights to ensure tree-indicated relations.

If seed sets $\langle \mathcal{S}_0, \dots, \mathcal{S}_m \rangle$ in V indicate objects $\langle O_0, \dots, O_m \rangle$, then $\bar{\mathcal{S}}_1 = \bigcup_{i \in \mathcal{L}} \{i\} \times \mathcal{S}_i$ indicates object O in \mathcal{N} , while $\bar{\mathcal{S}}_0 = \mathcal{L} \times \mathcal{S}_0$ indicates its complement in \mathcal{N} .

Sets $\bar{\mathcal{S}}_0$ and $\bar{\mathcal{S}}_1$ are used to define ψ_{last} in \mathcal{N} .

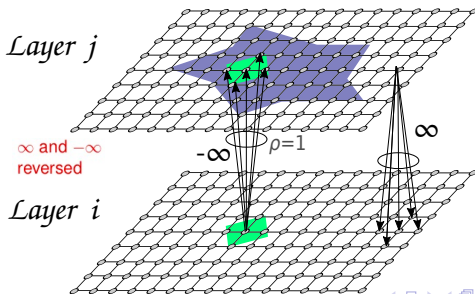
Inter-layer edges indicating **inclusions**

If O_j is the parent of O_i (i.e., $h(i) = j$),

we add all **edges** $\langle (i, c), (j, d) \rangle$ with $\|c - d\| \leq \rho$.

For $s = (i, c)$ and $t = (j, d)$ we define

$w_1(s, t) = w_0(t, s) = \infty$ and $w_0(s, t) = w_1(t, s) = -\infty$.



Inter-layer edges indicating **exclusions**

If O_i and O_j are siblings (i.e., $h(i) = h(j)$ and $i \neq j$),

we add all **edges** $\langle (i, c), (j, d) \rangle$ with $\|c - d\| \leq \rho$.

For $s = (i, c)$ and $t = (j, d)$ we define

$$w_1(s, t) = w_0(t, s) = w_0(s, t) = w_1(t, s) = \infty.$$

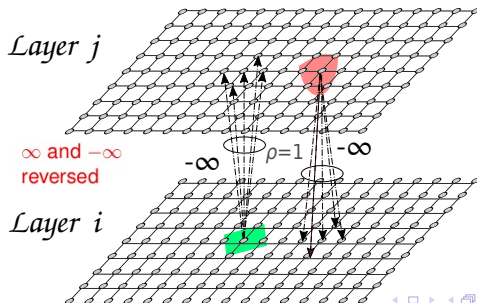


Illustration of the inter-layer arc construction

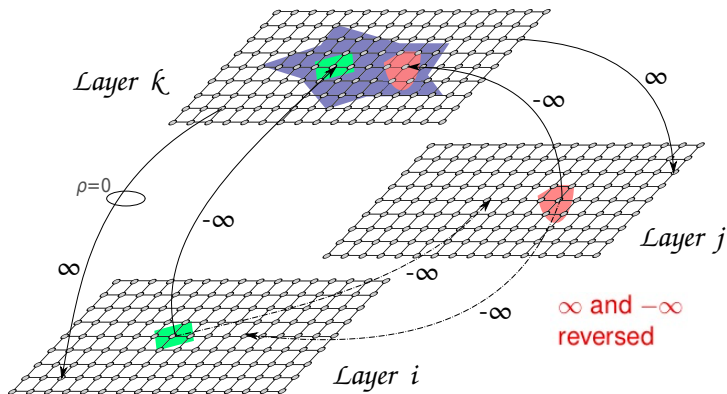


Figure: Illustration of the inter-layer arc construction, involving three objects O_i , O_j , and O_k , where O_k is the parent of two sibling objects, O_i and O_j , i.e., $h(i) = h(j) = k$.

HLOIFT Algorithm

Data: Weighted digraph \mathcal{N} ; ψ_{last} from image and sets $\bar{\mathcal{S}}_0, \bar{\mathcal{S}}_1$

Result: Array $\pi[\cdot]$ of paths, $\pi[t]$ being a path from a seed to t

```

1 foreach  $t \in \mathcal{N}$  do  $\pi[t] \leftarrow \langle t \rangle$  and  $S(t) \leftarrow 0$ ;
2  $Q \leftarrow \bar{\mathcal{S}}_0 \cup \bar{\mathcal{S}}_1$ 
3 while  $Q \neq \emptyset$  do
4   remove an element  $s$  of  $\max_{t \in Q} \psi_{\text{last}}(\pi[t])$  from  $Q$ 
5    $S(s) \leftarrow 1$ 
6   foreach  $x$  such that  $\langle s, x \rangle \in E$  and  $S(x) = 0$  do
7     if  $\psi_{\text{last}}(\pi[x]) < \psi_{\text{last}}(\pi[s] \wedge x)$  and
8        $[\pi[s]$  is from  $\bar{\mathcal{S}}_1$  or  $s$  and  $x$  are not siblings] then
9          $\pi[x] \leftarrow \pi[s] \wedge x$ 
10        if  $x \notin Q$  then insert  $t$  in  $Q$ 

```

Correctness of HLOIFT

Theorem (Leon, Ciesielski, Miranda, submitted)

An object O returned by **HLOIFT** generates objects $\langle O_0, \dots, O_m \rangle$ which are **consistent with the seeds** $\langle S_0, \dots, S_m \rangle$ and the **hierarchy** indicated by h .

Moreover, the graph cut $c(O)$ associated with O minimizes its l_∞ norm among all such objects, where

$$c(O) = \{ \langle s, t \rangle \in E : s \in O \ \& \ t \notin O \ \& \ s \text{ and } t \text{ are not siblings} \} \\ \cup \{ \langle s, t \rangle \in E : s, t \in O \ \& \ s \text{ and } t \text{ are siblings} \}.$$

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Experiment #1

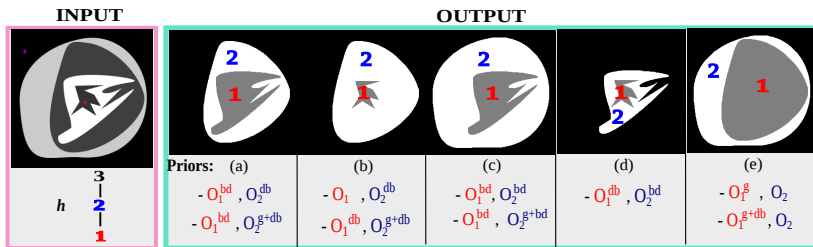
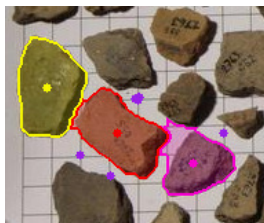


Figure: Example of two object segmentation by HLOIFT, where O_2 is parent of O_1 . Each object has different high-level priors –db: polarity from dark to bright pixels, bd: polarity from bright to dark pixels and g: geodesic star convexity prior. We used $\rho = 1.5$. Only two seeds.

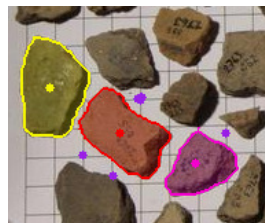
Experiment #2



Input image



$\rho = 0$



$\rho = 2$

Figure: Example showing how changing the ρ value from 0 to 2 can improve the archaeological fragment segmentation by HLOIFT, avoiding a result with touching objects.

Experiment #3

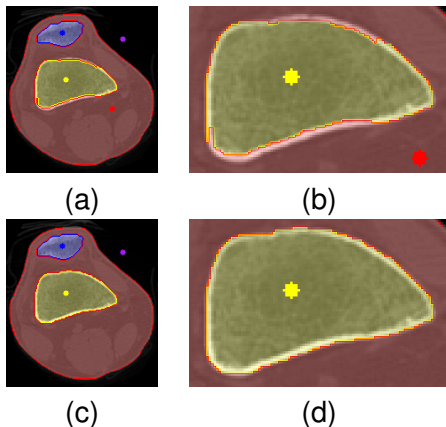
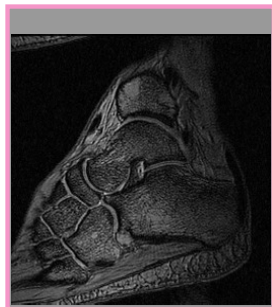


Figure: Knee segmentation composed of three objects in a CT image. (a-b) Result by IFT where the O_1 is mixing bright & dark boundaries. (c-d) An improved result is obtained by HLOIFT with boundary polarity from bright to dark pixels, requiring fewer seeds.

Experiment #4

INPUT



OUTPUT

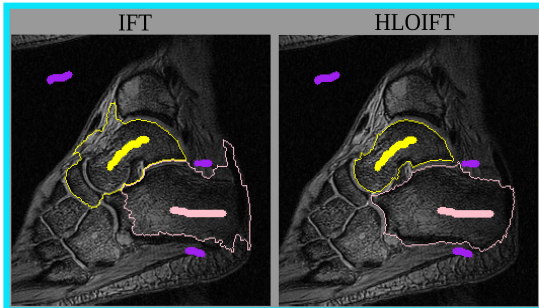


Figure: Talus (O_1) and calcaneus (O_2) segmentation. The two objects are sibling objects. For HLOIFT, we used $\rho = 0$, the geodesic star convexity and boundary polarity ($\alpha = -0.75$).

Exper. #5: CT, thoracic-abdominal axial cross section

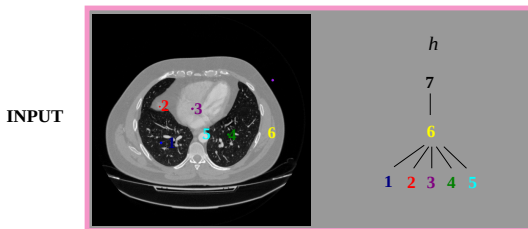
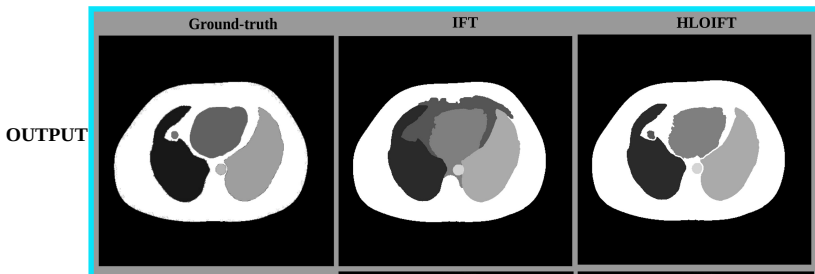


Figure: right lung (O_1), liver (O_2), heart (O_3), left lung (O_4), aorta (O_5) and the thoracic-abdominal region (O_6).



Experiment #6



(a)



(b)



(c)

Figure: Flower segmentation in two objects, the central part in cyan and the petals in yellow, using the inclusion relation. (a) The input image. (b) Result by the min-cut/max-flow algorithm in layered graphs. (c) Result by HLOIFT.

Efficiency: HLOIFT versus min-cut/max-flow

Image size (pixels)	Time of HLOIFT (ms)	Time of min-cut/max-flow (ms)
380 × 320	114.65	323.61
760 × 640	488.62	1,798.91
1520 × 1280	1,823.55	19,021.71

The running times for the flower segmentation by HLOIFT and the min-cut/max-flow algorithm in layered graphs using different image sizes.

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Summary

- We described efficient multi-object segmentation algorithm HLOIFT, which can use orientation, hierarchical relations between objects, and high-level priors for each object.
- We placed HLOIFT within a general framework of FC/IFT, which allows us to conclude its provable robustness on seed placements.
- We proved that the objects returned by HLOIFT are consistent with seeds placement and given hierarchy.
- We proved that the output of HLOIFT minimizes appropriate graph cut energy.

Credits

- K.C. Ciesielski, J.K. Udupa, A.X. Falcão, P.A.V. Miranda, “Fuzzy Connectedness image segmentation in Graph Cut formulation,” J. Math. Imaging Vision 44(3) (2012), 375-398
- K.C. Ciesielski, A.X. Falcão, P.A.V. Miranda, “Path-value functions for which Dijkstra’s algorithm returns optimal mapping,” J. Math. Imaging Vision 60(7) (2018), 1025-1036
- K.C. Ciesielski, Gabor T. Herman, T. Yung Kong, “General Theory of Fuzzy Connectedness Segmentations,” J. Math. Imaging Vision 55(3) (2016), 304-342;
- L.M.C. Leon, K.C. Ciesielski, P.A.V. Miranda, “Efficient Hierarchical Multi-Object Segmentation in Layered Graph,” (2018), submitted.

Thank you for your attention!