Less than 2^ω-many continuous functions that almost cover every continuous function

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Credits: This presentation is based on the papers

(1) K.C. Ciesielski and J.B. Seoane-Sepulveda, *Simultaneous small coverings by smooth functions under the covering property axiom*, submitted, available here.
(2) K.C. Ciesielski, J.L. Gamez-Merino, T. Natkaniec, and J.B. Seoane-Sepulveda, *On functions that are almost continuous and perfectly everywhere surjective but not Jones. Lineability and additivity*, Topology Appl. 235 (2018), 73–82, available here.

Will also rely on a material in the book

- K. Ciesielski and J. Pawlikowski, *Covering Property Axiom CPA. A combinatorial core of the iterated perfect set model*, Cambridge Univ. Press, 2004.

This presentation is available here.

Outline

The main theorem

For $X \subset \mathbb{R}$: $D^n(X)$—all n-times differentiable $f\colon X \to \mathbb{R}$;
$C^n(X)$—all $f \in D^n(X)$ with continuous nth derivatives $f^{(n)}$;
$D^0(X) = C^0(X)$; $A \subset^\star B$ denotes $|A \setminus B| < \mathfrak{c}$.

Theorem (Main Thm)

It is consistent with ZFC, it follows from the Covering Property Axiom CPA, that for every $n < \omega$ there exists a family $\mathcal{F}_n \subset C^n(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that

(i) $g \subset^\star \bigcup \mathcal{F}_n$ *for every* $g \in D^n(\mathbb{R})$.

Moreover, there is $\mathcal{F}_\infty \subset C^\infty(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that

(ii) $g \subset^\star \bigcup \mathcal{F}_\infty$ *for every* $g \in C^\infty(\mathbb{R})$.

Main Thm for $n = 0$: continuous functions

Main Thm for $n = 0$ can be stated: CPA implies

$$I_0\colon\ \exists \mathcal{F}_0 \in [C^0(\mathbb{R})]^{<\mathfrak{c}}\ \ \forall g \in C^0(\mathbb{R})\ \ \ g \subset^\star \bigcup \mathcal{F}_0$$

What I_0 means:

- Few functions from $\mathcal{F}_0$ cover* **every** continuous function
- Few functions from $\mathcal{F}_0$ cover* **every** level $\mathbb{R} \times \{y\}$
- $\subset^\star$ in I_0 cannot be $\subset$, as $\mathcal{F}_0$ cannot cover $\mathbb{R}^2$
- $I_0 \implies \mathrm{cov}(\mathrm{Meager}) < \mathfrak{c}$
  Proof: Pick $y \in \mathbb{R} \setminus \bigcup_{f\in\mathcal{F}_0} f[\mathbb{Q}]$ and put $g = \mathbb{R} \times \{y\}$.
  Then $\mathbb{R}$ is a union of $|\mathcal{F}_0|$-many nowhere dense sets $[f = g]$ and $|g \setminus \bigcup \mathcal{F}_0|$-many singletons, while $|\mathcal{F}_0| + |g \setminus \bigcup \mathcal{F}_0| < \mathfrak{c}$.
- So, I_0 contradicts CH and MA.

I_0 and the size of $\mathfrak{c}$

Fact (Proved for this talk. Known?)

$I_0 \implies \exists \mathcal{F}_0 \in [\mathbb{R}^{\mathbb{R}}]^{<\mathfrak{c}}\ \forall y \in \mathbb{R}\ \ \mathbb{R} \times \{y\} \subset^\star \bigcup \mathcal{F}_0 \implies \mathfrak{c} = |\mathcal{F}_0|^+$

Proof: Let $\kappa = |\mathcal{F}_0|^+$ and assume $\mathfrak{c} > \kappa$. Put $B = \mathbb{R}^2 \setminus \bigcup \mathcal{F}_0$, $B^y = \{x\colon \langle x, y\rangle \in B\}$, and note that $|B^y| < \mathfrak{c}$ for all $y \in \mathbb{R}$.

Claim: There is $Y \in [\mathbb{R}]^\kappa$ with $|\bigcup_{y\in Y} B^y| < \mathfrak{c}$.

Pr. If $\mathrm{cof}(\mathfrak{c}) > \kappa$, then any $Y \in [\mathbb{R}]^\kappa$ works. If $\mathrm{cof}(\mathfrak{c}) \leq \kappa$, choose cofinal $L \in [\mathfrak{c}]^\kappa$; there is $\lambda \in L$ with $Z_\lambda = \{y \in \mathbb{R}\colon |B^y| \leq \lambda\}$ of cardinality $> \kappa$. (Otherwise $\mathfrak{c} = |\bigcup_{\lambda\in L} Z_\lambda| \leq \kappa$.) Then any $Y \in [Z_\lambda]^\kappa$ works. □

Now, by Claim, there are $x_0 \in \mathbb{R} \setminus \bigcup_{y\in Y} B^y$ and $y_0 \in Y \setminus \{f(x_0)\colon f \in \mathcal{F}_0\}$. Then $\langle x_0, y_0\rangle \notin B \cup \bigcup \mathcal{F}_0 = \mathbb{R}^2$, a contradiction.

Outline

1. The theorem and its place in ZFC
2. Implications for (partial) maps between Polish spaces
3. Main Theorem for differentiable functions
4. Proof of Main Thm for $n = 0$
5. Sketch of proof of Main Thm for differentiable maps
6. Open problem

Small cover* of $C(X)$, $X \subset \mathbb{R}$ arbitrary.

$Borel(X,Y)$—all Borel measurable $f\colon X \to Y$

Fact

CPA implies that if $\mathcal{F}_0$ is as in I_0, then
$B^\star$: *$g \subset^\star \bigcup \mathcal{F}_0$ for every $X \subset \mathbb{R}$ and $g \in Borel(X,\mathbb{R})$.*

Proof: Kuratowski: there is Borel extension $G\colon \mathbb{R} \to \mathbb{R}$ of g.

KC & Pawlikowski: CPA implies that

$$B\colon \forall G \in Borel(\mathbb{R},\mathbb{R})\ \exists \mathcal{F}_G \in [C^0(\mathbb{R})]^{\omega_1} \text{ s.t. } G \subset \bigcup \mathcal{F}_G.$$

So, $g \subset G \subset \bigcup \mathcal{F}_G \subset^\star \bigcup \mathcal{F}_0$. □

Note: $B^\star \implies B$.

Small cover$^\star \subset C_K(X,Y)$, X and Y Polish spaces

Let $C_K(X,Y) = \bigcup\{C(K,Y)\colon K \subset X \text{ is compact }\}$

Corollary (Proved just for this talk.)

CPA implies that for every Polish X and Y the following holds

I_{XY}: $\exists \mathcal{F}_{X,Y} \in [C_K(X,Y)]^{<\mathfrak{c}}\ \forall g \in Borel(X,Y)\ g \subset^\star \bigcup \mathcal{F}_{X,Y}$

We can also have $\mathcal{F}_{X,Y} \subset Borel(X,Y)$ and,
if Y is an absolute extensor, $\mathcal{F}_{X,Y} \subset C(X,Y)$.

Proof: KC & Pawlikowski: CPA implies that there are compact 0-dimensional $\{X_\xi\}_{\xi<\omega_1}$ and $\{Y_\xi\}_{\xi<\omega_1}$ covering X and Y.

By previous Fact, for all $\zeta, \xi < \omega_1$ (embedding X_ξ, $Y_\zeta \hookrightarrow \mathbb{R}$)

- $\exists \mathcal{F}_{\zeta\xi} \in [C^0(\mathbb{R})]\ \forall g \in Borel(X_\zeta, Y_\xi)\quad g \subset^\star \bigcup \mathcal{F}_{\zeta\xi}$

Then $\mathcal{F}_{X,Y} = \{f \cap (X_\zeta \times Y_\xi)\colon \zeta, \xi < \omega_1\ \&\ f \in \mathcal{F}_{\zeta\xi}\}$ works. $\square$

More on small coverings of $C(X, Y)$

I_{XY}: $\exists \mathcal{F}_{X,Y} \in [C_K(X, Y)]^{<\mathfrak{c}}\ \forall g \in Borel(X, Y)\ g \subset^\star \bigcup \mathcal{F}_{X,Y}$

CPA implies: for every Polish X and Y, Y absolute extensor,

$$(\star)\quad \exists \mathcal{F}_{X,Y} \in [C(X, Y)]^{<\mathfrak{c}}\ \forall g \in C(X, Y)\ g \subset^\star \bigcup \mathcal{F}_{X,Y}$$

This fails for $X = [0, 1] \times 2^\omega$ and $Y = 2^\omega$.

Problem (Not investigated so far)

For what other spaces X and Y $(\star)$ or I_{XY} hold?

Outline

Families $\mathcal{F}_n$ for $n > 0$

Main Thm for $n \in \mathbb{N}$ can be stated: CPA implies

I_n: $\exists \mathcal{F}_n \in [C^n(\mathbb{R})]^{<\mathfrak{c}} \ \forall g \in D^n(\mathbb{R}) \quad g \subset^\star \bigcup \mathcal{F}_n$

Clearly I_n implies

J_n: $\forall g \in D^n(\mathbb{R})\ \exists \mathcal{F}_g \in [C^n(\mathbb{R})]^{<\mathfrak{c}}\ s.t.\ g \subset \bigcup \mathcal{F}_g$

CPA $\Longrightarrow$ J_n was first "proved" by KC & Pawlikowski [CPA book]

For $n > 1$ their proof was incorrect!

Thus, the proof from submitted paper is the first correct one.

Can we prove stronger versions of I_n, $n \in \mathbb{N}$?

$I_n = I(D^n, C^n)$: $\exists \mathcal{F}_n \in [C^n(\mathbb{R})]^{<\mathfrak{c}}\ \forall g \in D^n(\mathbb{R})\quad g \subset^\star \bigcup \mathcal{F}_n$

Fact: $I(C^{n-1}, D^n)$: $\exists \mathcal{F} \in [D^n(\mathbb{R})]^{<\mathfrak{c}}\ \forall g \in C^{n-1}(\mathbb{R})\quad g \subset^\star \bigcup \mathcal{F}_n$ is false for all $n \in \mathbb{N}$.

Pr: For $n=1$, there is $g_1 \in$ "$D^1(\mathbb{R})$"$\subset C^0(\mathbb{R})$ with $g_1' = \infty$ on a perfect P; so $|[f = g] \cap P| \le \omega$ for every $f \in D^0(\mathbb{R})$.

For $n=2$ use $g_2 = \int g_1, \ldots$ $\square$.

Fact: I_n^*: $\forall X \subset \mathbb{R}\ \exists \mathcal{F} \in [C^n(\mathbb{R})]^{<\mathfrak{c}}\ \forall g \in D^n(X)\quad g \subset^\star \bigcup \mathcal{F}_g$ is false for $n>1$.

Pr. Put $n=2$ and $\mathfrak{C}$—the Cantor ternary set. There is (simple) $f \in C^1(\mathbb{R})$ such that $g \restriction \mathfrak{C} \in D^2(\mathfrak{C})$ and $|[f = g] \cap \mathfrak{C}| < \omega$ for every $f \in D^2(\mathbb{R})$. So, $g \restriction \mathfrak{C}$ contradicts I_2^*. $\square$

Cor to Main Thm: CPA implies I_0^* & I_1^*. Argument: time permitting.

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The structure of CPA ($\mathrm{CPA}_{\mathrm{prism}}$ part that we use)

For $0<\alpha<\omega_1$ let Φ_α—all continuous 1-1 maps $f\colon \mathfrak{C}^\alpha\to\mathfrak{C}^\alpha$ s.t.

$$f(x)\restriction\xi=f(y)\restriction\xi \;\Leftrightarrow\; x\restriction\xi=y\restriction\xi \quad \text{for all } \xi<\alpha \text{ and } x,y\in\mathfrak{C}^\alpha$$

and $\mathbb{P}_\alpha=\{f[\mathfrak{C}^\alpha]\colon f\in\Phi_\alpha\}$. Also let $\mathbb{P}=\bigcup_{0<\alpha<\omega_1}\mathbb{P}_\alpha$.

$\mathrm{Perf}(X)$—all $P\subset X$ homeomorphic to $\mathfrak{C}$

Prism in X—any $P\in\mathrm{Perf}(X)$ with (implicit) continuous injection h from an $E\in\mathbb{P}$ onto P.

Subprism of a prism P given by $h\colon E\to P$—any $Q=h[E']$, with $E'\in\mathbb{P}$, $E'\subset E$.

$\mathcal{E}\subset\mathrm{Perf}(X)$ is $\mathcal{F}_{\mathrm{prism}}$-dense provided for every prism P in $\mathrm{Perf}(X)$ there exists a subprism Q of P with $Q\in\mathcal{E}$.

$\mathrm{CPA}_{\mathrm{prism}}$ $\mathfrak{c}=\omega_2$ and for every Polish space X and every $\mathcal{F}_{\mathrm{prism}}$-dense family $\mathcal{E}\subset\mathrm{Perf}(X)$ there is $\mathcal{E}_0\subset\mathcal{E}$ such that $|\mathcal{E}_0|\le\omega_1$ and $|X\setminus\bigcup\mathcal{E}_0|\le\omega_1$.

A simple reduction of I_0

If

$I_0(Z)$: $\exists \mathcal{F}_Z \in [C^0(\mathbb{R})]^{<\mathfrak{c}}\ \ \forall g \in C^0(Z)\ \ g \subset^\star \bigcup \mathcal{F}_Z$

holds for every compact perfect $Z \subset \mathbb{R}$, then

$\mathcal{F}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{F}_{[-n,n]}$ satisfies I_0.

Thus it is enough to show that, every compact perfect $Z \subset \mathbb{R}$,

CPA implies $I_0(Z)$.

We will apply CPA to Polish space $Z \times C(Z)$,

$C(Z)$ considered with the uniform convergence topology.

$\mathcal{F}_{\rm prism}$-density lemma

Lemma

For every Polish spaces X and Y, the family

$$\mathcal{E} = \{P \in \mathrm{Perf}(X \times Y)\colon \ \textit{either } \pi_1 \restriction P \textit{ or } \pi_2 \restriction P \textit{ is one-to-one}\}$$

is $\mathcal{F}_{\rm prism}$-dense, where π_1 and π_2 are the projections of $X \times Y$ onto X and Y, respectively.

For $X = Y = \mathbb{R}$ the lemma is proved in [CPA book].

The generalization is straightforward.

Lemma will be used to $X \times Y = Z \times C(Z)$,
with $Z \subset \mathbb{R}$ compact perfect.

$\mathrm{CPA}_{\mathrm{prism}}$ implies $I_0(Z)$

$$\mathcal{E}=\{P\in \mathrm{Perf}(Z\times C(Z))\colon \text{ either } \pi_1\restriction P \text{ or } \pi_2\restriction P \text{ is one-to-one}\}$$

is $\mathcal{F}_{\mathrm{prism}}$-dense. So, by $\mathrm{CPA}_{\mathrm{prism}}$,

there is $\mathcal{E}_0\subset\mathcal{E}$ such that $|\mathcal{E}_0|\le\omega_1$ and $|Z\times C(Z)\setminus\bigcup\mathcal{E}_0|\le\omega_1$.

Let $\mathcal{F}=\{P\in\mathcal{E}_0\colon \pi_1\restriction P \text{ is one-to-one}\}$.

So, for every $P\in\mathcal{F}$, $P\in C(\pi_1[P],C(Z))$ is a continuous map,

$f_P\colon \pi_1[P]\to\mathbb{R}$ defined as $f_P(x)=P(x)(x)$ is continuous and,

by **Tietze's** Extension Theorem, can be extended to $\hat{f}_P\in C(\mathbb{R})$.

Claim: $\mathcal{F}_Z=\{\hat{f}_P\colon P\in\mathcal{F}\}$ satisfies $I_0(Z)$. (Clearly $|\mathcal{F}_Z|\le\omega_1$.)

Why $\mathcal{F}_Z = \{\hat{f}_P \colon P \in \mathcal{F}\}$ satisfies $I_0(Z)$?

$\mathcal{E}_0 \subset \{P \in \mathrm{Perf}(Z \times C(Z)) \colon$ either $\pi_1 \restriction P$ or $\pi_2 \restriction P$ is one-to-one$\}$

$|Z \times C(Z) \setminus \bigcup \mathcal{E}_0| \leq \omega_1$, $\mathcal{F} = \{P \in \mathcal{E}_0 \colon \pi_1 \restriction P$ is one-to-one$\}$

Fix $g \in C(Z)$ and note that $|(Z \times \{g\}) \setminus \bigcup \mathcal{F}| \leq \omega_1$.

Fix $x \in Z$ s.t. $\langle x, g\rangle \in \bigcup \mathcal{F}$ and $P \in \mathcal{F}$ with $\langle x, g\rangle \in P$.

It is enough to show that $\langle x, g(x)\rangle \in f_P$, as $f_P \subset \hat{f}_P$.

Indeed, $f_P(x) = P(x)(x) = g(x)$, as $P(x) = g$ by $\langle x, g\rangle \in P$.

So, $g \restriction \pi_1[(Z \times \{g\}) \cap \bigcup \mathcal{F}] \subset \bigcup \mathcal{F}_Z$, as needed. □

Outline

Reduction of I_n, $n>0$, to simpler cases

It is enough to prove that, for every $Z=[a,b]$,

$I_n^*(Z)$: $\exists \mathcal{F}_Z \in [C^n(\mathbb{R})]^{<\mathfrak{c}}\ \ \forall g \in D^n(Z)\ \ \ g \subset^\star \bigcup \mathcal{F}_Z$

Idea: repeat $n=0$ argument for space $Z \times D^n(Z)$

Problem: $D^n(Z)$ is not a Polish space

Solution: show, under CPA, the following two statements

$I_n(Z)$: $\exists \mathcal{F}_Z \in [C^n(\mathbb{R})]^{<\mathfrak{c}}\ \ \forall g \in C^n(Z)\ \ \ g \subset^\star \bigcup \mathcal{F}_Z$

J_n: $\forall g \in D^n(\mathbb{R})\ \exists \mathcal{F}_g \in [C^n(\mathbb{R})]^{<\mathfrak{c}}$ s.t. $g \subset \bigcup \mathcal{F}_g$

This is good, since both these imply $I_n^*(Z)$.

Also, $C^n(Z)$ is Polish, with metric $\rho(f,g)=\sum_{i\le n}\|f^{(i)}-g^{(i)}\|_\infty$.

$\mathrm{CPA}_{\mathrm{prism}}$ implies $I_n(Z)$, $n > 0$ (brief sketch)

The argument is quite similar to that for $n = 0$, after you prove

$$\mathcal{E}_n = \{P \in \mathrm{Perf}(Z \times C^n(Z)) \colon \text{ either } \pi_2 \restriction P \text{ is 1-1 or } \pi_1 \restriction P \text{ is 1-1 and } \exists f_P \in C^n(\mathbb{R}) \forall g \in C^n(Z) \ \ g \restriction P^g \subset^\star f_P\}$$

is $\mathcal{F}_{\mathrm{prism}}$-dense, where $P^g = \{x \in Z \colon \langle x, g \rangle \in P\}$.

The actual condition that ensures the additional requirement

$$(*) \ \ \exists f_P \in C^n(\mathbb{R}) \forall g \in C^n(Z) \ \ g \restriction P^g \subset^\star f_P$$

is delicate and heavily relies on Whitney's Extension Theorem, a differentiable analog of Tietze's Extension Theorem.

Whitney's Extension Theorem, one variable case

Theorem (Whitney's Extension Theorem)

Let $P \subset \mathbb{R}$ be perfect, $n \in \mathbb{N}$, and $f\colon P \to \mathbb{R}$. There exists an extension $\bar{f} \in C^n(\mathbb{R})$ of f if, and only if,

(W_n) *$f \in C^n(P)$ and the map $q_{f^{(i)}}^{n-i}\colon P^2 \to \mathbb{R}$ is continuous for every $i \le n$,*

where $q_f^n\colon P^2 \to \mathbb{R}$ is defined as

$$q_f^n(a,b) = \begin{cases} \dfrac{T_b^n f(b) - T_a^n f(b)}{(b-a)^n} & \text{if } a \neq b, \\ 0 & \text{if } a = b \end{cases}$$

and $T_a^n f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!}(x-a)^i$ is the n-th degree Taylor polynomial of f at a.

$\mathrm{CPA}_{\mathrm{prism}}$ implies J_n, $n>0$ (brief sketch)

J_n: $\forall g\in D^n(\mathbb{R})\ \exists\mathcal{F}_g\in[C^n(\mathbb{R})]^{<\mathfrak{c}}$ $s.t.\ g\subset\bigcup\mathcal{F}_g$

Fix an $n\in\mathbb{N}$ and a $g\in D^n(\mathbb{R})$. For $Q\in\mathrm{Perf}(\mathbb{R})$ let $f=g\restriction Q$ and $\varphi^n_{g\restriction Q}\colon Q^2\to\mathbb{R}$ be defined as
$\varphi^n_{g\restriction Q}(a,b)=\sum_{k=0}^{n}|q^{n-k}_{f^{(k)}}(a,b)|+\sum_{k=0}^{n}|q^{n-k}_{f^{(k)}}(b,a)|$.

Since $\varphi^n_{g\restriction Q}$ is symmetric,

$$\mathcal{E}_g=\{Q\in\mathrm{Perf}(\mathbb{R})\colon g\restriction Q\in C^n(Q)\ \&\ \varphi^n_{g\restriction Q}\in C(Q^2)\}$$

is $\mathcal{F}_{\mathrm{prism}}$-dense. (Proved in [CPA book].)

By $\mathrm{CPA}_{\mathrm{prism}}$, there is $\mathcal{E}_0\in[\mathcal{E}]^{\omega_1}$ with $|\mathbb{R}\setminus\bigcup\mathcal{E}_0|\le\omega_1$.

So, $g\subset^\star\bigcup_{Q\in\mathcal{E}_0}g\restriction Q$. We need to show that each $g\restriction Q$ can be extended to an $f_Q\in C^n(\mathbb{R})$.

Smooth extendability of $g \restriction Q$

$$\varphi^n_{g\restriction Q}(a,b)=\sum_{k=0}^n |q^{n-k}_{(g\restriction Q)^{(k)}}(a,b)|+\sum_{k=0}^n |q^{n-k}_{(g\restriction Q)^{(k)}}(b,a)|$$

is continuous and $g\in D^n(\mathbb{R})$. By Whitney's Extension Theorem, need each $q^{n-k}_{(g\restriction Q)^{(k)}}(a,b)$ continuous.This follows from continuity of $\Psi(a,b)=\sum_{k=0}^n |q^{n-k}_{(g\restriction Q)^{(k)}}(a,b)|$.

Why $\Psi(a,b)$ is continuous?

(1) $\Psi(a,\cdot)$ is continuous, as $g\in D^n(\mathbb{R})$—classic-like argument for Taylor polynomial.

(2) $\varphi^n_{g\restriction Q}(a,b)=\Psi(a,b)+\Psi(b,a)$ is separately continuous—as $\varphi^n_{g\restriction Q}$ and $\Psi(a,\cdot)$ are continuous.

Lemma: (1) + (2) + a bit more $\Longrightarrow$ Ψ is continuous.

I_1^*: $\forall X \subset \mathbb{R}\ \exists \mathcal{F} \in [C^1(\mathbb{R})]^{<\mathfrak{c}}\ \forall g \in D^1(X)\ \ g \subset^\star \bigcup \mathcal{F}_g$

Theorem (Seems previously unknown)

For $X \subset \mathbb{R}$ with no isolated points, $g \in C(X)$, and a continuous extension $\bar{g}$ of g onto G_δ-set $G \supset X$: if $g \in D^n(X)$, then $\bar{g} \restriction B \in D^n(B)$ for some Borel $B \supset X$.

Proof of I_1^*: Can assume that X has no isolated points.

Choose Borel $B \supset X$ and $\bar{g} \in D^1(B)$ extension of g;

By CPA, there is $\mathcal{P} \in [\mathrm{Perf}(\mathbb{R})]^{\omega_1}$ with $B = \bigcup \mathcal{P}$. For each $P \in \mathcal{P}$,

by Jarník's theorem, there is an extension $g_P \in D^1(\mathbb{R})$ of $\hat{g} \restriction P$,

so, by Main Thm, there is $\mathcal{F}_P \in [C^1(\mathbb{R})]^{\omega_1}$ with $\hat{g} \restriction P \subset^\star \bigcup \mathcal{F}_P$.

Then $\mathcal{F}_g = \bigcup_{P \in \mathcal{P}} \mathcal{F}_P$ is as needed. □

Outline

Open problem

The proof of last theorem is based on the

Lemma

For every $X \subset \mathbb{R}$ with no isolated points and $g \in C(X)$ the set $\mathrm{Dif}(g)$ of points of differentiability of g is a Borel subset of X of class $G_{\delta\sigma\delta}$.

Problem

What is the lowest Borel rank of the set $\mathrm{Dif}(g)$ in the Lemma? For $X = \mathbb{R}$ the answer is $F_{\sigma\delta}$, as shown by Zahorski in 1941.

That is all!

Thank you for your attention!