Hierarchical segmentation in a directed graph setting which optimizes a graph cut energy

Krzysztof Chris Ciesielski

Department of Mathematics, West Virginia University
and
MIPG, Department of Radiology, University of Pennsylvania

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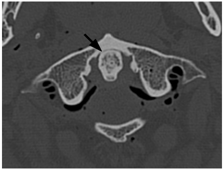
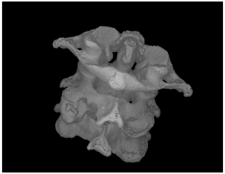
Outline

Outline

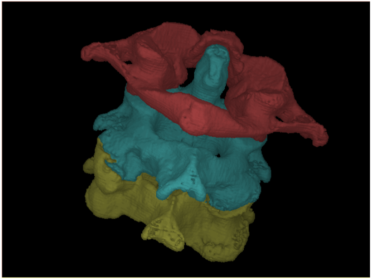
Image segmentation example 1: CT, cervical spine



A slice of an original 3D image



Surface rendition of segmented three vertebrae, together



Color surface rendition of the segmented three vertebra

Example 2: CT, thoracic-abdominal axial cross section

INPUT

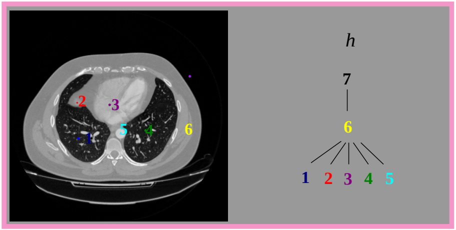


Figure: right lung (O_1), liver (O_2), heart (O_3), left lung (O_4), aorta (O_5) and the thoracic-abdominal region (O_6).

OUTPUT

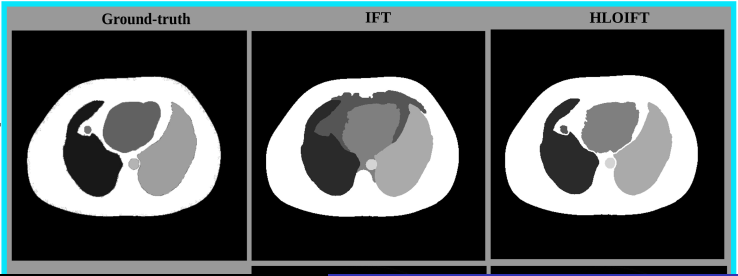


Image segmentation — formal setting

- An *image* is a map f from a set V (of spels) into $\mathbb{R}^k$

  The value $f(c)$ represents image intensity at c, a k-dimensional vector each component of which indicates a measure of some aspect of the signal, like color.

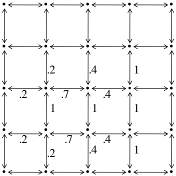
- *Segmentation problem:* Given an image $f\colon V \to \mathbb{R}^k$,

  find a “desired” family $\{O_1, \ldots, O_M\}$ of subsets of V.

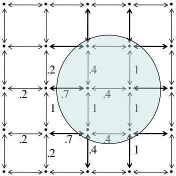
- We will assume the objects are indicated by disjoint sets S_i of seeds, imposing that $S_i \subset O_i$.

Image, its graph, and graph cut

An image, with intensity map $f \colon V \to \mathbb{R}^k$



Its graph $G = \langle V, E \rangle$, with some edge weights



Object O and its graph cut edges $c(O)$ in bold

- Vertices $v \in V$ are image pixels. Direct edges: all $\langle c, d \rangle, \langle d, c \rangle \in E$, with $c, d \in V$ nearby (e.g. 4 adjacency).
- Edge weights: $w(\langle c, d \rangle) =$ some function of $f(c) - f(d)$.
- Graph cut of O: $c(O) = \{\langle c, d \rangle \in E \colon c \in O \;\&\; d \notin O\}$.
  Only in one direction!

Graph cut measures: ℓ_p-norms, $1 \le p \le \infty$

Assuming $\langle c, d\rangle \in E \iff \langle d, c\rangle \in E$ and $w(\langle c, d\rangle) \ge 0$

ℓ_p-norm of $c(O)$ is defined as

$$\varepsilon_p(O) \stackrel{\text{def}}{=} \|w \restriction c(O)\|_p = \begin{cases} \left(\sum_{e\in c(O)} w(e)^p\right)^{1/p} & \text{if } p < \infty \\ \max_{e\in c(O)} w(e) & \text{if } p = \infty. \end{cases}$$

Standard analysis fact: $\|w\|_p \to_{p\to\infty} \|w\|_\infty$ for any map w.

Known algorithms minimizing ℓ_p-norms of graph cut

$p = 1$: Minimization solved by classic min-cut/max-flow algorithm.

Graph Cut, GC, delineation algorithm minimizes ε_1.

$p = \infty$: Minimization solved by (versions of) Dijkstra algorithm.

ε_∞ minimized objects are returned by the algorithms:
Power Watershed, PW [C. Couprie *et al*, 2011]
Relative Fuzzy Connectedness, RFC, Iterative RFC, IRFC, Image Foresting Transform, IFT, [Ciesielski, Udupa, Falcão, Miranda, 2012].

$p = 2$: Random Walker, RW, algorithm [Grady, 2006].

Fact: Inclusion-minimal ℓ_p-normed minimized delineations converge, as $p \to \infty$ to ℓ_∞-normed minimized delineation.

This talk's Main Algorithm, HLOIFT, minimizes ℓ_∞-norm of cut.

Outline

Paths and Optimal Path Forest OPF

- Fix directed graph $G = \langle V, E\rangle$ (with edge weight map w)
- *Path (in G)*: $p = \langle v_0, \ldots, v_\ell\rangle$ s.t. $\langle v_j, v_{j+1}\rangle \in E$ for $j < \ell$;
  p is from $S \subset V$ to $v \in V$ when $v_0 \in S$ and $v_\ell = v$;
  $p\hat{}w = \langle v_0, \ldots, v_\ell, w\rangle$; Π_G – all paths in G.
- Path cost function: any map $\psi\colon \Pi_G \to \mathbb{R}$.
- A path p (from $S \subset V$) to v is ψ-optimal provided

$$\psi(p) = \max\{\psi(q)\colon q \text{ is a path (from } S\text{) to } v\}.$$

- Jarník-Prim-Dijkstra algorithm DA for ψ and $S \subset V$ tries to find (S-rooted) forest, OPF, composed of ψ-optimal paths.
- HLOIFT is a DA for appropriate path cost map and graph.

Dijkstra Algorithm, DA, *aiming* to find ψ-optimal forest

Data: $G = \langle V, E \rangle$ and a path cost map $\psi \colon \Pi_G \to \mathbb{R}$
Result: an array $\pi[\,]$ of paths, aiming for being ψ-optimal

foreach $v \in V$ **do** $\pi[v] \leftarrow \langle v \rangle$
$Q \leftarrow V$
while $Q \neq \emptyset$ **do**
    remove an element w of $\max_{u \in Q} \psi(\pi[u])$ **from** Q
    foreach x *such that* $\langle w, x \rangle \in E$ **do**
        if $\psi(\pi[x]) < \psi(\pi[w]\hat{\ }x)$ **then** $\pi[x] \leftarrow \pi[w]\hat{\ }x$

DA is very efficient: quasi-linear w.r.t. the size of the graph.

For what path cost ψ DA works properly?

Studied in JMIV paper [Ciesielski, Falcão, Miranda, Sept. 2018]

correcting errors of TPAMI paper [Falcão, Stolfi, Lotufo, 2004].

If w is an edge weight map for undirected graph $G = \langle V, E \rangle$, then DA works properly for:

- FC/IFT: $\psi_{\min}(\langle v_0, \ldots, v_\ell \rangle) = \min_{1 \le j \le \ell} w(v_{j-1}, v_j)$ for $\ell > 0$

  $\psi_{\min}(\langle v_0 \rangle) = \infty$ if $v_0 \in S$, $\psi_{\min}(\langle v_0 \rangle) = -\infty$ if $v_0 \notin S$
- $\psi_{\text{sum}}(\langle v_0, \ldots, v_\ell \rangle) = -\sum_{1 \le j \le \ell} w(v_{j-1}, v_j)$ for $\ell > 0$

  $\psi_{\text{sum}}(\langle v_0 \rangle) = \infty$ if $v_0 \in S$, $\psi_{\text{sum}}(\langle v_0 \rangle) = -\infty$ if $v_0 \notin S$
- HLOIFT uses DA with $\psi_{\min}$ and oriented w, a problem!

DA with oriented variant of $\psi_{\min}$

In JMIV paper [Ciesielski, Herman, Kong, 2016]

we studied DA with ith object O_i having its oriented weights w_i and

$\psi^*_{\min}(\langle v_0, \ldots, v_\ell \rangle) = \min_{1 \le j \le \ell} w_i(v_{j-1}, v_j)$ with v_0 a seed of O_i.

Theorem (Ciesielski, Herman, Kong, 2016)

For $\psi^*_{\min}$ as above

- *The output of DA is completely robust under (unaffected by) small (within CORE sets) seed changes.*
- *The output of DA has a nice characterization in terms of path strength competition.*

However, for $\psi^*_{\min}$, the forest returned by DA need not be optimal. Also, in general, no minimality of a cut for $\psi^*_{\min}$.

Outline

$\psi^*_{\min}$ for which DA returns delineation with optimal cut

Let $\psi^\star_{\min}$ denotes $\psi^*_{\min}$ in object/background setting such that

$w_1(c,d) = w_0(d,c)$ for all $\langle c,d\rangle \in E$.

Theorem (preliminary; & Leon, Ciesielski, Miranda, submitted)

If object O is an output of DA run with $\psi^\star_{\min}$, then the graph cut

$$c(O) = \{\langle c,d\rangle \in E \colon c \in O \;\&\; d \notin O\}$$

minimizes the ℓ_∞ norm $\varepsilon_\infty(O) \stackrel{\text{def}}{=} \max_{\langle c,d\rangle \in c(O)} w_1(c,d)$ among all objects satisfying the constrains.

Assumption $w_1(c,d) = w_0(d,c)$ is needed to ensure that incorporating $\langle c,d\rangle$ in a path from either object or background influences the path strength the same way.

Oriented Image Foresting Transform algorithm OIFT

Is OIFT a DA run with ψ^*_{min}? Close, but formally not.

Assume that $w_1(c,d) = w_0(d,c)$ for all $\langle c,d\rangle \in E$ and let

$\psi_{\text{last}}(\langle v_0,\ldots,v_\ell\rangle) = w_i(v_{\ell-1},v_\ell)$ when $\ell > 0$ and v_0 a seed of O_i.

$\psi_{\text{last}}(\langle v_0\rangle) = \infty$ when v_0 a seed and $\psi_{\text{last}}(\langle v_0\rangle) = -\infty$ otherwise.

Definition

OIFT is a DA run with ψ_{last} as above.

Theorem (preliminary result: OIFT as DA with ψ^*_{min})

Any output of OIFT is an output of a particular implementation of DA with ψ^*_{min}.
Thus, a graph cut of any object returned by OIFT minimizes the ℓ_∞ norm among all objects satisfying the constrains.

Some properties of OIFT

- Can incorporate image brightness increase/decrease in weight function. If we like to favor transitions from bright to dark pixels when passing from object to the background, we can define, for some $\alpha \in (0, 1)$,

$$w_1(c, d) = \begin{cases} (1-\alpha)e^{-\|f(c)-f(d)\|} & \text{if } \|f(c)\| > \|f(d)\| \\ (1+\alpha)e^{-\|f(c)-f(d)\|} & \text{otherwise.} \end{cases}$$

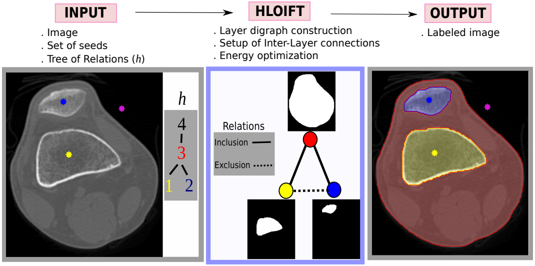
- Can incorporate shape constraints like geodesic star convexity [Mansilla, Jackowski, Miranda, 2013], geodesic band constraints [Braz, Miranda, 2014], Hedgehog Shape Prior, and other to be explored.

Outline

HLOIFT: input and output

HLOIFT is, essentially, OIFT algorithm run on a modified graph.

Input: Image, a tree representing inclusion/exclusion relations between the objects we seek, seeds representing the objects; $\rho \geq 0$ giving minimal distance between boundaries of objects.



Forming HLOIFT's graph

Let $f\colon V \to \mathbb{R}^k$ be an (n-dimensional) image containing objects $O_1, \ldots, O_m, O_{m+1} = V$. A hierarchy tree is indicated by a parent map h, with $h(i) = j$ meaning that O_j is a parent of O_i.

For every $i \in \mathcal{L} = \{1, \ldots, m\}$ let $\langle V, E_i, w_i \rangle$ be an edge weighted graph associated with image f and object O_i. The edges and weights can include other constrains, like shape.

HLOIFT weighted digraph is defined as $\langle \mathcal{L} \times V, E, w \rangle$, where its restriction to ith object layer, $\langle \{i\} \times V, E^i, w^i \rangle$, is an isomorphic copy of $\langle V, E_i, w_i \rangle$.

We still need to define inter-layer edges and their weights on the HLOIFT graph $\mathcal{N} = \mathcal{L} \times V$.

Let $p\colon \mathcal{N} \to V$ be a projection, $p(i, c) = c$.

Labeling of objects

HLOIFT, being essentially OIFT run on $\mathcal{N}$, returns a single object $O \subset \mathcal{N}$.

It encodes the objects and the background as

$$O_i = \{t \in V \colon (i,t) \in O\} = p[O \cap (\{i\} \times V)] \ \& \ O_0 = V \setminus \bigcup_{i \in \mathcal{L}} O_i.$$

This indicates how to define inter-layer edges and their weights to ensure tree-indicated relations.

If seed sets $\langle \mathcal{S}_0, \ldots, \mathcal{S}_m \rangle$ in V indicate objects $\langle O_0, \ldots, O_m \rangle$, then $\bar{\mathcal{S}}_1 = \bigcup_{i \in \mathcal{L}} \{i\} \times \mathcal{S}_i$ indicates object O in $\mathcal{N}$, while $\bar{\mathcal{S}}_0 = \mathcal{L} \times \mathcal{S}_0$ indicatess its complement in $\mathcal{N}$.

Sets $\bar{\mathcal{S}}_0$ and $\bar{\mathcal{S}}_1$ are used to define ψ_{last} in $\mathcal{N}$.

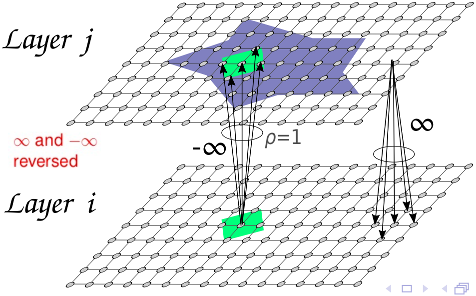
Inter-layer edges indicating inclusions

If O_j is the parent of O_i (i.e., $h(i) = j$),

we add all edges $\langle (i,c),(j,d) \rangle$ with $\|c - d\| \leq \rho$.

For $s = (i, c)$ and $t = (j, d)$ we define

$w_1(s,t) = w_0(t,s) = \infty$ and $w_0(s,t) = w_1(t,s) = -\infty$.



Inter-layer edges indicating exclusions

If O_i and O_j are siblings (i.e., $h(i) = h(j)$ and $i \neq j$),

we add all edges $\langle (i,c),(j,d) \rangle$ with $\|c - d\| \leq \rho$.

For $s = (i, c)$ and $t = (j, d)$ we define

$$w_1(s,t) = w_0(t,s) = w_0(s,t) = w_1(t,s) = \infty.$$

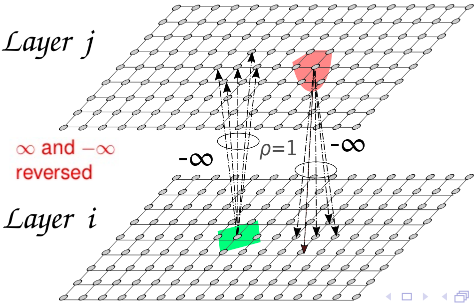


Illustration of the inter-layer arc construction

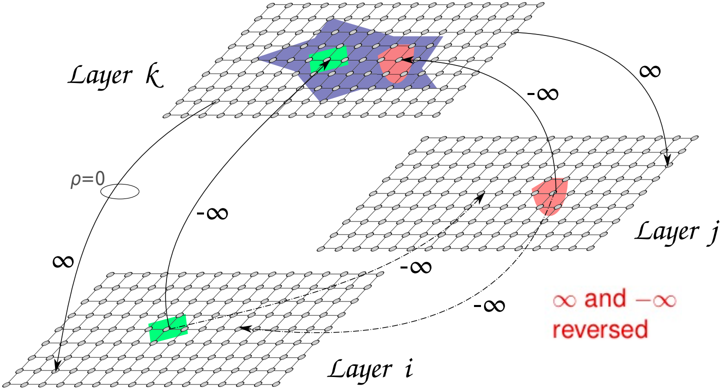


Figure: Illustration of the inter-layer arc construction, involving three objects O_i, O_j, and O_k, where O_k is the parent of two sibling objects, O_i and O_j, i.e., $h(i) = h(j) = k$.

HLOIFT Algorithm

Data: Weighted digraph $\mathcal{N}$; ψ_{last} from image and sets $\bar{\mathcal{S}}_0$, $\bar{\mathcal{S}}_1$
Result: Array $\pi[\,]$ of paths, $\pi[t]$ being a path from a seed to t

foreach $t \in \mathcal{N}$ **do** $\pi[t] \leftarrow \langle t \rangle$ and $S(t) \leftarrow 0$;
$Q \leftarrow \bar{\mathcal{S}}_0 \cup \bar{\mathcal{S}}_1$
while $Q \neq \emptyset$ **do**
    remove an element s of $\max_{t \in Q} \psi_{\text{last}}(\pi[t])$ **from** Q
    $S(s) \leftarrow 1$
    foreach x *such that* $\langle s, x \rangle \in E$ *and* $S(x) = 0$ **do**
        if $\psi_{\text{last}}(\pi[x]) < \psi_{\text{last}}(\pi[s]\hat{\ }x)$ *and*
            [$\pi[s]$ is from $\bar{\mathcal{S}}_1$ or s and x are not siblings] **then**
            $\pi[x] \leftarrow \pi[s]\hat{\ }x$
            if $x \notin Q$ **then** insert t in Q

Correctness of HLOIFT

Theorem (Leon, Ciesielski, Miranda, submitted)

An object O returned by HLOIFT generates objects $\langle O_0, \ldots, O_m \rangle$ which are consistent with the seeds $\langle \mathcal{S}_0, \ldots, \mathcal{S}_m \rangle$ and the hierarchy indicated by h.
Moreover, the graph cut $c(O)$ associated with O minimizes its ℓ_∞ norm among all such objects, where

$$\begin{aligned} c(O) &= \{\langle s,t\rangle \in E\colon s \in O \ \&\ t \notin O \ \&\ s \text{ and } t \text{ are not siblings}\} \\ &\cup \{\langle s,t\rangle \in E\colon s,t \in O \ \&\ s \text{ and } t \text{ are siblings}\}. \end{aligned}$$

Outline

Experiment #1

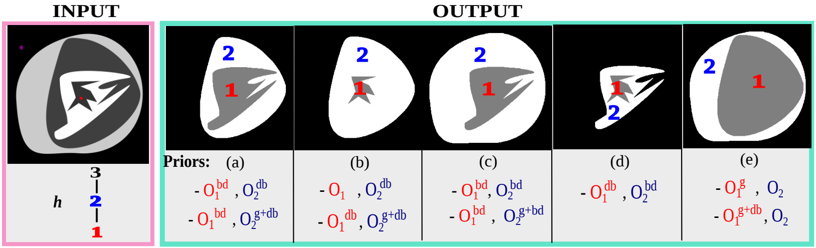
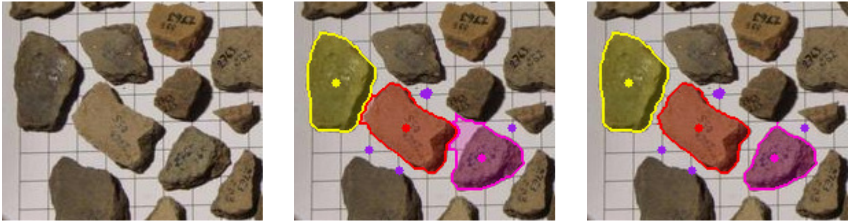


Figure: Example of two object segmentation by HLOIFT, where O_2 is parent of O_1. Each object has different high-level priors –db: polarity from dark to bright pixels, bd: polarity from bright to dark pixels and g: geodesic star convexity prior. We used $\rho = 1.5$. Only two seeds.

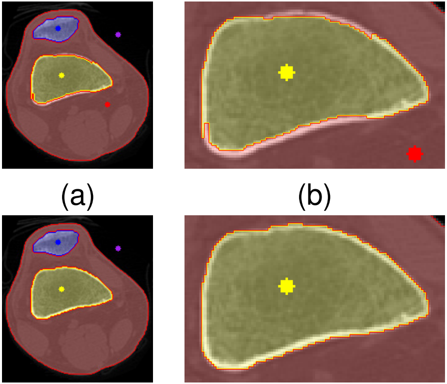
Experiment #2



Input image $\rho = 0$ $\rho = 2$

Figure: Example showing how changing the ρ value from 0 to 2 can improve the archaeological fragment segmentation by HLOIFT, avoiding a result with touching objects.

Experiment #3



(a) (b) (c) (d)

Figure: Knee segmentation composed of three objects in a CT image. (a-b) Result by IFT where the O_1 is mixing bright & dark boundaries. (c-d) An improved result is obtained by HLOIFT with boundary polarity from bright to dark pixels, requiring fewer seeds.

Experiment #4

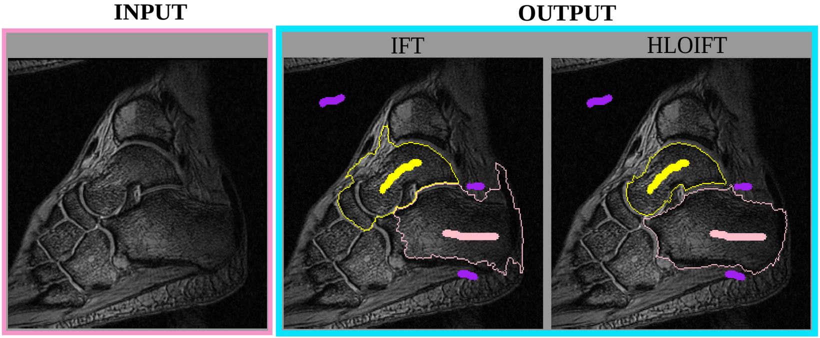


Figure: Talus (O_1) and calcaneus (O_2) segmentation. The two objects are sibling objects. For HLOIFT, we used $\rho = 0$, the geodesic star convexity and boundary polarity ($\alpha = -0.75$).

Exper. #5: CT, thoracic-abdominal axial cross section

INPUT

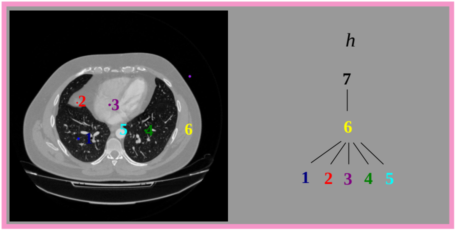
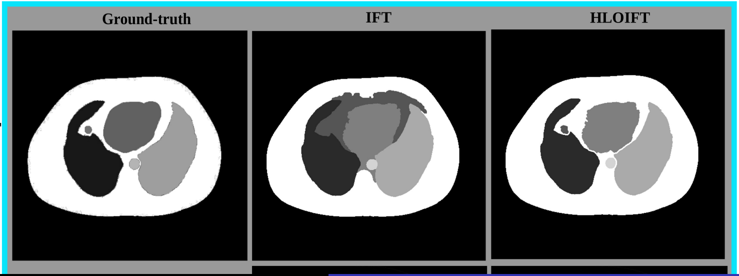


Figure: right lung (O_1), liver (O_2), heart (O_3), left lung (O_4), aorta (O_5) and the thoracic-abdominal region (O_6).

OUTPUT



Experiment #6

(a) (b) (c)

Figure: Flower segmentation in two objects, the central part in cyan and the petals in yellow, using the inclusion relation. (a) The input image. (b) Result by the min-cut/max-flow algorithm in layered graphs. (c) Result by HLOIFT.

Efficiency: HLOIFT versus min-cut/max-flow

Image size (pixels)	Time of HLOIFT (ms)	Time of min-cut/max-flow (ms)
380×320	114.65	323.61
760×640	488.62	1,798.91
1520×1280	1,823.55	19,021.71

The running times for the flower segmentation by HLOIFT and the min-cut/max-flow algorithm in layered graphs using different image sizes.

Outline

Summary

- We described efficient multi-object segmentation algorithm HLOIFT, which can use orientation, hierarchical relations between objects, and high-level priors for each object.
- We placed HLOIFT within a general framework of FC/IFT, which allows us to conclude its provable robustness on seed placements.
- We proved that the objects returned by HLOIFT are consistent with seeds placement and given hierarchy.
- We proved that the output of HLOIFT minimizes appropriate graph cut energy.

Credits

- K.C. Ciesielski, J.K. Udupa, A.X. Falcão, P.A.V. Miranda, "Fuzzy Connectedness image segmentation in Graph Cut formulation," J. Math. Imaging Vision 44(3) (2012), 375-398
- K.C. Ciesielski, A.X. Falcão, P.A.V. Miranda, "Path-value functions for which Dijkstra's algorithm returns optimal mapping," J. Math. Imaging Vision 60(7) (2018), 1025-1036
- K.C. Ciesielski, Gabor T. Herman, T. Yung Kong, "General Theory of Fuzzy Connectedness Segmentations," J. Math. Imaging Visionn 55(3) (2016), 304-342;
- L.M.C. Leon, K.C. Ciesielski, P.A.V. Miranda, "Efficient Hierarchical Multi-Object Segmentation in Layered Graph," (2018), submitted.

Thank you for your attention!