

Differentiability versus continuity: Restriction and extension theorems and monstrous examples

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Based on survey written with Juan B. Seoane-Sepúlveda

Talk 1 of special session on

Different levels of smoothness: Restriction, extension, and covering theorem

Summer Symposium in Real Analysis XLII, The White
Nights Symposium, Saint-Petersburg, Russia, June 10, 2018

Preamble: new (2017) results that are behind this talk

Example (New simple construction of a classic example)

There exists a differentiable nowhere monotone map $f: \mathbb{R} \rightarrow \mathbb{R}$.

Example (Greatly simplified construction of 2016 example)

There exists a differentiable auto-homeomorphism f of a compact perfect $X \subset \mathbb{R}$ with $f' \equiv 0$.

Theorem (C^1 interpolation thm, no Lebesgue measure needed)

For every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$:

- there is perfect $P \subset \mathbb{R}$ s.t. $f \upharpoonright P$ is Lipschitz;
- there is C^1 map $g: \mathbb{R} \rightarrow \mathbb{R}$ with $f \cap g$ uncountable.

Theorem (Simple proof of Whitney and Jarník extension thms)

If $Q \subset \mathbb{R}$ is closed, then any differentiable $f: Q \rightarrow \mathbb{R}$ has differentiable extension $F: \mathbb{R} \rightarrow \mathbb{R}$. This F is C^1 iff such extension exists iff a simple (new) condition for f holds.

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- 1 Continuity from differentiability: classical results
- 2 Continuity from differentiability: newer results
- 3 Differentiability from continuity: differentiable restrictions
- 4 Properties of differentiable maps on perfect $P \subset \mathbb{R}$
- 5 Differentiable extensions: Jarník and Whitney theorems
- 6 Bonus: Russian connection

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Continuity from differentiability: What is it to ask?

Clearly, if $F: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, then F is continuous.

For differentiable $G: \mathbb{C} \rightarrow \mathbb{C}$, G' is continuous (due to Cauchy.)

However, F' need not be continuous, e.g., for

$$F(x) := \begin{cases} x^2 \sin(x^{-1}) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

True question: *To what extent $f = F'$ must be continuous?*

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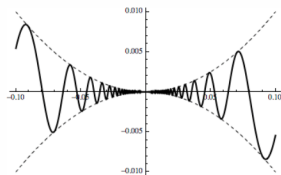
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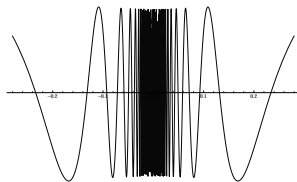
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Graph of F

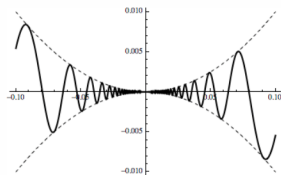


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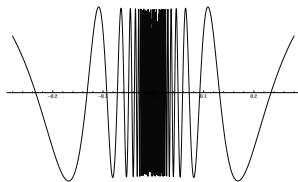
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Theorem (Darboux 1875)

Any derivative $f: \mathbb{R} \rightarrow \mathbb{R}$ has the intermediate value property (IVP), that is, for every $a < b$ and y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ with $f(x) = y$.

Since then, maps with IVP are called **Darboux functions**.

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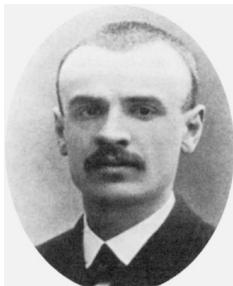
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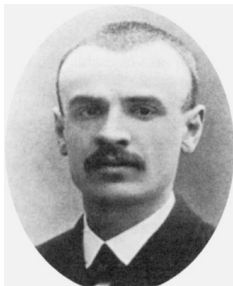


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Theorem (1899 dissertation of Baire)

The derivative of any differentiable $F: \mathbb{R} \rightarrow \mathbb{R}$ is Baire class one, that is, it is a pointwise limit of continuous functions. In particular, the set of points of continuity of F' (as for any Baire class one function) is a dense G_δ -set.

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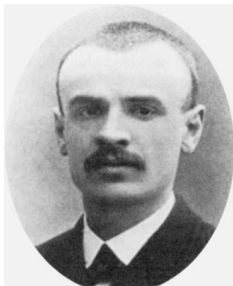


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Proof of previous theorem and a characterization

$F'(x) = \lim_{n \rightarrow \infty} F_n(x)$, with $F_n(x) := \frac{f(x+1/n) - f(x)}{1/n}$ continuous.

For any $g: \mathbb{R} \rightarrow \mathbb{R}$, $C_g := \{x: g \text{ is continuous at } x\}$ is a G_δ -set:

$C_g := \bigcap_{n=1}^{\infty} V_n$, where the open sets V_n are defined as

$V_n := \bigcup_{\delta > 0} \{x \in \mathbb{R}: |g(s) - f(g)| < 1/n \text{ for all } s, t \in (x - \delta, x + \delta)\}$.

If $g = \lim_{n \rightarrow \infty} g_n$, $g_n: \mathbb{R} \rightarrow \mathbb{R}$ continuous, then C_g contains a

dense G_δ -set $G := \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} U_N^n$, where each U_N^n is the interior of the closed set

$$\{x \in \mathbb{R}: |f_k(x) - f_m(x)| \leq 1/n \text{ for all } m, k \geq N\}.$$

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Let $G \subset \mathbb{R}$.

There exists a derivative f with $C_f = G$ iff G is a dense G_δ .

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Fixed point property

Theorem (Relatively new)

If $f = f_n \circ \cdots \circ f_1$, where each $f_i: [0, 1] \rightarrow [0, 1]$ is a derivative, then f has a fixed point.

For $n = 1$: easy exercise, as $h(x) = f(x) - x$ is Darboux.

For $n = 2$: proved independently in 2001 by Csörnyei, O'Neil & Preiss and by Elekes, Keleti & Prokaj.

For arbitrary n : Szuca 2003.

Open Problem

Must f as in the theorem have connected graph?

Yes for $n = 1$. Positive answer would imply the theorem.

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Baire classification of composition of the derivatives.

Let $f = f_n \circ \cdots \circ f_1$, where each f_i is a derivative.

Then f is Darboux.

Any Darboux Baire class one map has connected graph.

A natural question: must f be of Baire class 1? **NO**

Theorem (Andy Bruckner and K. Ciesielski 2018)

There exist derivatives $\varphi, \gamma: [-1, 1] \rightarrow [-1, 1]$ such that their composition $\psi := \varphi \circ \gamma$ is not of Baire class one.

We use $\gamma(x) := \cos(x^{-1})$ and φ Pompeiu's map, see below.

Problem (could be easy) Find derivatives f_i such that $f = f_n \circ \cdots \circ f_1$ is of Baire class not lower than n .

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Baire classification of composition of the derivatives.

Let $f = f_n \circ \cdots \circ f_1$, where each f_i is a derivative.

Then f is Darboux.

Any Darboux Baire class one map has connected graph.

A natural question: must f be of Baire class 1? **NO**

Theorem (Andy Bruckner and K. Ciesielski 2018)

There exist derivatives $\varphi, \gamma: [-1, 1] \rightarrow [-1, 1]$ such that their composition $\psi := \varphi \circ \gamma$ is not of Baire class one.

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Differentiable monster (# 1)

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; Aron, Gurariy & Seoane-Sepúlveda 2005; and many others)

There is differentiable $f: \mathbb{R} \rightarrow \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x: f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

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Arnaud Denjoy and Dimitrie Pompeiu



Arnaud Denjoy (1884–1974)



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A variant of Pompeiu function, of 1907

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Lemma (KC; small variation of Pompeiu's result)

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New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h: \mathbb{R} \rightarrow \mathbb{R}$ with $Z = \{x \in \mathbb{R}: h'(x) = 0\}$ being a dense G_δ -set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x - t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. So, $h' > 0$ on D .

Any t in residual $G = \bigcap_{d \in D} ((-d + Z) \cap (d - Z))$ works.

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Outline

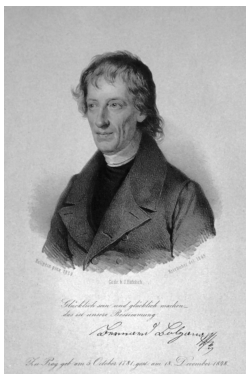
- 1 Continuity from differentiability: classical results
- 2 Continuity from differentiability: newer results
- 3 Differentiability from continuity: differentiable restrictions**
- 4 Properties of differentiable maps on perfect $P \subset \mathbb{R}$
- 5 Differentiable extensions: Jarník and Whitney theorems
- 6 Bonus: Russian connection

How much differentiability continuous map must have

None?

Example (Weierstrass 1886; Bolzano, unpublished, 1822)

There exists continuous $F: \mathbb{R} \rightarrow \mathbb{R}$ differentiable at no point.



Bernard Bolzano (1781-1848)



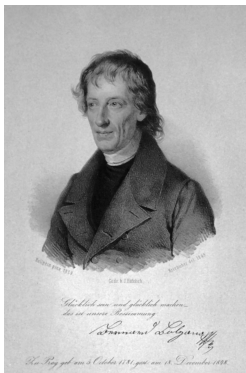
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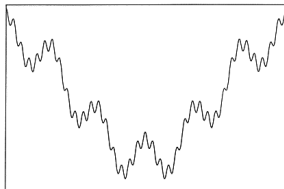
Weierstrass' Monster: $W(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(13^n \pi x)$



Teiji Takagi (1875–1960)



Bartel van der Waerden
(1903–1996)



$$F(x) = \sum_{n=0}^{\infty} 4^n \min\{|x - \frac{k}{8^n}| : k \in \mathbb{Z}\}$$

Weierstrass' Monster of
Takagi from 1903, and
van der Waerden, from 1930

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \upharpoonright Q$ is differentiable.

Remark

There are continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f \upharpoonright Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2): [0, 1] \rightarrow [0, 1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and \mathcal{I} -approximately differentiable. So it is as in the remark. □

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Some differentiability after all!

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New proof of differentiable restriction theorem

Goal: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is cont, then $f \upharpoonright Q$ is diff. for some perfect Q .

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For every continuous increasing $f: [a, b] \rightarrow \mathbb{R}$ there is perfect P such that $f \upharpoonright P$ is Lipschitz.

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Let $a < b$ and \mathcal{J} be a family of open intervals with $\bigcup \mathcal{J} \subset (a, b)$.

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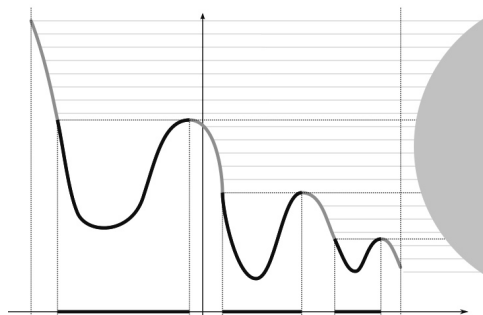


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The points in the set $U \cap (a, b)$ are those lying in the shadow.

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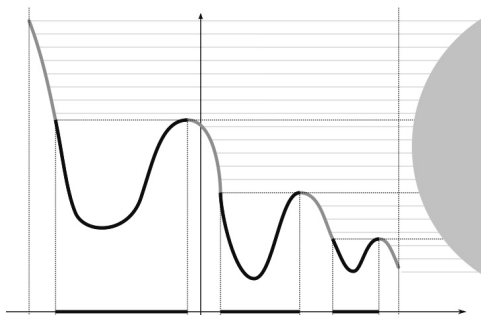


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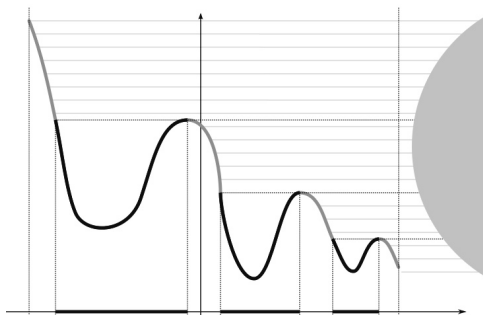


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By Fact (ii), $\sum_{\mathcal{J} \in \mathcal{J}} \ell(f[\mathcal{J}]) \leq f(b) - f(\bar{a})$. So,

$$\sum_{\mathcal{J} \in \mathcal{J}} \ell(\mathcal{J}) \leq \frac{1}{L} \sum_{\mathcal{J} \in \mathcal{J}} \ell(f[\mathcal{J}]) \leq \frac{f(b)-f(\bar{a})}{L} < b - \bar{a}, \text{ and by Fact (i),}$$

$P \neq \emptyset$. To get $P \setminus X \neq \emptyset$ increase slightly \mathcal{J} .

End of proof of differentiable restriction theorem

Goal: If $f: \mathbb{R} \rightarrow \mathbb{R}$ is cont, then $f \upharpoonright Q$ is diff. for some perfect Q .

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Proof of differentiable restriction theorem.

f is Lipschitz on some perfect P : proved above for somewhere monotone f ; otherwise f is constant on some perfect set.

For function $f \upharpoonright P$ use Morayne theorem to find perfect $Q \subset P$ such that the quotient map for $f \upharpoonright Q$ is uniformly continuous. Then Q is as needed. \square

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Outline

- 1 Continuity from differentiability: classical results
- 2 Continuity from differentiability: newer results
- 3 Differentiability from continuity: differentiable restrictions
- 4 Properties of differentiable maps on perfect $P \subset \mathbb{R}$**
- 5 Differentiable extensions: Jarník and Whitney theorems
- 6 Bonus: Russian connection

Differentiable monster (# 2)

Are differentiable $f: P \rightarrow \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good? **Not at all!**

Example (Ciesielski & Jasinski 2016; simplified by KC in 2017)

There exists differentiable auto-homeomorphism f of a compact perfect subset \mathfrak{X} of the Cantor ternary set \mathfrak{C} such that $f' \equiv 0$.

Counterintuitive, as f is shrinking at every $x \in \mathfrak{X}$

($|f(x) - f(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)

but it maps compact \mathfrak{X} **onto** itself. Also

Theorem (Edelstein 1962, almost contradicting above thm)

If $f: X \rightarrow X$ is LC and X is compact, then f has a periodic point,

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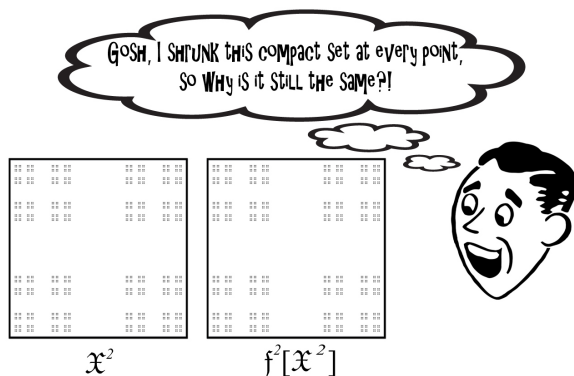


Figure: The result of the action of $f^2 = \langle f, f \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

Definition of f with $f' \equiv 0$, Monster # 2

$f = h \circ \sigma \circ h^{-1}$, where $h: 2^\omega \rightarrow \mathbb{R}$ is embedding and $\sigma: 2^\omega \rightarrow 2^\omega$ is the “add one and carry” adding machine:

$$\sigma(s) = \begin{cases} \langle 0, 0, 0, \dots \rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0, 0, \dots, 0, 1, s_{k+1}, s_{k+2}, \dots \rangle & \text{if } s_k = 0 \text{ \& } s_i = 1 \text{ for } i < k. \end{cases}$$

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E.g. $N(101101) = (1001101)_2$

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Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)} N(s \upharpoonright n)$,

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as it fails only for $s = \langle s_0, \dots, s_{n-2}, s_{n-1}, \dots \rangle = \langle 1, \dots, 1, 0, \dots \rangle$.

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To see $f'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t.

$0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega : s_i \neq t_i\} > k$. Then,

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Dynamical system f

Every orbit $\{x, f(x), f^2(x), \dots\}$ of f is dense in \mathfrak{X} .

So, f is a minimal dynamical system. **Must it be?**

Theorem (KC & JJ 2016: YES, essentially)

*If $f: X \rightarrow X$ is onto, PC, and X is infinite compact, then there is a **perfect** $P \subset X$ s.t. $f \upharpoonright P$ is a minimal dynamical system,*

where f is **pointwise contractive, PC**, if for every $x \in X$ there is open $U \ni x$ and $L \in [0, 1)$ s.t. $|f(x) - f(y)| \leq L|x - y|$ for all $y \in U$.

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Outline

- 1 Continuity from differentiability: classical results
- 2 Continuity from differentiability: newer results
- 3 Differentiability from continuity: differentiable restrictions
- 4 Properties of differentiable maps on perfect $P \subset \mathbb{R}$
- 5 Differentiable extensions: Jarník and Whitney theorems**
- 6 Bonus: Russian connection

Notation

For $J = (a, a + h)$ let $I_J = [a + h/3, a + 2h/3]$, middle third of J .

For closed $Q \subset \mathbb{R}$ and $f: Q \rightarrow \mathbb{R}$ let

$\hat{Q} = Q \cup \bigcup \{I_J : J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}$,

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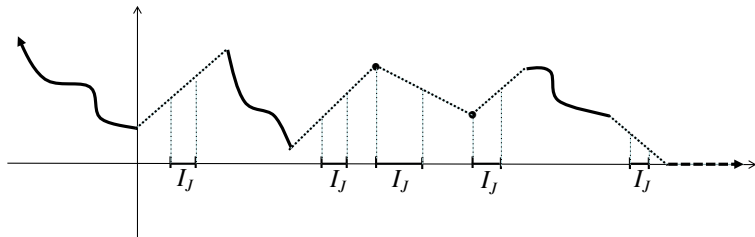


Figure: The linear interpolation \bar{f} of f , represented by thick curves.

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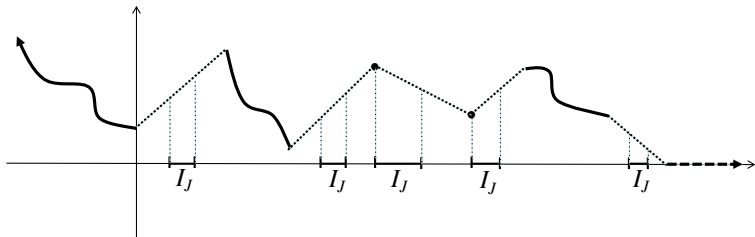


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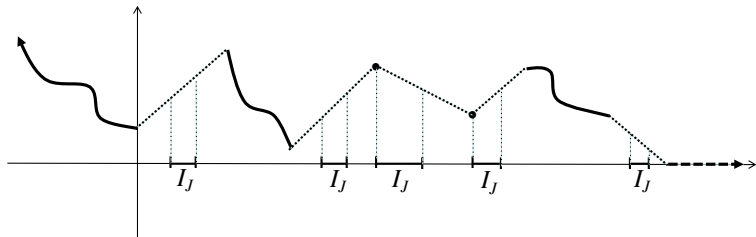


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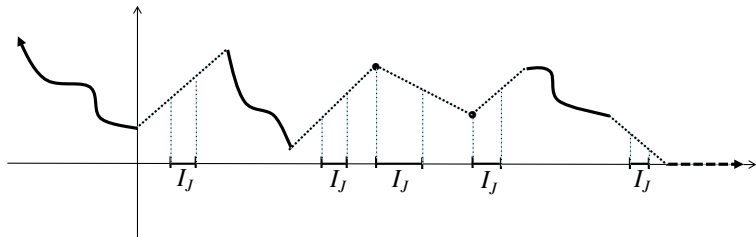


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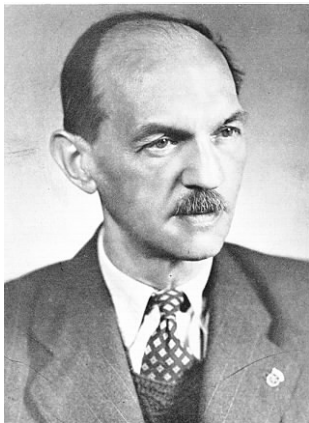
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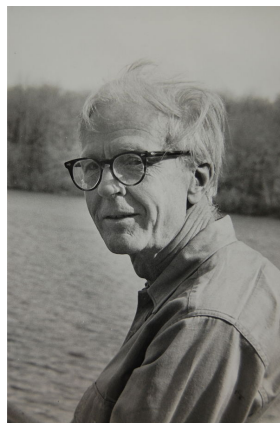
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Theorem (Jarník and Whitney thms, version of **MC&KC 2017**)

If $Q \subset \mathbb{R}$ is closed, then any differentiable $f: Q \rightarrow \mathbb{R}$ has differentiable extension $F: \mathbb{R} \rightarrow \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f} = \bar{f} \upharpoonright \hat{Q}$ is continuously differentiable.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f: \mathbb{R} \rightarrow \mathbb{R}$ there is C^1 map $g: \mathbb{R} \rightarrow \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \upharpoonright Q$ is uniformly continuous.

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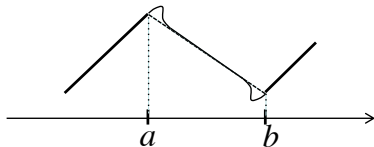


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

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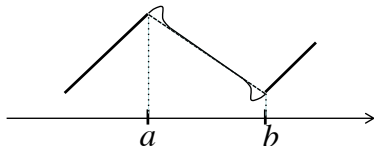


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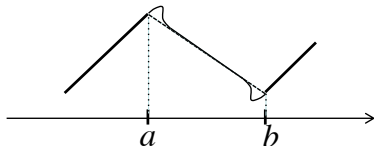


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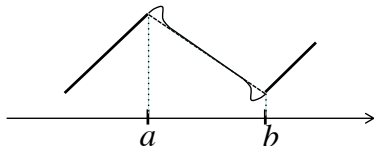


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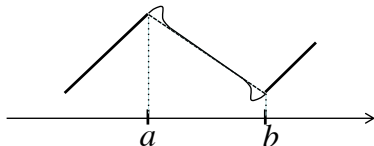


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For every closed set $P \subseteq \mathbb{R}$ and differentiable $f: P \rightarrow \mathbb{R}$, there exists a differentiable extension $F: \mathbb{R} \rightarrow \mathbb{R}$ of f such that F is nowhere monotone on $\mathbb{R} \setminus P$. In particular, if P is nowhere dense in \mathbb{R} , then \hat{f} is monotone on no interval.

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Differentiable extensions of f , Monster # 2

By Jarník's theorem, our $f: \mathfrak{X} \rightarrow \mathfrak{X}$ can be extended to differentiable $F: \mathbb{R} \rightarrow \mathbb{R}$. Can such F be C^1 ?

Theorem (KC & JJ 2016: No)

If $f: X \rightarrow \mathbb{R}$ is differentiable with $|f'| < 1$ on X and f has a C^1 extension, then $X \not\subseteq f[X]$.

Can such F can be bad? **Yes, very bad!**

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Differentiable monster (#3)

Example (Ciesielski & Cheng-Han Pan (Ph.D. student) 2018)

There exists everywhere differentiable nowhere monotone function $F: \mathbb{R} \rightarrow \mathbb{R}$ (i.e., Monster #1) such that $F \upharpoonright \mathfrak{X} = f$ (i.e., Monster #2).

So #3, as #1 + #2 = #3

Proof.

Use previous theorem to f . □

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Outline

- 1 Continuity from differentiability: classical results
- 2 Continuity from differentiability: newer results
- 3 Differentiability from continuity: differentiable restrictions
- 4 Properties of differentiable maps on perfect $P \subset \mathbb{R}$
- 5 Differentiable extensions: Jarník and Whitney theorems
- 6 Bonus: Russian connection

Challenge for the audience

Recognize, from a photo, the following well known Russian mathematician.

Do not tell the name aloud, as yet!

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Here is the photo



???

Here is the photo, with hint to educated in languages



Никола́й Никола́евич Лу́зин

Mystery unraveled



Nikolai Luzin (1883-1950)

Testing continuity of $f: \mathbb{R}^2 \rightarrow \mathbb{R}$: old result

Theorem (Luzin 1948, textbook in russian)

For every function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, if $f(x, h(x))$ is continuous for every $h \in C(\mathbb{R})$, then f is continuous.

For $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is \mathcal{H} -continuous if $f \upharpoonright h$ is continuous for every $h \in \mathcal{H}$.

Fact. If $h \in C(\mathbb{R})$, then $f \upharpoonright h$ is continuous if, and only if, $f(x, h(x))$ is continuous.

Luzin's result: every $C(\mathbb{R})$ -continuous $f(x, y)$ is continuous.

Q. For what other classes $\mathcal{H} \subset \mathbb{R}^{\mathbb{R}}$,

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Testing continuity of $f: \mathbb{R}^2 \rightarrow \mathbb{R}$: new results

For $X \subset \mathbb{R}^2$, $T(X)$ all translations of X .

Theorem (Ciesielski & Rosenblatt 2014)

The statement \mathcal{H} -continuity implies continuity

- *is true for $\mathcal{H} = "C^1"$ (derivative can be infinite);*
- *is false for $\mathcal{H} = D^1$;*
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