Local contractions and fixed point theorems

Krzysztof Chris Ciesielski[1] and Jakub Jasinski[2]

[1]West Virginia University
Morgantown, WV
and
University of Pennsylvania
Philadelphia, PA

[2]University of Scranton,
Scranton, PA

Real Functions Seminar
University of Gdańsk
Gdańsk, November 21 2017

The talk is based on the following papers:

- K.C. Ciesielski and J. Jasinski, *An auto-homeomorphism of a Cantor set with derivative zero everywhere*, J. Math. Anal. Appl. **434** (2016), 1267–1280.
- K.C. Ciesielski and J. Jasinski, *On fixed points of locally and pointwise contracting maps*, Topology Appl. **204** (2016), 70–78.
- K.C. Ciesielski and J. Jasinski, *Fixed point theorems of locally and pointwise contracting maps*, Canadian J. of Math. (2017) accepted.

Let $\langle X, d\rangle$ be a metric space. We compare various classes of continuous self-maps $f : X \to X$. All of these self-maps are proved to have fixed or periodic points for spaces X with certain topological properties and appeared in literature before 2016. We will assume X to be

1. complete
2. complete and connected
3. complete and rectifiably path connected
4. complete and d-convex
5. compact
6. compact and connected
7. compact and rectifiably path connected
8. compact and d-convex

Definition (#1)

A function $f : X \to X$ is called Contractive, (C), if there exists a constant $0 \le \lambda < 1$ such that for any **two** elements $x, y \in X$ we have $d(f(x), f(y)) \le \lambda d(x, y)$.

Theorem (Banach, 1922)

If (X, d) is a complete metric space and $f : X \to X$ is (C), then f has a unique fixed point, that is, there exists a unique $\xi \in X$ such that $f(\xi) = \xi$.

Historical Overview - Global Classics

Definition (#2)

A function $f : X \to X$ is called Shrinking, $(\mathbb{S})$, if for any two elements $x, y \in X, x \neq y$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

If $\langle X, d\rangle$ is compact and $f : X \to X$ is $(\mathbb{S})$, then it has a unique fixed point.

Proof.

Let $\phi : X \to \mathbb{R}, \phi(x) = d(x, f(x))$. ϕ is continuous so it attains a minimum at some point $\xi \in X$. Then $\phi(\xi) = 0$ so $f(\xi) = \xi$. □

Definition (#3)

A function $f : X \to X$ is called Locally Shrinking, (LS), if for any element $z \in X$ there exists an $\varepsilon_z > 0$ such that $f \restriction B(z, \varepsilon)$ is shrinking, i.e. for any two $x \neq y \in B(z, \varepsilon_z)$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

Let $\langle X, d \rangle$ be compact and let $f : X \to X$.

(i) *If f is (LS) and X is connected, then f has a unique fixed point.*

(ii) *If f is (LS), then f has a periodic point.* ♠

Historical Overview - Local Classics/Recent

Definition (#6)

A function $f : X \to X$ is called Uniformly Locally Contracting, (ULC), if there exist a $\lambda \in [0, 1)$ and an $\varepsilon > 0$ such that for every $z \in X$ the restriction $f \restriction B(z, \varepsilon)$ is contractive with the same $\lambda_z = \lambda$.

Theorem

Assume that $\langle X, d\rangle$ is complete and that $f : X \to X$ is (ULC)

(i) (Edelstein, 1961) *If X is connected, then f has a unique fixed point.*

(ii) (C & J, 2016) *If X has a finite number of components , then f has a periodic point.*

Historical Overview - Local (Pointwise) Classics

Definition (#4)

A function $f : X \to X$ is called Pointwise Contracting, (PC), if for every $z \in X$ there exists a $\lambda_z \in [0,1)$ and an $\varepsilon_z > 0$ such that $d(f(x), f(z)) \leq \lambda_z d(x, z)$ for any element $x \in B(z, \varepsilon_z)$.

Definition (#5)

A function $f : X \to X$ is called uniformly Pointwise Contracting, (uPC), if there exists a $\lambda \in [0,1)$ such that for every $z \in X$ there exists an $\varepsilon_z > 0$ with $d(f(x), f(z)) \leq \lambda d(x, z)$ for any element $x \in B(z, \varepsilon_z)$.

Theorem (Hu and Kirk, 1978)

If $\langle X, d \rangle$ is a rectifiably path connected complete metric space and a map $f : X \to X$ is (uPC), then f has a unique fixed point.

Hu & Kirk's proof was based on a Holmes' Proposition:

Theorem (Holmes 1975)

For compact X (PC) implies (LC).

Example (Jungck 1982)

Let
$A = \{(x, 0) : 0 \leq x \leq 1\}, B = \{(x, \frac{x}{4}) : 0 \leq x \leq \frac{1}{2}\}, X = A \cup B.$

$$g(x, y) = \begin{cases} (\frac{x}{2}, \frac{x}{4}) & \text{if } (x, y) \in A \\ (0, 0) & \text{if } (x, y) \in B. \end{cases}$$

- $g : X \to X$ is $\left(\frac{5}{8}\right) - (\mathrm{PC})$ on the compact subspace $X \subseteq \mathbb{R}$
- Let $0 < \varepsilon < \frac{1}{2}$ and $0 < x < \frac{\varepsilon}{2}$. Take $p = (x, 0)$ and $q = \left(x, \frac{x}{4}\right)$. Then $d(p, q) = \frac{x}{4}$ and $d(g(p), g(q)) = d(\left(\frac{x}{2}, \frac{x}{8}\right), (0, 0)) > \frac{x}{2} > d(p, q)$. g is not (LC) at $(0, 0)$.

Historical Overview - Local Classics/Recent

Definition (#5)

A function $f : X \to X$ is called uniformly Pointwise Contracting, (uPC), if there exists a $\lambda \in [0, 1)$ such that for every $z \in X$ there exists an $\varepsilon_z > 0$ with $d(f(x), f(z)) \leq \lambda d(x, z)$ for any element $x \in B(z, \varepsilon_z)$.

Theorem (Hu and Kirk, 1978; proof corrected by Jungck 1982)

If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f\colon X \to X$ is (uPC), then f has a unique fixed point.

Theorem (C & J, Top. and its App. 204 2016 70-78)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If $f\colon X \to X$ is (PC), then f has a unique fixed point.

Local Recent

Theorem (C & J, Top. and its App. 204 2016 70-78)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If $f\colon X \to X$ is (PC), then f has a unique fixed point.

Example (Some "connectedness" is needed. C & J, J. Math. Anal. Appl. 434 2016 1267 - 1280)

There exists a compact (Cantor-like) set $\mathfrak{X} \subset \mathbb{R}$ and a (PC) self-map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ without periodic points. In fact $\mathfrak{f}$ is differentiable autohomeomorphism of $\mathfrak{X}$ with $\mathfrak{f}' \equiv 0$, so $\mathfrak{f}$ is (uPC) with any $\lambda \in (0, 1)$.

The Ten Contracting/Shrinking Properties

Global Properties. $f : X \to X$ is

(C) contractive if

$$\exists \lambda \in [0,1) \forall x, y \in X \, (d(f(x), f(y)) \leq \lambda d(x,y)),$$

(S) shrinking if

$$\forall x \neq y \in X \, (d(f(x), f(y)) < d(x,y)).$$

Clearly (C) $\implies$ (S).

Each global property gives rise to two groups of local properties, one named local and the other group named pointwise:

The Ten Contracting/Shrinking Properties

Local Properties:

(LC) f is *locally contractive if* $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\, (d(f(x), f(y)) \leq \lambda_z d(x,y))$,

(LS) *f is locally shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x \neq y \in B(z, \varepsilon_z)\, (d(f(x), f(y)) < d(x,y))$,

Pointwise Properties (we fix y=z):

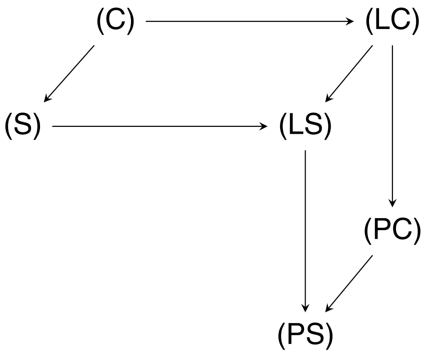
(PC) f is *pointwise contractive if* $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\, (d(f(x), f(z)) \leq \lambda_z d(x,z))$,

(PS) *f is pointwise shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\, (d(f(x), f(z)) < d(x,z))$,

Clearly $(\text{Locally}) \Longrightarrow (\text{Pointwise})$.

The Ten Contracting/Shrinking Properties

The following implications follow from the definitions:



The Ten Contracting/Shrinking Properties

Local properties can be made stronger by requiring uniformity, i.e. that the same $\lambda \in [0,1)$ and/or the same $\varepsilon > 0$ work for all $z \in X$.

Local Properties:

(LC) *f is locally contractive if* $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\,(d(f(x), f(y)) \leq \lambda_z d(x,y))$,

(uLC) *f is (weakly) uniformly locally contractive if* $\exists \lambda \in [0,1) \forall z \in X \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\,(d(f(x), f(y)) \leq \lambda d(x,y))$,

(ULC) *f is (strongly) Uniformly locally contractive if* $\exists \lambda \in [0,1) \exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\,(d(f(x), f(y)) \leq \lambda d(x,y))$,

(LS) *f is locally shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\,(d(f(x), f(y)) < d(x,y))$,

(ULS) *f is Uniformly locally shrinking if* $\exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\,(d(f(x), f(y)) < d(x,y))$.

The Ten Contracting/Shrinking Properties

Similarly, pointwise properties can be made stronger by requiring uniformity, i.e. that the same $\lambda \in [0,1)$ and/or the same $\varepsilon > 0$ works for all $z \in X$.

Pointwise Properties:

(PC) *f is pointwise contractive if* $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\,(d(f(x), f(z)) \le \lambda_z d(x,z))$,

(uPC) *f is (weakly) uniformly pointwise contractive if* $\exists \lambda \in [0,1) \forall z \in X \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\,(d(f(x), f(z)) \le \lambda d(x,z))$,

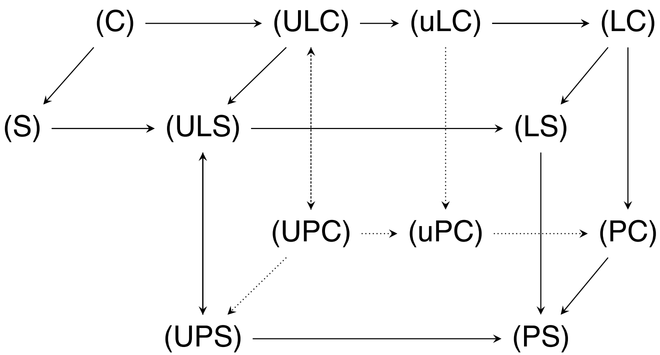
(UPC) *f is (strongly) Uniformly pointwise contractive if* $\exists \lambda \in [0,1) \exists \varepsilon > 0 \forall z \in X \forall x \in B(z, \varepsilon)\,(d(f(x), f(z)) \le \lambda d(x,z))$,

(PS) *f is pointwise shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\,(d(f(x), f(z)) < d(x,z))$,

(UPS) *f is Uniformly pointwise shrinking if* $\exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\,(d(f(x), f(y)) < d(x,y))$.

The 10 Contracting/Shrinking Properties ... or is it 12?

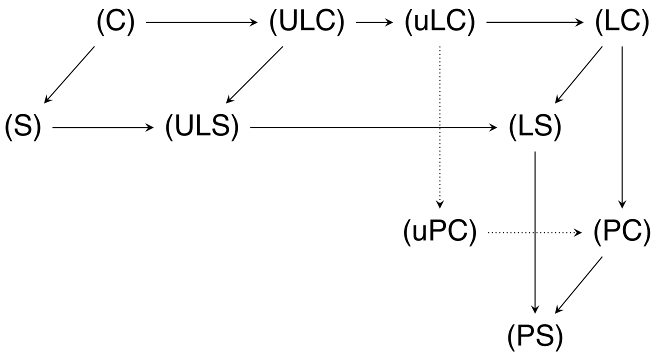
The following implications follow from the definitions:



Remark: (ULS)=(UPS) and (ULC)=(UPC). Any (λ, ε)-(UPC) function is $(\lambda, \frac{\varepsilon}{2})$-(ULC) and (ε)-(UPS) is $\left(\frac{\varepsilon}{2}\right)$-(ULS).

The Ten Contracting/Shrinking Properties

The following diagram

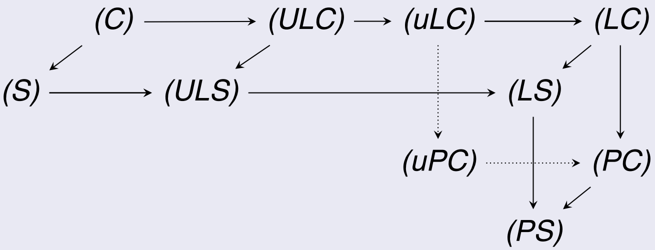


shows the essential classes and implications that hold for any metric space X.

Fixed and Periodic Points

Theorem (Complete Spaces)

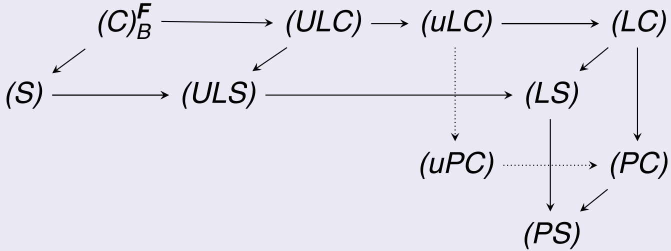
Assume X is complete. (A) No combination of any of the properties shown imply any other property, unless the graph forces such implication.



Fixed and Periodic Points

Theorem (Complete Spaces)

Assume X is complete. (A) No combination of any of the properties shown imply any other property, unless the graph forces such implication. (B) Neither does any combination of them imply the existence of a fixed (or even periodic) point unless it contains property (C).



Fixed and Periodic Points

Theorem (Complete Spaces cont.)

Specifically, there exist 9 *complete spaces X with self-maps $f : X \to X$ without periodic points witnessing the following:*

- (PC): (PC) $\not\Leftarrow$ (S)
- (uPC): (uPC) $\not\Leftarrow$ (S)&(LC)
- (LS): (LS) $\not\Leftarrow$ (uPC)
- (ULS): (ULS) $\not\Leftarrow$ (uLC)
- (S): (S) $\not\Leftarrow$ (ULC)
- (LC): (LC) $\not\Leftarrow$ (S)&(uPC)
- (uLC): (uLC) $\not\Leftarrow$ (S)&(LC)&(uPC)
- (ULC): (ULC) $\not\Leftarrow$ (S)&(uLC)
- (C): (C) $\not\Leftarrow$ (S)&(ULC)

Fixed and Periodic Points, blue does not imply yellow

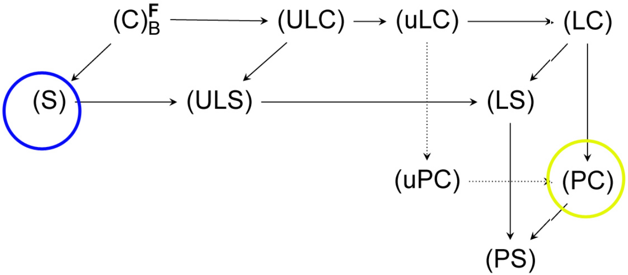
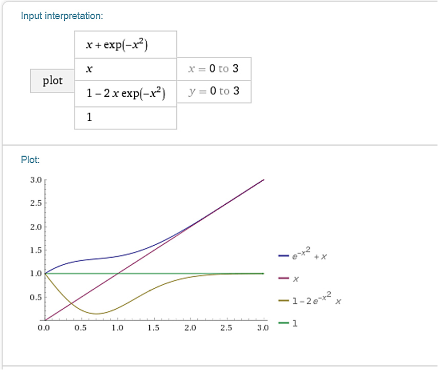


Figure: (PC) $\not\Rightarrow$ (S).

Remark: f is (PC) iff $D^* f(z) = limsup_{x \to z} \frac{d(f(x), f(z))}{d(x,z)} < 1$ for all $z \in X$.
So if f is *differentiable* on X, then f is (PC) iff $|f'(x)| < 1$ for all $x \in X$.

Take $X = [0, \infty)$ and $f(x) = x + e^{-x^2}$ so $f'(x) = 1 - 2xe^{-x^2}$.

$X = [0, \infty)$ and $f(x) = x + e^{-x^2}$ so $f'(x) = 1 - 2xe^{-x^2}$.



We have $f'(0) = 1$ so not-(PC) at $z = 0$. Also $f'[(0, \infty)] \subseteq (0, 1)$ so f is (S) by the MVT. For all $x \in [0, \infty)$, $f(x) > x$ so no periodic points.

Fixed and Periodic Points, blue does not imply yellow

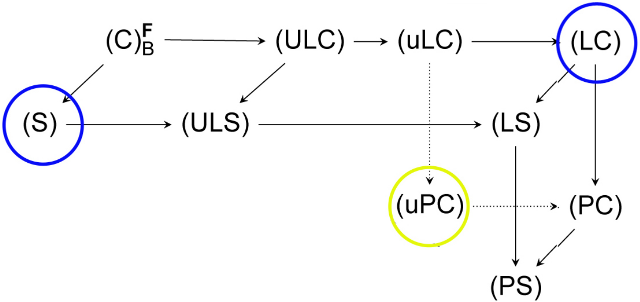
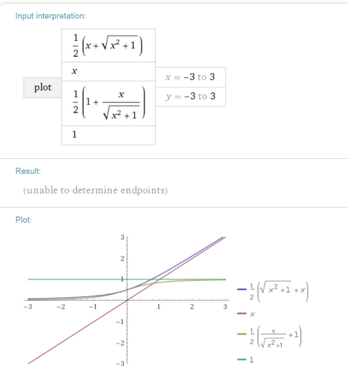


Figure: (uPC) $\not\Leftarrow$ (S)&(LC). Take $X = \mathbb{R}$ and $f(x) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right)$.

Then $f'(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$.



For any $a \in \mathbb{R}$, $f'[(-\infty, a]] = (0, c]$ for some $c < 1$ so MVT gives $(\mathrm{S})\&(\mathrm{LC})$. $\lim_{x \to \infty} f'(x) = 1$ so $\neg(\mathrm{uPC})$.

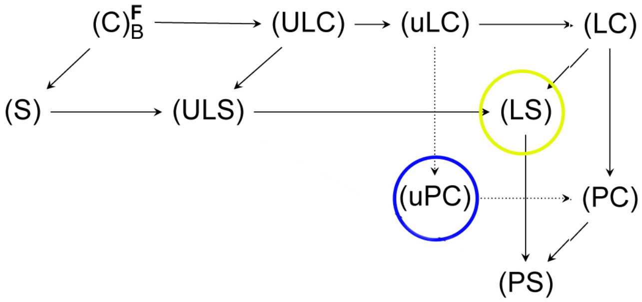


Figure: $(\mathrm{LS}) \not\Leftarrow (\mathrm{uPC})$ There exists a compact perfect set $\mathfrak{X} \subseteq \mathbb{R}$ and an autohomeomorphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ with $\mathfrak{f}' \equiv 0$. So $\mathfrak{f}$ is (uPC) with any $\lambda \in (0, 1)$ and $\mathfrak{f}$ has no periodic points, [C & J, 2015] so it is not (LS) by the Edelstein's Theorem ♠.

Fixed and Periodic Points, blue does not imply yellow

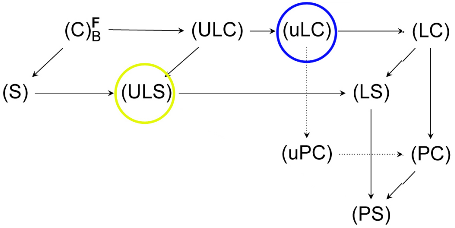


Figure: $\text{(ULS)} \not\Leftarrow \text{(uLC)}$ For $0 < n < \omega$ let $a_n = n$, $b_n = n + \frac{1}{n}$, take $X = \{a_n.b_n : n < \omega\}$ and $f = id_X$. X is complete. f is (uLC) because every point is isolated and not (ULS). Indeed, for any $\varepsilon > 0$ take $n > \frac{1}{\varepsilon}$ so

$$\varepsilon > \frac{1}{n} = |b_n - a_n| = |f(b_n) - f(a_n)|.$$

What if we require compact X? Connected X?

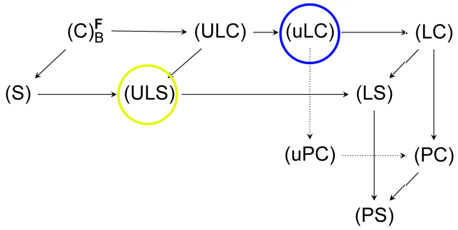


Figure: (ULS) $\not\Leftarrow$ (uLC), Connected X. Take two increasing sequences: $0 < \beta_n \nearrow 1$ and $0 = a_0 < a_1 < ... \nearrow \infty$, $I_n = [a_n, a_{n+1}]$, such that $|I_{2n}| = |I_{2n+1}| = \frac{1}{n+1}$. Define metrics $\rho_n(x,y) = |I_n| \left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}$ on I_n and *"make"* a metric ρ on $X = \bigcup_{n<\omega} I_n = [0,\infty)$ so that $f : X \to X$, mapping linearly and increasingly I_n onto I_{n+1} has needed properties. For $x \leq y$, $n < m$

$$\rho(x,y) = \begin{cases} \rho_n(x,y) & \text{if } x,y \in I_n \\ \rho_n(x,a_{n+1}) + |a_m - a_{n+1}| + \rho_m(a_m,y) & \text{if } x \in I_n, y \in I_m \end{cases}$$

A closer look at the metric rho

For $n < \omega$, $\beta_n \in (0,1)$ and for $x, y \in I_n$, $\rho_n(x,y) = |I_n| \left(\frac{|x-y|}{|I_n|} \right)^{\beta_n}$

- ρ_n is a metric on I_n
- ρ_n is topologically equivalent to the standard metric, $|x-y|$
- ρ_n is complete

$$\rho(x,y) = \begin{cases} \rho_n(x,y) & \text{if } x, y \in I_n \\ \rho_n(x, a_{n+1}) + |a_m - a_{n+1}| + \rho_m(a_m, y) & \text{if } x \in I_n, y \in I_m \end{cases}$$

- ρ is a metric on $[0, \infty)$
- ρ is topologically equivalent to the standard metric, $|x-y|$
- ρ is complete
- ρ is path connected (*not* rectifiably path connected)

It follows:

- $X = \bigcup_{n<\omega} I_n = [0,\infty)$ so $f : X \to X$ is *not* (ULS) because $|I_{2n}| = |I_{2n+1}|$
- f is (uLC). Take any $\lambda \in (0,1)$. It suffices to show that $f_n = f \restriction I_n$ is $(\lambda)-(\mathrm{uLC})$ for any $n < \omega$. By linearlity for $x, y \in I_n$ we have

$$\frac{|f_n(x) - f_n(y)|}{|I_{n+1}|} = \frac{|x-y|}{|I_n|}.$$

so

$$\rho_{n+1}(f_n(x), f_n(y)) = |I_{n+1}| \left(\frac{|x-y|}{|I_n|}\right)^{\beta_{n+1}} =$$
$$\frac{|I_{n+1}|}{|I_n|} \left(\frac{|x-y|}{|I_n|}\right)^{\beta_{n+1}-\beta_n} \rho_n(x,y).$$

Notice that f_n is Lipschitz with constant $\frac{|I_{n+1}|}{|I_n|}$.

A closer look at the metric rho

To have $(\lambda)-(\mathrm{uLC})$ we need $\frac{|I_{n+1}|}{|I_n|}\left(\frac{|x-y|}{|I_n|}\right)^{\beta_{n+1}-\beta_n}\leq\lambda$ and

equivalently: $\frac{|I_{n+1}|}{|I_n|}\left(\frac{|x-y|}{|I_n|}\right)^{\beta_{n+1}-\beta_n}\leq\lambda$

$$\left(\frac{|x-y|}{|I_n|}\right)^{\beta_{n+1}-\beta_n}\leq\frac{|I_n|}{|I_{n+1}|}\lambda$$

$$\left(\frac{|x-y|}{|I_n|}\right)\leq\left(\frac{|I_n|}{|I_{n+1}|}\lambda\right)^{\frac{1}{\beta_{n+1}-\beta_n}}$$

$$\left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}\leq\left(\frac{|I_n|}{|I_{n+1}|}\lambda\right)^{\frac{\beta_n}{\beta_{n+1}-\beta_n}}$$

$$|I_n|\left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}\leq|I_n|\left(\frac{|I_n|}{|I_{n+1}|}\lambda\right)^{\frac{\beta_n}{\beta_{n+1}-\beta_n}}$$

$$\rho_n(x,y)=|I_n|\left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}\leq|I_n|\left(\frac{|I_n|}{|I_{n+1}|}\lambda\right)^{\frac{\beta_n}{\beta_{n+1}-\beta_n}}.$$

A closer look at the metric rho

It follows that f_n is $(\lambda, \varepsilon) - (ULC)$ with $\varepsilon = \frac{1}{2}|I_n| \left(\frac{|I_n|}{|I_{n+1}|}\lambda \right)^{\frac{\beta_n}{\beta_{n+1} - \beta_n}}$.
ε depends on n so f is only $\mathrm{(uLC)}$.

Fixed and Periodic Points, blue does not imply yellow

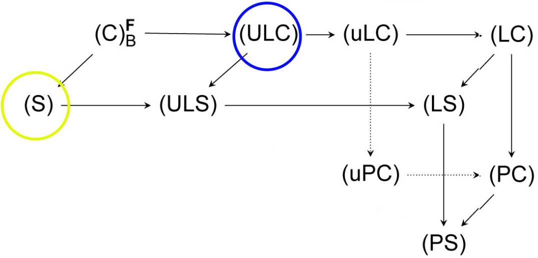


Figure: $(S) \not\Leftarrow (ULC)$ Remetrization.

Fixed and Periodic Points, blue does not imply yellow

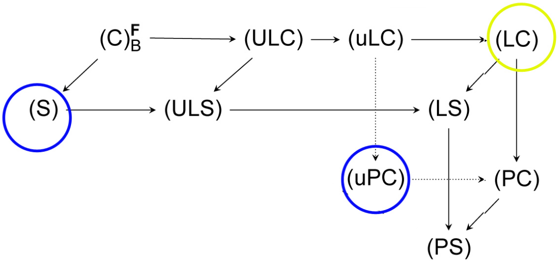


Figure: (LC) $\not\Leftarrow$ (S)&(uPC) Remetrization.

Fixed and Periodic Points, blue does not imply yellow

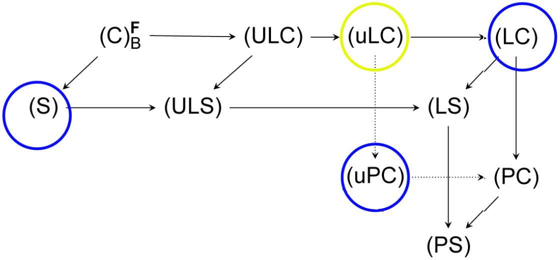


Figure: (uLC) $\not\Leftarrow$ (S)&(LC)&(uPC) Remetrization.

Fixed and Periodic Points, blue does not imply yellow

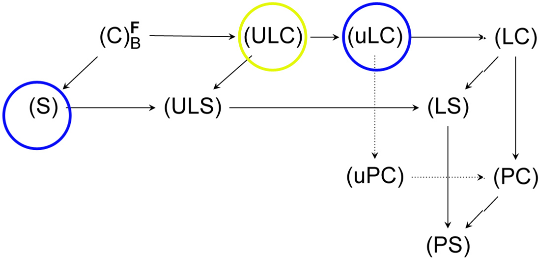


Figure: $(\mathrm{ULC}) \not\Leftarrow (\mathrm{S})\&(\mathrm{uLC})$ Remetrization.

Fixed and Periodic Points, blue does not imply yellow

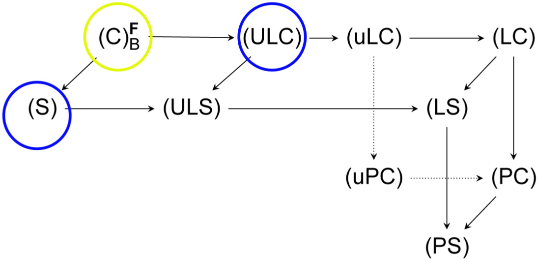
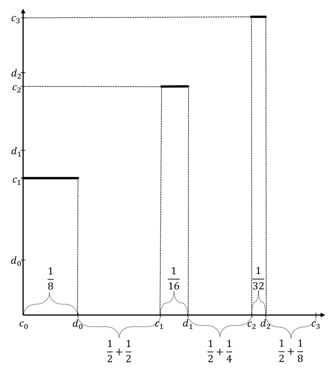


Figure: $(C) \not\Leftarrow (S)\&(ULC)$ We have the following ...

Fixed and Periodic Points, blue does not imply yellow

Example (A (S)&(ULC)&not(C) map f without periodic points)

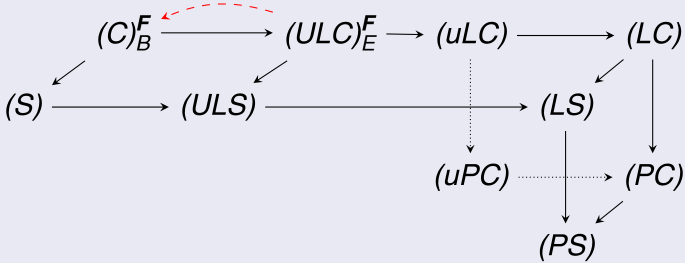
Define sequences $\langle c_n \rangle$ and $\langle d_n \rangle$: $c_0 = 0$, $d_n = c_n + 2^{-(n+3)}$ and $c_{n+1} = d_n + \frac{1}{2} + 2^{-(n+1)}$. Set $X = \bigcup_{n<\omega}[c_n, d_n]$ and let $f : X \to X$, $f(x) = c_{n+1}$ for $x \in [c_n, d_n]$. We have



Fixed and Periodic Points - Connected Spaces

Theorem (Connected Spaces)

Assume X is complete and connected. (A) No combination of any of the properties shown imply any other property, unless the graph forces such implication. (B) Neither does any combination imply the exitance of a periodic point unless it contains (C) *or* (ULC).



What is the red arrow?

For $\varepsilon > 0$ let X be ε*-chainable*, i.e. for every $p, q \in X$ there exists a finite sequence $s = \langle x_0, x_1, \ldots, x_n \rangle$, referred to as an ε*-chain from* p *to* q, such that $x_0 = p$, $x_n = q$, and $d(x_i, x_{i+1}) \leq \varepsilon$ for every $i < n$. The length of the ε-chain s is defined as $\mathfrak{l}(s) = \sum_{i<n} d(x_{i+1}, x_i)$.

Remark

Any connected space is ε-chainable for any $\varepsilon > 0$.

For $x, y \in X$ let $D_\varepsilon(x, y) = \inf\{\mathfrak{l}(s) \colon s \text{ is an } \varepsilon\text{-chain from } x \text{ to } y\}$.

- D_ε is a metric on X, topologically equivalent to the original metric d and if $\langle X, d \rangle$ is complete then so is $\langle X, D_\varepsilon \rangle$.
- If $f \colon \langle X, d \rangle \to \langle X, d \rangle$ is (η, λ)-(ULC) for some $\eta > \varepsilon$, then $f \colon \langle X, D_\varepsilon \rangle \to \langle X, D_\varepsilon \rangle$ is (C) with constant λ.

An (S) & (ULC) but not(C) map f

Example

Define $d : [0,\infty)^2 \to [0,\infty)$ by $d(x,y) = \ln(1+|x-y|)$ and $f\colon [0,\infty) \to [0,\infty)$ by $f(x) = x/2$.

- d is a complete, rectifiebly path connected metric on $[0,\infty)$ topologically equivalent to the standard metric.
- For any $x \in [0,\infty)$ and $z > 0$, $\frac{d(f(x),f(x+z))}{d(x,x+z)} = \frac{\ln(1+z/2)}{\ln(1+z)} < 1$. Therefore, f is (S).
- $\lim_{z\to\infty} \frac{d(f(x),f(x+z))}{d(x,x+z)} = \lim_{z\to\infty} \frac{1+z}{2+z} = 1$ so f is not (C).
- $\lim_{z\to 0} \frac{d(f(x),f(x+z))}{d(x,x+z)} = \lim_{z\to 0} \frac{1+z}{2+z} = \frac{1}{2}$ so, there exists an $\varepsilon > 0$ such that $\frac{d(f(x),f(x+z))}{d(x,x+z)} < \frac{3}{4}$ for every $x \geq 0$ and $z \in (0,\varepsilon)$. But this means that f is $\left(\frac{\varepsilon}{2}, \frac{3}{4}\right)$-(ULC).

Rectifiably Path Connected Spaces

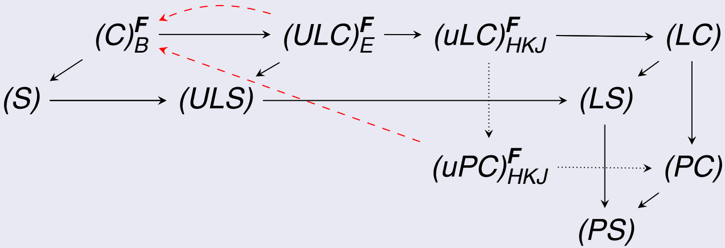
Definition

A metric space X is *rectifiably path connected* provided any two points $x, y \in X$ can be connected in X by a path $p\colon [0,1] \to X$ of finite length $\ell(p)$,that is, by a continuous map p satisfying $p(0) = x$ and $p(1) = y$, and having a finite *length* $\ell(p)$ defined as the supremum over all numbers (sums):

$$\sum_{i=1}^{n} d(p(t_i), p(t_{i-1})) \text{ with } 0 < n < \omega \text{ and } 0 = t_0 < t_1 < \cdots < t_n = 1.$$

Theorem (Rectifiably Path Connected Spaces)

Assume X is complete and rectifiably path connected. (A) No combination of any of the properties shown imply any other property, unless the graph forces such implication. (B) Neither does any combination imply the exitance of a periodic point unless it contains (C), (ULC), (uLC) *or* (uPC).



What is the other red arrow?

Recall,

Theorem (Hu and Kirk, 1978; proof corrected by Jungck 1982)

If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f\colon X \to X$ is (uPC), then f has a unique fixed point.

Proof.

Define $D_0\colon X^2 \to [0, \infty)$, $D_0(x, y) = \inf\{\ell(p)\colon p$ is an rectifiable path from x to $y\}$.

- D_0 is a metric on X topologically equivalent to d. If $\langle X, d\rangle$ is complete, then so is $\langle X, D_0\rangle$.
- If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (λ)-(uPC), then $f\colon \langle X, D_0\rangle \to \langle X, D_0\rangle$ is (C) with the contraction constant λ.

□

Definition

A metric space $\langle X, d\rangle$ is *d-convex* provided for any distinct points $x, y \in X$ there exists a path $p\colon [0,1] \to X$ from x to y such that

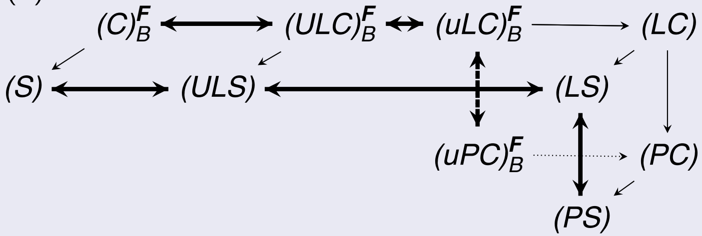
$$d(p(t_1), p(t_3)) = d(p(t_1), p(t_2)) + d(p(t_2), p(t_3))$$

whenever $0 \leq t_1 < t_2 < t_3 \leq 1$.

Fixed and Periodic Points - Connected Spaces

Theorem (d-convex Spaces)

Assume X is complete and d-convex. Jungck (1982) showed $(\mathrm{uPC}) \Rightarrow (\mathrm{C})$ *with the same* λ*. A modified argument shows that* $(\mathrm{PS}) \Rightarrow (\mathrm{S})$*.*

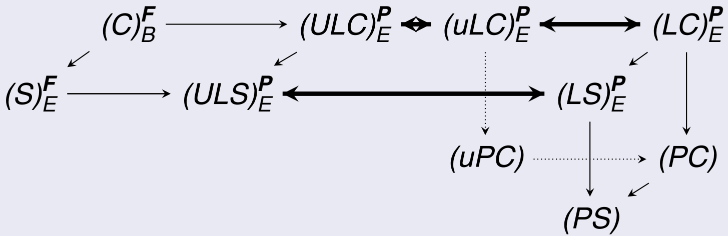


(A) No combination of any of the properties shown imply any other property, unless the graph forces such implication.
(B)Neither does any combination imply the existence of a periodic point unless it contains (C)=(ULC)=(uLC)=(uPC)*.*

Fixed and Periodic Points - Compact Spaces

Theorem (Compact Spaces)

Assume $\langle X, d\rangle$ is compact. Ding and Nadler (2002) and C&J 2015 showed $(\mathrm{LC}) \Rightarrow (\mathrm{ULC})$ *and* $(\mathrm{LS}) \Rightarrow (\mathrm{ULS})$.

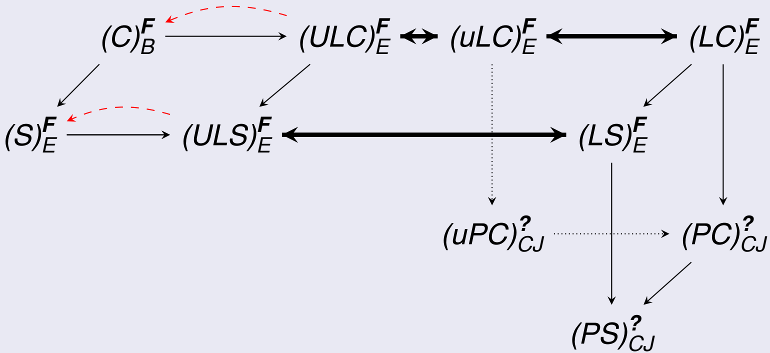


(A) No combination of any of the properties shown imply any other property, unless the diagram forces such implication.
(B)Neither does any combination imply the existence of a fixed or periodic unless indicated on the diagram.

Fixed and Periodic Points - Compact Spaces

Theorem (Compact Connected Spaces)

Assume X is compact and connected.

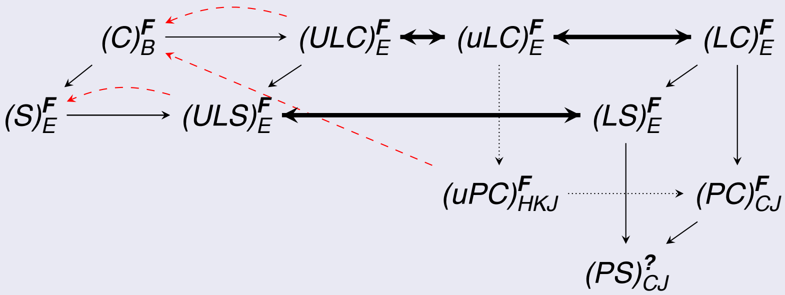


(A) No combination of any of the properties shown imply any other property, unless the diagram forces such implication. (B) ???

Fixed and Periodic Points - Compact Spaces

Theorem (Compact Rectifiably Path Connected Spaces)

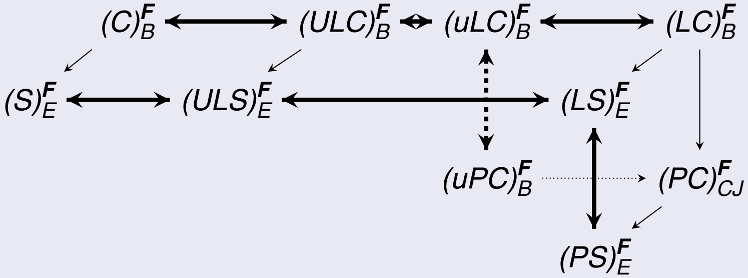
Assume X is compact and rectifiably path connected.



(A) No combination of any of the properties shown imply any other property, unless the diagram forces such implication. (B) ?

Theorem (Compact d-Convex Spaces)

Assume X is compact and d-convex.



No combination of any of the properties shown imply any other property, unless the diagram forces such implication.

Open Problems

1. Assume that $\langle X, d\rangle$ is compact and either connected or path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), must f have either fix or periodic point? What if f is (PC)? or (uPC)?
2. Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), does it imply that f has a fixed or periodic point?

Proof Outline

Recall,

Theorem (C & J, 2016)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If $f: X \to X$ is (PC), then f has a unique fixed point.

PROOF (outline). For $x, y \in X$ and a rectifiable path $p : [a, b] \to X, p(a) = x, p(b) = y$ let $\ell(p) = sup\{\sum_{i<n} d(t_i, t_{i+1}) : n < \omega$ and $a = t_0 < t_1 < ... < t_n = b\}$.
Define $D_0 : X^2 \to [0, \infty)$,
$D_0(x, y) = \inf\{\ell(p) : p$ is a rectifiable path from x to $y\}$.
We need to show:

(1) D_0 is a metric on X;
(2) $\langle X, D_0\rangle$ is complete;
(3) There exists $\bar{x} \in X$ such that $D_0(\bar{x}, f(\bar{x})) = L = \inf\{D_0(x, f(x)) : x \in X\}$;
(4) $L = 0$.

(3)

Even when $\langle X, d\rangle$ is compact, $\langle X, D_0\rangle$ does not need to be. Let X be the Topologist's Sine Curve with arc. Then $\langle X, d\rangle$ with standard metric from $\mathbb{R}^2$, is compact but $\langle X, D_0\rangle$ is not. It's actually homeomorphic with $[0, \infty)$. So (3) is not obvious.

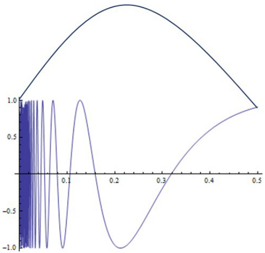


Figure: Topologist's Sine Curve with arc.

Proof of (3)

(3) There exists $\bar{x} \in X$ such that $D_0(\bar{x}, f(\bar{x})) = L = \inf\{D_0(x, f(x)) : x \in X\}$.

Let $\langle x_n \in X \colon n < \omega \rangle$ be a sequence with $\lim_{n \to \infty} D_0(x_n, f(x_n)) = L$. We have:

Theorem (Menger 1930)

In a metric space X, if there is a rectifiable path in X from x to y, then there is a geodesic, i.e. a path with minimal length ℓ, in X from x to y.

so for every $n < \omega$ there exists a path $p_n \colon [0, 1] \to X$ from x_n to $f(x_n)$ with range $P_n \subseteq X$ and $\ell(p_n) = D_0(x_n, f(x_n))$.

We have the following:

Theorem (Myers 1945)

Let $\langle X, d\rangle$ be a compact metric space and, for any $n < \omega$, let $p_n \colon [0,1] \to X$ be a rectifiable path such that $\ell(p_n \restriction [0,t]) = t\ell(p_n)$ for any $t \in [0,1]$. If $L = \liminf_{n\to\infty} \ell(p_n) < \infty$, then there exists a subsequence $\langle p_{n_k} : k < \omega\rangle$ converging uniformly to a rectifiable path $p \colon [0,1] \to X$ with $\ell(p) \leq L$.

WLOG, by reparametrizing our p_n, we can assume that for any $t \in [0,1]$, $\ell(p_n \restriction [0,t]) = t\ell(p_n)$.
So by the Myers' Theorem there exists a subsequence $\langle p_{n_k} : k < \omega\rangle$ converging uniformly to a rectifiable path $p \colon [0,1] \to X$ with $\ell(p) \leq L$.
Take $\bar{x} = p(0) = \lim_{k\to\infty} p_{n_k}(0) = \lim_{k\to\infty} x_{n_k}$, then p is from $\bar{x}$ to $p(1) = \lim_{k\to\infty} p_{n_k}(1) = \lim_{k\to\infty} f(x_{n_k}) = f(\bar{x})$.
So, $D_0(\bar{x}, f(\bar{x})) \leq \ell(p) \leq L$, that is, $\bar{x}$ satisfies (3).

Historical Overview - Local Classics

Definition (#3)

A function $f : X \to X$ is called Locally Shrinking, (LS), if for any element $z \in X$ there exists an $\varepsilon_z > 0$ such that $f \upharpoonright B(z, \varepsilon)$ is shrinking, i.e. for any two $x \neq y \in B(z, \varepsilon_z)$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

Let $\langle X, d\rangle$ be compact and let $f : X \to X$.

(i) *If f is (LS), then f has a periodic point.* ♠

(ii) *If f is (LS) and X is connected, then f has a unique fixed point.*

Proof.

(i) follows from (ii).
For (ii) define a new metric $D_0(x, y) = min\{\mathfrak{l}(s) : s$ is an ε-chain from x to $y\}$ and show that f is (S) on $\langle X, D_0\rangle$.(!) □

Definition (#5)

A function $f : X \to X$ is called uniformly Pointwise Contracting, (uPC), if there exists a $\lambda \in [0, 1)$ such that for every $z \in X$ there exists an $\varepsilon_z > 0$ with $d(f(x), f(z)) \leq \lambda d(x, z)$ for any element $x \in B(z, \varepsilon_z)$.

Theorem (Hu and Kirk, 1978; proof corrected by Jungck, 1982)

If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f\colon X \to X$ is (uPC), then f has a unique fixed point.

Proof.

For $x, y \in X$ and a we define $D(x, y) = inf\{l(p) : p : [0, 1] \to X$ rectifiable path from x to $y\}$.

- If $\langle X, d\rangle$ is complete than so is $\langle X, d\rangle$
- If $f : \langle X, d\rangle \to \langle X, d\rangle$ is $(\lambda) -$ u-PC then $f : \langle X, D\rangle \to \langle X, D\rangle$ is (C) with the constant λ. (!)

Theorem (Hu and Kirk, 1978; proof corrected by Jungck, 1982)

If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f\colon X \to X$ is (uPC), then f has a unique fixed point.

Proof.

The assumptions on $\langle X, d\rangle$ and on f imply that there exists a complete metric D_0 on X,

$$D_0(x, y) = inf\{l(p) : p \text{ is a rectifiable path from } x \text{ to } y\}$$

such that f is (C), when X is considered with the metric D. So, by the Banach Theorem, f has a unique fixed point. □

Theorem (C & J, 2015)

There exists a perfect compact set $\mathfrak{X} \subseteq \mathbb{R}$ and autohomeomorphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ with $\mathfrak{f}'(x) = 0$ for all $x \in X$.It follows that $\mathfrak{f}$ is $\lambda - (\mathrm{uPC})$ with any $\lambda \in [0, 1)$. Moreover, $\langle \mathfrak{X}, \mathfrak{f} \rangle$ is a minimal dynamical system so $\mathfrak{f}$ has no periodic points.

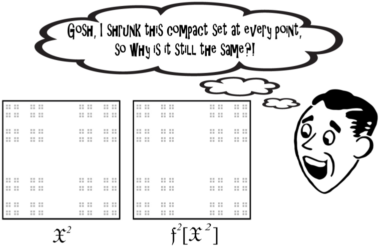


Figure: Action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2$.