Higher level differentiability: Generalized Ulam-Zahorski problem and small coverings by smooth maps

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Based on survey written with Juan B. Seoane–Sepúlveda
Talk 2 of special session on
Different levels of smoothness: Restriction, extension, and covering theorem

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Outline

1. Extensions to *n*-times differentiable functions
2. Generalized Ulam-Zahorski interpolation problem
3. $\mathbb{SC}$: Simultaneous $\mathbb{S}$mall $\mathbb{C}$overings by smooth functions
4. $\mathbb{SC}$: Reduction to three results
5. $\mathbb{SC}$: Restriction theorems, prism density, and $\mathrm{CPA}_{\mathrm{prism}}$
6. $\mathbb{SC}$: Proof of Theorem A
7. Open problems

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Notation

D^n: is the class of all n-times differentiable $f\colon \mathbb{R} \to \mathbb{R}$

C^n: all $f \in D^n$ with continuous nth derivative $f^{(n)}$

For perfect $P \subset \mathbb{R}$, a D^n map $f\colon P \to \mathbb{R}$, and $a \in P$ let $T^n_a f(x)$ denote the n-th degree Taylor polynomial of f at a:

$$T^n_a f(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

and the "nth quotient" map $q^n_f\colon P^2 \to \mathbb{R}$ is given by

$$q^n_f(a,b) := \begin{cases} \dfrac{T^n_b f(b) - T^n_a f(b)}{(b-a)^n} & \text{if } a \neq b, \\ 0 & \text{if } a = b. \end{cases}$$

Whitney's Extension Theorem for one variable

$$q_f^n(a,b) := \frac{T_b^n f(b) - T_a^n f(b)}{(b-a)^n}$$

Theorem (Case $P \subset \mathbb{R}$ of theorem of Whitney 1934)

Let $P \subset \mathbb{R}$ be perfect, $n \in \mathbb{N}$, and $f\colon P \to \mathbb{R}$.

There exists a C^n extension $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ of f if, and only if,

(W_n) f is C^n and $q_{f^{(i)}}^{n-i}\colon P^2 \to \mathbb{R}$ is continuous for every $i \le n$.

Necessity of (W_n) is clear, as $\bar{f}$ satisfies it.

Sufficiency is not easy, even in the simple case of $P \subset \mathbb{R}$.

Our submitted paper with Seoane–Sepúlveda contains a detailed 4-page proof of this sufficiency.

Higher order of Jarník's Extension Theorem?

Open Problem

Is there an analogous characterization of functions $f\colon P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is perfect, that admit D^n extensions $\bar{f}\colon \mathbb{R} \to \mathbb{R}$?

Any f admitting D^n extension $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ must satisfy

(V_n): f is D^n and (W_{n-1}) from Whitney's Extension Theorem

For $n = 1$, (V_n) is sufficient, by Jarník's Extension Theorem.

For $n = 2$, (V_n) is not sufficient ($\mathfrak{C}$ is the Cantor ternary set):

Example (Ciesielski & Seoane–Sepúlveda, 2018)

There exists a C^1 function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f' \restriction \mathfrak{C} \equiv 0$ and for no perfect set $P \subset \mathfrak{C}$ there is a C^2 extension $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ of $f \restriction P$.

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About Generalized Ulam-Zahorski Problem

For $\mathcal{F},\mathcal{G}\subset\mathbb{R}^{\mathbb{R}}$, usually $\mathcal{F}\subsetneq\mathcal{G}$, it is the statement

UZ$(\mathcal{G},\mathcal{F})$: $\forall g\in\mathcal{G}\ \exists f\in\mathcal{F}$ *with uncountable* $f\cap g$.

Zahorski 1948, solving 1940 problem of Ulam: $\neg$ UZ$(C^0,\text{analytic})$

Zahorski asked: does UZ(C^0,C^∞) hold?

What about UZ$(\mathcal{G},\mathcal{F})$ for other classes of differentiable maps?

Agronsky, Bruckner, Laczkovich, Preiss 1985: UZ(C^0,C^1) holds

Olevskiĭ 1994: UZ(C^1,C^2) holds, but $\neg$ UZ(C^0,C^2) & $\neg$ UZ(C^2,C^3),

solving all UZ(C^n,C^m) problems.

Strong D^n-C^n interpolation theorem

Theorem (Ciesielski and Seoane–Sepúlveda, 2018)

For every $n \in \mathbb{N}$, perfect $P \subset \mathbb{R}$, and D^n map $f \colon \mathbb{R} \to \mathbb{R}$ there is a C^n map $g \colon \mathbb{R} \to \mathbb{R}$ for which $[f = g] \cap P$ is uncountable. In particular, $\mathrm{UZ}(D^n, C^n)$ holds.

Proof: Short but a bit tricky.

Using Whitney's Extension Theorem.

Special case of a result discussed latter.

All $\mathrm{UZ}(\mathcal{G},\mathcal{F})$ problems for $\mathcal{F},\mathcal{G}\in\mathbb{D}=\bigcup_{n<\omega}\{D^n, C^n\}$

Corollary

For every $n\in\mathbb{N}$ with $n\geq 2$:

(a) *C^1 is the smallest $\mathcal{F}\in\mathbb{D}$ for which $\mathrm{UZ}(C^0,\mathcal{F})$ holds.*

(b) *If $\mathcal{F}\in\mathbb{D}$ is the smallest for which $\mathrm{UZ}(D^1,\mathcal{F})$ holds, then $\mathcal{F}\in\{C^1,C^2\}$.*

(c) *C^2 is the smallest $\mathcal{F}\in\mathbb{D}$ for which $\mathrm{UZ}(C^1,\mathcal{F})$ holds.*

(d) *C^n is the smallest $\mathcal{F}\in\mathbb{D}$ for which $\mathrm{UZ}(D^n,\mathcal{F})$ holds.*

(e) *C^n is the smallest $\mathcal{F}\in\mathbb{D}$ for which $\mathrm{UZ}(C^n,\mathcal{F})$ holds.*

Proof.

(a) By previous theorem $\mathrm{UZ}(C^0, D^2)$ implies $\mathrm{UZ}(C^0, C^2)$.
Since $\neg\,\mathrm{UZ}(C^0, C^2)$, then so is $\neg\,\mathrm{UZ}(C^0, D^2)$.
Arguments for (b)-(e) are similar. □

Remaining Ulam-Zahorski interpolation problem

Open Problem

Does UZ(D^1, D^2) hold?

Notice, that if UZ(D^1, D^2) holds, then so does UZ(D^1, C^2).

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The main covering theorem

$A \subset^\star B$ will mean "$B \setminus A$ has cardinality $\leq \omega_1$."

$\mathrm{CPA}_{\mathrm{prism}}$ is a simple part of the covering property axiom CPA consistent with ZFC.

Theorem (Ciesielski and Seoane–Sepúlveda 2018)

$\mathrm{CPA}_{\mathrm{prism}}$ implies that for every $\nu \in \omega \cup \{\infty\}$ there exists a family $\mathcal{F}_\nu \subset C^\nu(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that

(i) *$g \subset^\star \bigcup \mathcal{F}_\nu$ for every $g \in D^\nu(\mathbb{R})$.*

Moreover, for $n \in \{0, 1\}$, and only such n, we also have

(ii) *$g \subset^\star \bigcup \mathcal{F}_n$ for every $g \in D^n(X)$, where $X \subset \mathbb{R}$ is arbitrary.*

What remains of this talk revolves around this theorem.

No (obvious) expansions of the Main Theorem

Main Theorem states that: For $n < \omega$ & $k < 2$, $\mathrm{CPA}_{\text{prism}}$ implies

$$I_n = I(D^n, C^n)\colon\ \exists \mathcal{F}_n \in [C^n(\mathbb{R})]^{<\mathfrak{c}}\ \forall g \in D^n(\mathbb{R})\quad g \subset^\star \bigcup \mathcal{F}_n$$

$$I_k^*\colon\ \exists \mathcal{F}_k \in [C^k(\mathbb{R})]^{<\mathfrak{c}}\ \forall X \subset \mathbb{R}\ \forall g \in D^k(X)\quad g \subset^\star \bigcup \mathcal{F}_k$$

This cannot be expanded, as

Fact: $I(C^{n-1}, D^n)$ is false for all $n \in \mathbb{N}$.

Proof: For $n = 1$, there is $g_1 \in$ "$D^1(\mathbb{R})$"$\subset C^0(\mathbb{R})$ with $g_1' = \infty$ on a perfect P; so $|[f = g] \cap P| \leq \omega$ for every $f \in D^0(\mathbb{R})$.
For $n = 2$ use $g_2 = \int g_1$, etc. . . . $\square$.

Fact: I_k^* is false for $k > 1$.

Proof: Put $k = 2$ and $\mathfrak{C}$—the Cantor ternary set. There is (simple) $f \in C^1(\mathbb{R})$ such that $g \restriction \mathfrak{C} \in D^2(\mathfrak{C})$ and $|[f = g] \cap \mathfrak{C}| < \omega$ for every $f \in D^2(\mathbb{R})$. So, $g \restriction \mathfrak{C}$ contradicts I_2^*. $\square$

Main Theorem for $n = 0$: independence of ZFC

Main Theorem for $n = 0$ can be stated as: $\mathrm{CPA}_{\mathrm{prism}}$ implies

I_0: $\exists \mathcal{F}_0 \in [C^0(\mathbb{R})]^{<\mathfrak{c}} \ \forall g \in C^0(\mathbb{R}) \quad g \subset^\star \bigcup \mathcal{F}_0$

What I_0 means:

- Few functions from $\mathcal{F}_0$ cover* **every** continuous function
- Few functions from $\mathcal{F}_0$ cover* **every** level $\mathbb{R} \times \{y\}$
- $\subset^\star$ in I_0 cannot be $\subset$, as $\mathcal{F}_0$ cannot cover $\mathbb{R}^2$
- $I_0 \implies \mathrm{cov}(\mathrm{Meager}) < \mathfrak{c}$
  Proof: Pick $y \in \mathbb{R} \setminus \bigcup_{f \in \mathcal{F}_0} f[\mathbb{Q}]$ and put $g = \mathbb{R} \times \{y\}$.
  Then $\mathbb{R}$ is a union of $|\mathcal{F}_0|$-many nowhere dense sets $[f = g]$ and $|g \setminus \bigcup \mathcal{F}_0|$-many singletons, while $|\mathcal{F}_0| + |g \setminus \bigcup \mathcal{F}_0| < \mathfrak{c}$.
- So, I_0 contradicts CH and MA.

I_0 and the size of $\mathfrak{c}$

Fact (Proved for this talk. Not in the papers. Known?)

$I_0 \implies \exists \mathcal{F}_0 \in [\mathbb{R}^\mathbb{R}]^{<\mathfrak{c}}\ \forall y \in \mathbb{R}\ \ \mathbb{R} \times \{y\} \subset^\star \bigcup \mathcal{F}_0 \implies \mathfrak{c} = |\mathcal{F}_0|^+$

Proof: Let $\kappa = |\mathcal{F}_0|^+$ and assume $\mathfrak{c} > \kappa$. Put $B = \mathbb{R}^2 \setminus \bigcup \mathcal{F}_0$, $B^y = \{x \colon \langle x, y\rangle \in B\}$, and note that $|B^y| < \mathfrak{c}$ for all $y \in \mathbb{R}$.

Claim: There is $Y \in [\mathbb{R}]^\kappa$ with $|\bigcup_{y \in Y} B^y| < \mathfrak{c}$.

Proof. If $\operatorname{cof}(\mathfrak{c}) > \kappa$, then any $Y \in [\mathbb{R}]^\kappa$ works. If $\operatorname{cof}(\mathfrak{c}) \le \kappa$, choose cofinal $L \in [\mathfrak{c}]^\kappa$; there is $\lambda \in L$ with $Z_\lambda = \{y \in \mathbb{R} \colon |B^y| \le \lambda\}$ of cardinality $> \kappa$. (Otherwise $\mathfrak{c} = |\bigcup_{\lambda \in L} Z_\lambda| \le \kappa$.) Then any $Y \in [Z_\lambda]^\kappa$ works. □

Now, by Claim, there are $x_0 \in \mathbb{R} \setminus \bigcup_{y \in Y} B^y$ and $y_0 \in Y \setminus \{f(x_0) \colon f \in \mathcal{F}_0\}$. Then $\langle x_0, y_0 \rangle \notin B \cup \bigcup \mathcal{F}_0 = \mathbb{R}^2$, a contradiction.

Families $\mathcal{F}_n$ for $n > 0$

I_n: $\exists \mathcal{F}_n \in [C^n(\mathbb{R})]^{<\mathfrak{c}}\ \forall g \in D^n(\mathbb{R})\quad g \subset^\star \bigcup \mathcal{F}_n$

Clearly I_n implies

J_n: $\forall g \in D^n(\mathbb{R})\ \exists \mathcal{F}_g \in [C^n(\mathbb{R})]^{<\mathfrak{c}}$ $s.t.$ $g \subset \bigcup \mathcal{F}_g$

$\mathrm{CPA}_{\mathrm{prism}} \implies J_n$ was first “proved” by KC & Pawlikowski [CPA book]

For $n > 1$ their proof was incorrect!

Thus, the proof from submitted paper is the first correct one.

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Main Theorem via three key theorems

Theorem (A)

$\mathrm{CPA}_{\rm prism}$ *implies that for every* $\nu \in \omega \cup \{\infty\}$ *and every compact interval* $I = [a, b] \subset \mathbb{R}$ *there exists a family* $\mathcal{F}^I_\nu \subset C^\nu(\mathbb{R})$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that* $g \subset^\star \bigcup \mathcal{F}^I_\nu$ *for every* $g \in C^\nu(I)$.

Theorem (B)

$\mathrm{CPA}_{\rm prism}$ *implies that for every* $n \in \mathbb{N}$ *and* $g \in D^n(\mathbb{R})$ *there exists a family* $\mathcal{F}_g \subset C^n(\mathbb{R})$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that* $g \subset \bigcup \mathcal{F}_g$.

Theorem (C)

$\mathrm{CPA}_{\rm prism}$ *implies that for every* $n \in \{0, 1\}$ *and* $g \in D^n(X)$ *with* $X \subset \mathbb{R}$ *there exists a family* $\mathcal{F}_g \subset C^n(\mathbb{R})$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that* $g \subset \bigcup \mathcal{F}_g$.

Proof of Main Theorem from Theorems A-C

Fix $\nu \in \omega \cup \{\infty\}$. Put $\mathcal{F}_\nu = \bigcup_{n=1}^{\infty} \mathcal{F}_\nu^{[-n,n]}$, with $\mathcal{F}_\nu^{[-n,n]}$ from Thm A.

Choose a $g \in D^\nu(X)$ such that $X \subset \mathbb{R}$ and $X = \mathbb{R}$ unless $\nu < 2$.

We need to show that $g \subset^\star \bigcup \mathcal{F}_\nu$.

There is an $\mathcal{F}_g \in [C^\nu(\mathbb{R})]^{\leq\omega_1}$ such that $g \subset \bigcup \mathcal{F}_g$.

For $\nu < 2$ follows from Thm C,
for $\nu = \infty$ this is justified by $\mathcal{F}_g = \{g\} \subset D^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$,
while for the remaining cases this follows from Thm B.

For each $n \in \mathbb{N}$ and $f \in \mathcal{F}_g$ we have
$f \restriction [-n,n] \subset^\star \bigcup \mathcal{F}_\nu^{[-n,n]} \subset \bigcup \mathcal{F}_\nu$. So, we have needed

$$g \subset \bigcup \mathcal{F}_g = \bigcup_{f \in \mathcal{F}_g} \bigcup_{n=1}^{\infty} f \restriction [-n,n] \subset^\star \bigcup \mathcal{F}_\nu$$

Borel extensions of maps from $D^\nu(X)$, $X \subset \mathbb{R}$ arbitrary

Theorem (KC & Seoane–Sepúlveda 2018; known earlier?)

Let $X \subset \mathbb{R}$ be with no isolated points and $\nu \in \omega \cup \{\infty\}$. For every $g \in D^\nu(X)$ there exist Borel $B \supset X$ and $\bar{g} \in D^\nu(B)$ extending g.

Proof based on the lemmas. (Known for $X = \mathbb{R}$.)

Lemma (KC & Seoane–Sepúlveda 2018; known earlier?)

For every $X \subset \mathbb{R}$ with no isolated points and $g \in C(X)$ the set $\mathrm{Dif}(g)$ of points of differentiability of g is an $F_{\sigma\delta}$ subset of X.

Lemma (KC & Seoane–Sepúlveda 2018; known earlier?)

For every $X \subset \mathbb{R}$ with no isolated points and $g \in C(X)$ if $f \in D^1(X)$, then the derivative $f' \colon X \to \mathbb{R}$ is of Baire class 2.

Proof of Theorem C

$\forall n<2,\ X\subset\mathbb{R},\ g\in D^n(X)\ \exists\ \mathcal{F}_g\in[C^n(\mathbb{R})]^{\le\omega_1}$ with $g\subset\bigcup\mathcal{F}_g$.

Proof.

Fix $n<2,\ X\subset\mathbb{R},\ g\in D^n(X)$.
We can assume that that X has no isolated points.
Then, there exist Borel $B\supset X$ and $\bar g\in D^\nu(B)$ extending g.
By $\mathrm{CPA}_{\rm prism}$, see [CPA book], there exists a family $\mathcal{P}$ of cardinality $\le\omega_1$ of compact subsets of B such that $B=\bigcup\mathcal{P}$.
For every $P\in\mathcal{P}$, we have $\bar g\restriction P\in D^n(P)$ and

- there exists an extension $g_P\in D^n(\mathbb{R})$ of $\bar g\restriction P$

—by Tietze (for $n=0$) or Jarník (for $n=1$) extension theorem.
So, by Theorem B, there is $\mathcal{F}_P\in[C^n(\mathbb{R})]^{\le\omega_1}$ with $g_P\subset\bigcup\mathcal{F}_P$.
Then, $\mathcal{F}_g=\bigcup_{P\in\mathcal{P}}\mathcal{F}_P$ is as needed, since

$$g\subset\bar g=\bigcup_{P\in\mathcal{P}}\bar g\restriction P\subset\bigcup_{P\in\mathcal{P}}g_P\subset\bigcup_{P\in\mathcal{P}}\bigcup\mathcal{F}_P=\bigcup\mathcal{F}_g.$$

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Restriction theorems expressed in density language

$\mathrm{Perf}(X)$—all $P \subset X$ homeomorphic to $\mathfrak{C}$.

1). **Theorem on D^n-C^n interpolation:** For any $f \in D^n(\mathbb{R})$ the family

$$\mathcal{E}_f := \{Q \in \mathrm{Perf}(\mathbb{R})\colon f \restriction Q \text{ is extendable to } g \in C^n(\mathbb{R})\}$$

is $\mathrm{Perf}(\mathbb{R})$-*dense*: every $P \in \mathrm{Perf}(\mathbb{R})$ contains a $Q \in \mathcal{E}_f$.

2) For $n = 1$ and $f \in C^0(\mathbb{R})$, $\mathcal{E}_f$ is not $\mathrm{Perf}(\mathbb{R})$-dense.

We need more structure: sets $P \in \mathrm{Perf}(\mathbb{R})$ of positive measure; then $Q \in \mathcal{E}_f^n$ contained in P can have also positive measure.

3) To prove Theorem C and state $\mathrm{CPA}_{\rm prism}$ we need the notion of $\mathcal{F}_{\rm prism}$-density, where sets $P \in \mathrm{Perf}(\mathbb{R})$ come with more structure, and "good" $Q \in \mathcal{E}_f^n$ contained in P retain part of it.

$\mathcal{F}_{\text{prism}}$-density and $\text{CPA}_{\text{prism}}$

Based on a family $\mathbb{P}$ of perfect subsets of $\mathfrak{C}^\alpha$, $0 < \alpha < \omega_1$, containing all cubes $\prod_{\xi<\alpha} P_\xi$, $P_\xi \in \text{Perf}(\mathfrak{C})$.
($\mathbb{P}$—all sets $f[\mathfrak{C}^\alpha]$, where $f\colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ is continuous 1-1 s.t.

$$f(x) \restriction \xi = f(y) \restriction \xi \ \Leftrightarrow\ x \restriction \xi = y \restriction \xi \quad \text{for all } \xi < \alpha \text{ and } x, y \in \mathfrak{C}^\alpha.$$

This definition will not be used in this talk.)

Prism in X—any $P \in \text{Perf}(X)$ with (implicit) continuous injection h from an $E \in \mathbb{P}$ onto P.

Subprism of a prism P given by $h\colon E \to P$—any $Q = h[E']$, with $E' \in \mathbb{P}$, $E' \subset E$.

$\mathcal{E} \subset \text{Perf}(X)$ is $\mathcal{F}_{\text{prism}}$-dense provided for every prism P in $\text{Perf}(X)$ there exists a subprism Q of P with $Q \in \mathcal{E}$.

$\text{CPA}_{\text{prism}}$: $\mathfrak{c} = \omega_2$ and for every Polish space X and every $\mathcal{F}_{\text{prism}}$-dense family $\mathcal{E} \subset \text{Perf}(X)$ there is $\mathcal{E}_0 \in [\mathcal{E}]^{\le\omega_1}$ with $X \subset^\star \bigcup \mathcal{E}_0$.

Proof of Theorem B

Th B: $\text{CPA}_{\text{prism}} \Rightarrow \forall n \in \mathbb{N},\ f \in D^n(\mathbb{R})\ \exists \mathcal{F}_f \in [C^n(\mathbb{R})]^{\leq\omega_1}\ f \subset \bigcup \mathcal{F}_f$.

Proposition (discussed in the next slide)

$\mathcal{E} := \{Q \in \text{Perf}(\mathbb{R})\colon f \restriction Q$ is extendable to $g_Q \in C^n(\mathbb{R})\}$ is $\mathcal{F}_{\text{prism}}$-dense for every $f \in D^n(\mathbb{R})$.

Proof of Theorem B.

By $\text{CPA}_{\text{prism}}$ used with $X = \mathbb{R}$ and $\mathcal{F}_{\text{prism}}$-dense $\mathcal{E}$

there is an $\mathcal{E}_0 \in [\mathcal{E}]^{\leq\omega_1}$ with $\mathbb{R} \subset^\star \bigcup \mathcal{E}_0$.

Then $\hat{\mathcal{F}}_f = \{g_Q\colon Q \in \mathcal{E}_0\}$ has cardinality $\leq \omega_1$ and $f \subset^\star \bigcup \hat{\mathcal{F}}_f$.

An extension $\mathcal{F}_f$ of $\hat{\mathcal{F}}_f$ by ω_1 constant maps gives $f \subset \bigcup \mathcal{F}_f$. □

Sketch of the proof of Proposition

$\mathcal{E}_f = \{Q \colon f \restriction Q$ is extendable to $g_Q \in C^n(\mathbb{R})\}$ is $\mathcal{F}_{\text{prism}}$-dense.

Proof.

Fix a prism P and define symmetric $\varphi_f^n \colon P^2 \setminus \Delta \to \mathbb{R}$ as

$$\varphi_f^n(a,b) = \sum_{k=0}^n |q_{f(k)}^{n-k}(a,b)| + \sum_{k=0}^n |q_{f(k)}^{n-k}(b,a)|.$$

By lemma from [CPA book] there is a subprism Q of P such that $\varphi_f^n \colon Q^2 \setminus \Delta \to [-\infty, \infty]$ is uniformly continuous.
So, it has a continuous extension to Q^2. This extension is 0 on the diagonal, as $f \in D^n(\mathbb{R})$. (This is proved with two lemmas.)
So, $f \restriction Q$ satisfies assumptions of Whitney's Extension thm. □

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Polish space structures we need

Th A: $\mathrm{CPA}_{\mathrm{prism}} \Rightarrow \forall \nu \leq \omega\ \exists \mathcal{F}^{I}_{\nu} \in [C^{\nu}(\mathbb{R})]^{\leq \omega_1}\ \forall g \in C^{\nu}(I)\ g \subset^{\star} \bigcup \mathcal{F}^{I}_{\nu}$.

Why $C^{\nu}(I)$ rather than $C^{\nu}(\mathbb{R})$ or $D^{\nu}(\mathbb{R})$?

$C^{\nu}(\mathbb{R})$ and $D^{\nu}(\mathbb{R})$ are not Polish, $C^{\nu}(I)$ is, with metric

$$\rho(f,g) = \sum_{i<\nu} \|f^{(i)} - g^{(i)}\|_{\infty}$$

We use $\mathrm{CPA}_{\mathrm{prism}}$ with Polish space $I \times C^{\nu}(I)$

Family $\mathcal{F}_0^I$

Th A: $\text{CPA}_{\text{prism}} \Rightarrow \forall \nu \leq \omega\ \exists \mathcal{F}_\nu^I \in [C^\nu(\mathbb{R})]^{\leq\omega_1}\ \forall g \in C^\nu(I)\ g \subset^\star \bigcup \mathcal{F}_\nu^I$.

Lemma (From [CPA book])

$$\mathcal{E}_0 := \{P \in \text{Perf}(I \times C(I)) \colon \ \textit{either } \pi_1 \restriction P \textit{ or } \pi_2 \restriction P \textit{ is 1-to-1}\}$$

is $\mathcal{F}_{\text{prism}}$*-dense.*

So, by $\text{CPA}_{\text{prism}}$, there is $\mathcal{E} \in [\mathcal{E}_0]^{\leq\omega_1}$ with $I \times C(I) \subset^\star \bigcup \mathcal{E}$.

If $P \in \mathcal{F} := \{P \in \mathcal{E} \colon \pi_1 \restriction P \text{ is 1-to-1}\}$, then $P \in C(\pi_1[P], C(I))$ and $f_P \colon \pi_1[P] \to \mathbb{R}$ defined as $f_P(x) = P(x)(x)$ is continuous, so extendable to $\hat{f}_P \in C(\mathbb{R})$.

Claim: $\mathcal{F}_0^I := \{\hat{f}_P \colon P \in \mathcal{F}\}$ is as needed.

$\mathcal{F}_0^I := \{\hat{f}_P \colon P \in \mathcal{F}\}$ satisfies Theorem A for $n = 0$

$\mathcal{E}_0 = \{P \in \mathrm{Perf}(I \times C(I))\colon$ either $\pi_1 \restriction P$ or $\pi_2 \restriction P$ is 1-to-1$\}$

$\mathcal{E} \in [\mathcal{E}_0]^{\leq\omega_1}$ with $I \times C(I) \subset^\star \bigcup \mathcal{E}$, $\mathcal{F} = \{P \in \mathcal{E} \colon \pi_1 \restriction P$ is 1-to-1$\}$

Fix $g \in C(I)$. Need $g \subset^\star \bigcup \mathcal{F}_0^I$. Note that $I \times \{g\} \subset^\star \bigcup \mathcal{F}_0^I$.

Fix $x \in I$ s.t. $\langle x, g\rangle \in \bigcup \mathcal{F}_0^I$ and $P \in \mathcal{F}_0^I$ with $\langle x, g\rangle \in P$.

It is enough to show that $\langle x, g(x)\rangle \in f_P$, as $f_P \subset \hat{f}_P$.

Indeed, $f_P(x) = P(x)(x) = g(x)$, as $P(x) = g$ by $\langle x, g\rangle \in P$.

So, $g \restriction \pi_1[(I \times \{g\}) \cap \bigcup \mathcal{F}] \subset \bigcup \mathcal{F}_I$, as needed. $\square$

Theorem A for $\nu > 0$

f_P with $P \in \mathcal{E}_0$ need not to have C^n extension.
We need

Lemma (with proof similar to one needed for Theorem C)

$$\mathcal{E}_\infty = \{P \in \mathcal{E}_0\colon\ \textit{if } \pi_1 \restriction P \textit{ is 1-1, then } \exists \hat{f}_P \in C^0(\mathbb{R}) \textit{ extending } f_P$$
$$\textit{s.t. } \forall n < \omega \textit{ either } f_P^{(n)} \equiv \pm\infty \textit{ or } \hat{f}_P \in C^n(\mathbb{R})\}$$

is $\mathcal{F}_{\rm prism}$-dense.

Fact For $P \in \mathcal{E}_\infty$ and $g \in C^n(I)$, if $g \cap f_P$ is uncountable,

then $\hat{f}_P \in C^n(\mathbb{R})$ and $g \cap f_P \subset^\star \hat{f}_P$.

Pr. If $Q \in \mathrm{Perf}(P)$ & $g = f_P$ on Q, then, on Q, $f_P^{(n)} \equiv g^{(n)} \not\equiv \pm\infty$.

Proof of Theorem A, general case

$\mathcal{E}_\infty \subset \mathcal{E}_0$ is $\mathcal{F}_{\rm prism}$-dense and for any $P \in \mathcal{E}_\infty$ and $g \in C^n(I)$, if $g \cap f_P$ is uncountable, then $\hat{f}_P \in C^n(\mathbb{R})$ and $g \cap f_P \subset^\star \hat{f}_P$.

By $\mathrm{CPA}_{\rm prism}$, there is $\mathcal{E} \in [\mathcal{E}_\infty]^{\le\omega_1}$ with $I \times C(I) \subset^\star \bigcup \mathcal{E}$.

Put $\mathcal{F} := \{P \in \mathcal{E} \colon \pi_1 \restriction P \text{ is 1-to-1}\}$, $\mathcal{F}_0^I := \{\hat{f}_P \colon P \in \mathcal{F}\}$, and

$\mathcal{F}_\nu^I := \{\hat{f}_P \in C^\nu(\mathbb{R}) \colon P \in \mathcal{F}\}$.

Fix $g \in C^\nu(I)$. Need $g \subset^\star \bigcup \mathcal{F}_\nu^I$. Indeed

$$g \subset^\star \bigcup_{P \in \mathcal{E}} g \cap f_P \subset^\star \bigcup_{\hat{f}_P \in \mathcal{F}_\nu^I} g \cap \hat{f}_P \subset \bigcup \mathcal{F}_\nu^I.$$

Outline

1. Extensions to *n*-times differentiable functions
2. Generalized Ulam-Zahorski interpolation problem
3. $\mathbb{SC}$: Simultaneous $\mathbb{S}$mall $\mathbb{C}$overings by smooth functions
4. $\mathbb{SC}$: Reduction to three results
5. $\mathbb{SC}$: Restriction theorems, prism density, and $\mathrm{CPA}_{\mathrm{prism}}$
6. $\mathbb{SC}$: Proof of Theorem A
7. Open problems

Open problems (from both lectures)

- Let $f = f_n \circ \cdots \circ f_1$, where each $f_i \colon [0,1] \to [0,1]$ is a derivative. Must f have a connected graph?
- Characterize, for $n > 1$, all maps $f \colon P \to \mathbb{R}$, where $P \subset \mathbb{R}$ is closed (or just perfect), that admit D^n extensions $\bar{f} \colon \mathbb{R} \to \mathbb{R}$.
- D^1-D^2 interpolation problem: *Does every $f \in D^1(\mathbb{R})$ admits $g \in D^2(\mathbb{R})$ with uncountable $f \cap g$?* This is equivalent to D^1-C^2 interpolation problem.

That is all!

Thank you for your attention!