Differentiability versus continuity: Restriction and extension theorems and monstrous examples

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Based on survey written with Juan B. Seoane–Sepúlveda
Talk 1 of special session on
Different levels of smoothness: Restriction, extension, and covering theorem

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Preamble: new (2017) results that are behind this talk

Example (New simple construction of a classic example)

There exists a differentiable nowhere monotone map $f\colon \mathbb{R} \to \mathbb{R}$.

Example (Greatly simplified construction of 2016 example)

There exists a differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect $\mathfrak{X} \subset \mathbb{R}$ with $\mathfrak{f}' \equiv 0$.

Theorem (C^1 interpolation thm, no Lebesgue measure needed)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$:

- *there is perfect $P \subset \mathbb{R}$ s.t. $f \restriction P$ is Lipschitz;*
- *there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.*

Theorem (Simple proof of Whitney and Jarník extension thms)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.This F is C^1 iff such extension exists iff a simple (new) condition for f holds.

No familiarity

with Lebesgue measure

is needed to follow any proof

behind this talk

Outline

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1. Continuity from differentiability: classical results
2. Continuity from differentiability: newer results
3. Differentiability from continuity: differentiable restrictions
4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$
5. Differentiable extensions: Jarník and Whitney theorems
6. Bonus: Russian connection

Continuity from differentiability: What is it to ask?

Clearly, if $F\colon \mathbb{R} \to \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G\colon \mathbb{C} \to \mathbb{C}$, G' is continuous (due to Cauchy.)

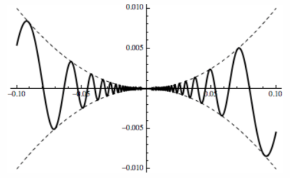
However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin\left(x^{-1}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

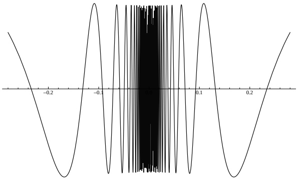
True question: *To what extend $f = F'$ must be continuous?*

About $F(x) = x^2 \sin\left(x^{-1}\right)$

This F appeared already in the 1881 paper of Vito Volterra (1860-1940)



Graph of F



Graph of F'

To what extend $f = F'$ must be continuous?

Jean-Gaston Darboux
(1842-1917)

Theorem (Darboux 1875)

Any derivative $f\colon \mathbb{R} \to \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called Darboux functions.

Baire result

René-Louis Baire
(1874-1932)

Theorem (1899 dissertation of Baire)

The derivative of any differentiable $F\colon \mathbb{R} \to \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a dense G_δ-set.

Proof of previous theorem and a characterization

$F'(x) = \lim_{n\to\infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$ continuous.

For any $g\colon \mathbb{R} \to \mathbb{R}$, $C_g := \{x\colon g \text{ is continuous at } x\}$ is a G_δ-set: $C_g := \bigcap_{n=1}^\infty V_n$, where the open sets V_n are defined as

$$V_n := \bigcup_{\delta>0} \{x \in \mathbb{R}\colon |g(s) - f(g)| < 1/n \text{ for all } s,t \in (x-\delta, x+\delta)\}.$$

If $g = \lim_{n\to\infty} g_n$, $g_n\colon \mathbb{R} \to \mathbb{R}$ continuous, then C_g contains a dense G_δ-set $G := \bigcap_{n=1}^\infty \bigcup_{N=1}^\infty U_N^n$, where each U_N^n is the interior of the closed set

$$\{x \in \mathbb{R}\colon |f_k(x) - f_m(x)| \le 1/n \text{ for all } m,k \ge N\}.$$

Theorem

Let $G \subset \mathbb{R}$.
There exists a derivative f with $C_f = G$ iff G is a dense G_δ.

Outline

Fixed point property

Theorem (Relatively new)

If $f = f_n \circ \cdots \circ f_1$, where each $f_i \colon [0,1] \to [0,1]$ is a derivative, then f has a fixed point.

For $n = 1$: easy exercise, as $h(x) = f(x) - x$ is Darboux.

For $n = 2$: proved independently in 2001
by Csörnyei, O'Neil & Preiss and by Elekes, Keleti & Prokaj.

For arbitrary n: Szuca 2003.

Open Problem

Must f as in the theorem have connected graph?

Yes for $n = 1$. Positive answer would imply the theorem.

Baire classification of composition of the derivatives.

Let $f = f_n \circ \cdots \circ f_1$, where each f_i is a derivative.

Then f is Darboux.

Any Darboux Baire class one map has connected graph.

A natural question: must f be of Baire class 1? NO

Theorem (*Andy* Bruckner and K. Ciesielski 2018)

There exist derivatives $\varphi, \gamma \colon [-1,1] \to [-1,1]$ such that their composition $\psi := \varphi \circ \gamma$ is not of Baire class one.

We use $\gamma(x) := \cos(x^{-1})$ and φ Pompeiu's map, see below.

Problem (could be easy) Find derivatives f_i such that $f = f_n \circ \cdots \circ f_1$ is of Baire class not lower than n.

Differentiable monster (# 1)

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; and many others)

There is differentiable $f \colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x \colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

Arnaud Denjoy and Dimitrie Pompeiu

Arnaud Denjoy (1884–1974)

Dimitrie Pompeiu (1873–1954)

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$*, with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$*.*

(ii) $h = g^{-1} \colon \mathbb{R} \nearrow \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$*.*

Pr. (i) Continuity follows from $|g(x)| \le \sum_{i=1}^{\infty} r^i (|x| + i + 1)$.

Differentiability requires $g'(x) = \sum_{i=1}^{\infty} r^i \frac{1}{3} \frac{1}{(x-q_i)^{2/3}}$. Easy when series $= \infty$. Other case follows from $0 < \frac{\psi_i(y) - \psi_i(x)}{y - x} \le 6\psi_i'(x)$.

(ii) and (iii) easily follow from (i).

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x-t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D.

Any t **in residual** $G = \bigcap_{d \in D} \big((-d + Z) \cap (d - Z)\big)$ works.

Clearly f is differentiable with $f'(x) = h'(x-t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t+d) = h'(d) - h'(t+d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d-t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

Outline

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F\colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

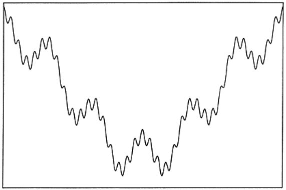
Bernard Bolzano (1781-1848)

Karl Weierstrass (1815–1897)

Weierstrass' Monster: $W(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x)$

Teiji Takagi (1875–1960)

Bartel van der Waerden (1903–1996)



$F(x) = \sum_{n=0}^{\infty} 4^n min\{|x - \frac{k}{8^n}|: k \in \mathbb{Z}\}$

Weierstrass' Monster of

Takagi from 1903, and

van der Waerden, from 1930

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

New proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Theorem (With new (2017/18) simple proof, by KC)

For every continuous increasing $f\colon [a,b] \to \mathbb{R}$ there is perfect P such that $f \restriction P$ is Lipschitz.

Proof based on the following results, due to Riesz:

Lemma (Rising sun lemma 1932, proof is an easy exercise)

If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \le g(d)$ for every component (c,d) of $U = \{x \in [a,b)\colon g(x) < g(y)$ for some $y \in (x,b]\}$.

Fact (Proved by induction)

Let $a < b$ and $\mathcal{J}$ be a family of open intervals with $\bigcup \mathcal{J} \subset (a,b)$.

(i) *If $[\alpha,\beta] \subset \bigcup \mathcal{J}$, then $\sum_{I\in\mathcal{J}} \ell(I) > \beta - \alpha$.*

(ii) *If $I \in \mathcal{J}$ are pairwise disjoint, then $\sum_{I\in\mathcal{J}} \ell(I) \le b - a$.*

Riesz' Rising sun lemma

If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \leq g(d)$ for every component (c,d) of $U = \{x \in [a,b)\colon g(x) < g(y) \text{ for some } y \in (x,b]\}$.

Frigyes Riesz (1880-1956)

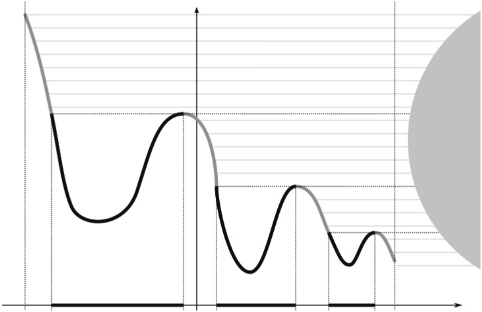


Illustration of the Rising Sun Lemma

The points in the set $U \cap (a,b)$ are those lying in the shadow.

Proof of Lipschitz restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Have: If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \le g(d)$ for every comp. (c,d) of $\{x \in [a,b)\colon g(x) < g(y)$ for some $y \in (x,b]\}$.

Sketch of proof. Fix $L > \frac{f(b)-f(a)}{b-a}$, put $g(t) = f(t) - Lt$, and

$$U = \{x \in [a,b)\colon g(y) > g(x) \text{ for some } y \in (x,b]\}.$$

f is Lipschitz on $P = [\bar{a}, b] \setminus U$ with constant L, where

$\bar{a} = \sup\{x\colon [a,x) \subset U)\}$. Fix $X = \{x_n\colon n \in \mathbb{N}\}$. Need $P \setminus X \neq \emptyset$.

If $\mathcal{J} =$ open components of U, then $\ell(f[J]) \ge L\ell(J)$ for $J \in \mathcal{J}$.

By Fact (ii), $\sum_{J\in\mathcal{J}} \ell(f[J]) \le f(b) - f(\bar{a})$. So,

$\sum_{J\in\mathcal{J}} \ell(J) \le \frac{1}{L}\sum_{J\in\mathcal{J}} \ell(f[J]) \le \frac{f(b)-f(\bar{a})}{L} < b - \bar{a}$, and by Fact (i),

$P \neq \emptyset$. To get $P \setminus X \neq \emptyset$ increase slightly $\mathcal{J}$.

End of proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Have: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Proof of differentiable restriction theorem.

f is Lipschitz on some perfect P: proved above for somewhere monotone f; otherwise f is constant on some perfect set.

For function $f \restriction P$ use Morayne theorem to find perfect $Q \subset P$ such that the quotient map for $f \restriction Q$ is uniformly continuous. Then Q is as needed. □

Outline

Differentiable monster (# 2)

Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- f is *locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

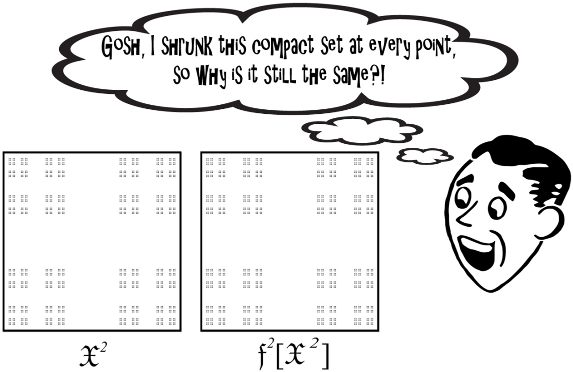


Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

Definition of $\mathfrak{f}$ with $\mathfrak{f}' \equiv 0$, Monster # 2

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma\colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s_i = 1 \text{ for all } i<\omega, \\ \langle 0,0,\ldots,0,1,s_{k+1},s_{k+2},\ldots\rangle & \text{if } s_k = 0 \ \&\ s_i = 1 \text{ for } i<k. \end{cases}$$

$h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

where $N(s \restriction 0) = 1$ and, for $n > 0$,

$$\begin{aligned} N(s\restriction n) &= \sum_{i<n-1} s_i 2^i + (1-s_{n-1})2^{n-1} + 2^n \\ &= (1(1-s_{n-1})s_{n-2}\ldots s_0)_2. \end{aligned}$$

E.g. $N(101101) = (1001101)_2$

Proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Dynamical system $\mathfrak{f}$

Every orbit $\{x, \mathfrak{f}(x), \mathfrak{f}^2(x), \ldots\}$ of $\mathfrak{f}$ is dense in $\mathfrak{X}$.

So, $\mathfrak{f}$ is a minimal dynamical system. Must it be?

Theorem (KC & JJ 2016: YES, essentially)

If $f\colon X \to X$ is onto, PC, and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamical system,

where *f is pointwise contractive, PC*, if for every $x \in X$ there is open $U \ni x$ and $L \in [0,1)$ s.t. $|f(x) - f(y)| \le L|x - y|$ for all $y \in U$.

Outline

Notation

For $J=(a,a+h)$ let $I_J=[a+h/3,a+2h/3]$, middle third of J.

For closed $Q\subset\mathbb{R}$ and $f\colon Q\to\mathbb{R}$ let

$\hat{Q}=Q\cup\bigcup\{I_J\colon J \text{ is a bounded connected component of } \mathbb{R}\setminus Q\}$,

$\bar{f}\colon\mathbb{R}\to\mathbb{R}$ — "the" linear interpolation of f, $\hat{f}=\bar{f}\restriction\hat{Q}$.

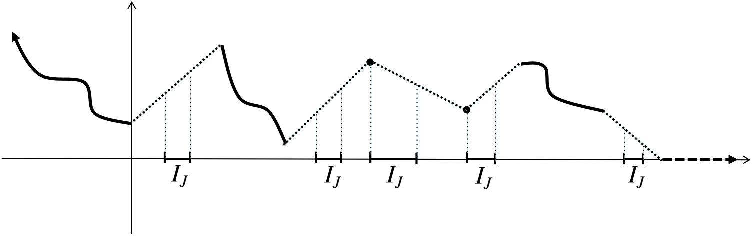


Figure: The linear interpolation $\bar{f}$ of f, represented by thick curves.

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

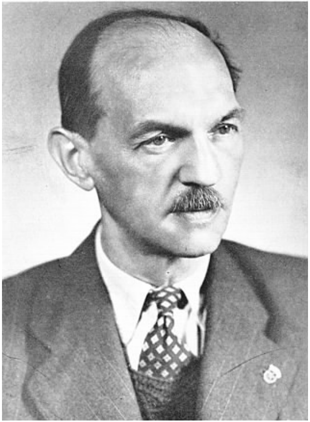
Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.

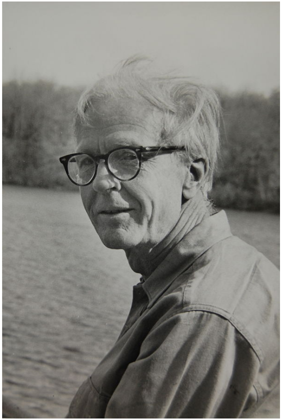
Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Vojtěch Jarník and Hassler Whitney



Vojtěch Jarník (1897–1970)



Hassler Whitney (1907-1989)

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \restriction \hat{Q}$ is continuously differentiable.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not an end-point of a connected component of $\mathbb{R} \setminus Q$.

The right extension: Small modification of $\bar{f}$: $F = \bar{f} + g$:

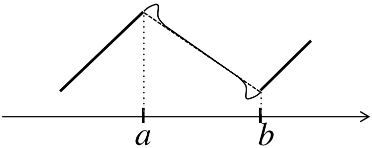


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

Details: elementary. Require some checking.

Differentiable extensions of $\mathfrak{f}$, Monster # 2

By Jarník's theorem, our $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{X}$ can be extended to differentiable $F \colon \mathbb{R} \to \mathbb{R}$. Can such F be C^1?

Theorem (KC & JJ 2016: No)

If $f \colon X \to \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? Yes, very bad!

Theorem (KC & Cheng-Han Pan (Ph.D. student) 2018)

For every closed set $P \subseteq \mathbb{R}$ and differentiable $f \colon P \to \mathbb{R}$, there exists a differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in $\mathbb{R}$, then $\hat{f}$ is monotone on no interval.

Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F\colon \mathbb{R} \to \mathbb{R}$ (i.e., Monster #1)
such that $F \restriction \mathfrak{X} = \mathfrak{f}$ (i.e., Monster #2).

So #3, as #1+ #2 = #3

Proof.

Use previous theorem to $\mathfrak{f}$. □

Outline

Challenge for the audience

Recognize, from a photo, the following well known Russian mathematician.

Do not tell the name aloud, as yet!

Here is the photo

???

Here is the photo, with hint to educated in languages

Николáй Николáевич Лýзин

Mistery unraveled

Nikolai Luzin (1883-1950)

Testing continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$: old result

Theorem (Luzin 1948, textbook in russian)

For every function $f\colon \mathbb{R}^2 \to \mathbb{R}$, if $f(x, h(x))$ is continuous for every $h \in C(\mathbb{R})$, then f is continuous.

For $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$, $f\colon \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{H}$-continuous if $f \restriction h$ is continuous for every $h \in \mathcal{H}$.

Fact. If $h \in C(\mathbb{R})$, then $f \restriction h$ is continuous if, and only if, $f(x, h(x))$ is continuous.

Luzin's result: every $C(\mathbb{R})$-continuous $f(x, y)$ is continuous.

Q. For what other classes $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$,

$\mathcal{H}$-continuity implies continuity?

Testing continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$: new results

For $X \subset \mathbb{R}^2$, $T(X)$ all translations of X.

Theorem (Ciesielski & Rosenblatt 2014)

The statement $\mathcal{H}$-continuity implies continuity

- *is true for $\mathcal{H} =$ "C^1" (derivative can be infinite);*
- *is false for $\mathcal{H} = D^1$;*
- *is false for $\mathcal{H} = T(h)$ for any continuous $h\colon \mathbb{R} \to \mathbb{R}$;*
- *is true for $\mathcal{H} = T(h)$ for some Baire 1 function $h\colon \mathbb{R} \to \mathbb{R}$.*

That is all!

Thank you for your attention!