Differentiable pointwise contractive auto-homeomorphism of a Cantor set

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Based on a joint work with Jakub Jasinski

see http://www.math.wvu.edu/˜kcies/publications.html

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Credits: This presentation is based on the papers

(1) K.C. Ciesielski, Pointwise shrinking self-surjection of a Cantor set, notes of Spring 2017.

(2) K.C. Ciesielski and J. Jasinski, *An auto-homeomorphism of a Cantor set with zero derivative everywhere*, J. Math. Anal. Appl. 434(2) (2016), 1267–1280.

- K.C. Ciesielski and J. Jasinski, *On closed subsets of $\mathbb{R}$ and of $\mathbb{R}^2$ admitting Peano functions*, Real Anal. Exchange 40(2) (2015), 309–317.
- K.C. Ciesielski and J. Jasinski, *On fixed points of locally and pointwise contracting maps*, Topology Appl. 204 (2016), 70–78;
- K.C. Ciesielski and J. Jasinski, *Fixed point theorems for maps with local and pointwise contraction properties*, 57 pages, Canad. J. Math., (2017), in print.

Outline

1. The example and why it seems paradoxical
2. New simple construction of the example
3. Pointwise shrinking maps and minimal dynamical systems
4. Open problems: Continuum theory?
5. The begining: Study of Peano-like functions

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1 The example and why it seems paradoxical

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The main result

Theorem ([KC & JJ], new short proof to be presented)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

(i) *$\mathfrak{f}$ is a minimal dynamical system (i.e., the $\mathfrak{f}$-orbit $O(x) = \{f^{(n)}(x)\colon n \in \omega\}$ of every $x \in \mathfrak{X}$ is dense in $\mathfrak{X}$);*

(ii) *$\mathfrak{f}$ can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$.*

Fact: $\mathfrak{f}' \equiv 0$ implies that $\mathfrak{f}$ is *pointwise contractive*:

(PC) for every $x \in \mathfrak{X}$ there are open $U \ni x$ and $\lambda_x \in [0,1)$ such that $|\mathfrak{f}(x) - \mathfrak{f}(y)| \leq \lambda_x |x - y|$ for any $y \in U$.

However, $\mathfrak{f}' \equiv 0$ does **not** imply that $\mathfrak{f}$ is *locally contractive*:

(LC) for every $x \in X$ there are open $U \ni x$ and $\lambda_x \in [0,1)$ s.t. $|f(y) - f(z)| \leq \lambda_x |y - z|$ for any $y, z \in U$.

$\mathfrak{f}$ seems paradoxical: topological angle

Our $\mathfrak{f}$ is PC but has neither fixed nor periodic points, while:

Theorem ([KC & JJ 2016]; variant of Hu and Kirk [HK 1978])

If $\langle X, d\rangle$ is compact rectifiable-path connected metric space. If $f\colon X \to X$ is PC, then f has a unique fixed point.

[HK]: without compactness, but f must be *uniformly PC, UPC*.

Theorem (Edelstein 1962, almost contradicting main example)

If $f\colon X \to X$ is LS and X is compact, then f has a periodic point.

- f is *locally shrinking, LS*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is *shrinking*, that is, $d(f(y), f(z)) < d(y, z)$ for every distinct $y, z \in U$.

$\mathfrak{f}$ seems paradoxical: real analysis angle

We have $\mathfrak{X} \subseteq \mathfrak{f}[\mathfrak{X}]$, while:

Fact: Assume that $X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$.

(i) $X \not\subseteq f[X]$ when X is a bounded closed interval and $|f'| \le \lambda < 1$ on X since then, by the Mean Value Theorem, $|f(y) - f(z)| \le \lambda|y - z|$ for every $y, z \in X$, so that the diameter of $f[X]$ is strictly smaller than the diameter of X. If $\mathfrak{f}' \equiv 0$, then f is constant.

(ii) $X \not\subseteq f[X]$ when X has a positive finite Lebesgue measure $m(X)$ and $|f'| \le \lambda < 1$ on X since then $m(f[X]) \le \lambda m(X)$.

(iii) $X \not\subseteq f[X]$ when $|f'| < 1$ on X and f can be extended to a **continuously** differentiable function $F\colon \mathbb{R} \to \mathbb{R}$. This has been proved by KC & JJ, *RAEx* **39**(1), 2014.

What is going on with $\mathfrak{f}$?

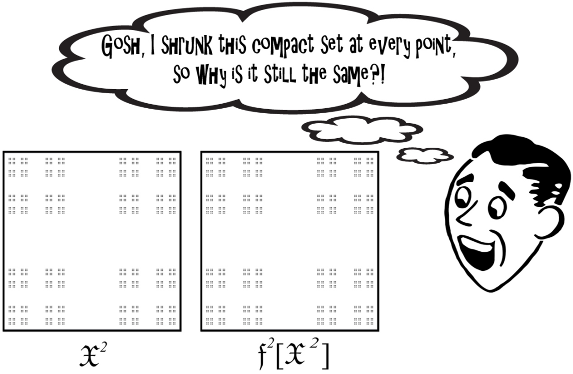

Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

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Format of the example

Example (Ciesielski & Jasinski 2016 example simplified)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

(i) $\mathfrak{f}$ is a minimal dynamical system

(ii) $\mathfrak{f}$ can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$.

(ii) follows from Jarník's theorem; (i) from the format of $\mathfrak{f}$:

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and $\sigma\colon 2^\omega \to 2^\omega$ is the "add one and carry" adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots\rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0,0,\ldots,0,1,s_{k+1},s_{k+2},\ldots\rangle & \text{if } s_k = 0 \text{ and } s_i = 1 \text{ for } i < k. \end{cases}$$

Definition of $h\colon 2^\omega \to \mathbb{R}$ with $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

$h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$, where $N(s \restriction 0) = 1$ and
$N(s \restriction n) = \sum_{i<n-1} s_i 2^i + (1 - s_{n-1}) 2^{n-1} + 2^n$ for $n > 0$.

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega\colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \overset{\text{(i)}}{\leq} |h(s) - h(t)| \overset{\text{(ii)}}{\leq} 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Proof. For $h_n(s) = \sum_{k<n} 2 s_k 3^{-(k+1)N(s\restriction k)}$ we have

$$h_n(s) + 2 s_n 3^{-(n+1)N(s\restriction n)} \leq h(s) \leq h_n(s) + (2 s_n + 1) 3^{-(n+1)N(s\restriction n)},$$

as $h(s) = h_n(s) + 2 s_n 3^{-(n+1)N(s\restriction n)} + 2 \sum_{k>n} 3^{-(k+1)N(s\restriction k)}$ and

$$0 \leq 2 \sum_{k>n} 3^{-(k+1)N(s\restriction k)} \leq 2 \sum_{i=1}^{\infty} 3^{-[(n+1)N(s\restriction n)+i]} = 3^{-(n+1)N(s\restriction n)}$$

So, (i): $|h(s) - h(t)| \geq 3^{-(n+1)N(s\restriction n)}$;
and (ii): $|h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}$.

Proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2 s_n 3^{-(n+1)N(s\restriction n)}$,

Have: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega \ \exists k < \omega \ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

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From shrinking maps to minimal dynamics

For a metric space $\langle X, d\rangle$ and a map $f\colon X \to X$

- *f is pointwise shrinking, PS*, if for every $x \in X$ there is open $U \ni x$ such that $d(f(x), f(y)) < d(x, y)$ for all $y \in U$, $y \neq x$.
- If $X \subset \mathbb{R}$ and $|f'| < 1$ everywhere, then f is *PS*.

Theorem (KC & JJ 2014)

If $f\colon X \to X$ is onto, PS, and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamical system.

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LS and X is compact, then f has a periodic point,

- *f is locally shrinking, LS*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is *shrinking*, that is, $d(f(y), f(z)) < d(y, z)$ for every distinct $y, z \in U$.

Sketch of proof

$\langle X, d\rangle$ is infinite compact, $f\colon X \to X$ is pointwise shrinking

Thm: There is perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamics.

This is proved by showing the following facts:

1. $T \subseteq X$ infinite compact & $T \subset f[T]$, imply T is uncountable. ($T \subset f[T]$ for no countable T of Cantor-Bendixon rank $\alpha < \omega_1$.)
2. $F_m = \{x \in P\colon f^{(m)}(x) = x\}$ is finite for every $m \in \mathbb{N}$.
3. For every orbit $O(x)$ of $x \in F = \bigcup_{m\in\mathbb{N}} F_m$, $f[B(O(x), \varepsilon)] \subseteq B(O(x), \varepsilon)$ for every small enough $\varepsilon > 0$.
4. There is open $U \supset F$ s.t. $T = X \setminus U$ is infinite & $T \subset f[T]$.
5. Find minimal P in $\{P \subset T\colon$ compact $\neq \emptyset$ s.t. $P \subset f[P]\}$. (Exists by Zorn's Lemma—Birkhoff's argument.) Such P is as needed.

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Can $\mathfrak{X}$ from main example be (path) connected?

Open Problem (Pr1)

Let $\langle X, d\rangle$ be compact & either **connected** or **path connected**.
If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is *PS*, must f have fix/periodic point?
What if f is *PC* or *uPC*, where

f is *pointwise contractive, PC*, if for every $x \in X$ there are open $U \ni x$ and $\lambda \in [0, 1)$ s.t. $d(f(x), f(y)) \le \lambda d(x, y)$ for all $y \in U$;

f is *uPC*, if there is $\lambda \in [0, 1)$ s.t. for every $x \in X$ there is open $U \ni x$ for which $d(f(x), f(y)) \le \lambda d(x, y)$ for all $y \in U$.

What is known on Problem Pr1

Pr1: For X compact & either **connected** or **path connected**, if $f\colon X \to X$ is *PS/PC/uPC*, must f have fix/periodic point?

- $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ shows that connectedness is essential;
- True, when X is rectifiably path connected and f is *PC*:

Theorem (KC & JJ 2016)

Assume that $\langle X, d\rangle$ is compact rectifiably path connected metric space. If $f\colon X \to X$ is PC, then f has a unique fixed point.

This is variant of 1978 theorem of Hu and Kirk 1978 (corrected by Jungck in 1982) proved without compactness of X, but with a stronger assumption that f is *uniformly PC, UPC*.

Compactness is essential: Hu and Kirk 1978 gave an example of path connected X and *uPC* map $f\colon X \to X$ with no periodic point.

One more problem

Open Problem (Pr2)

Let $\langle X, d\rangle$ be compact and **rectifiably path connected**.
If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is *PS*, must f have fix/periodic point?

Pr1 and Pr2 are the only open problems in our comprehensive study of ten classes of self-maps on metric spaces $\langle X, d\rangle$ with the local and pointwise (a.k.a. local radial) contraction properties.

The relations among the classes, assuming different topological properties of X, are represented as graphs, a sample of which is shown below.

Graph sample

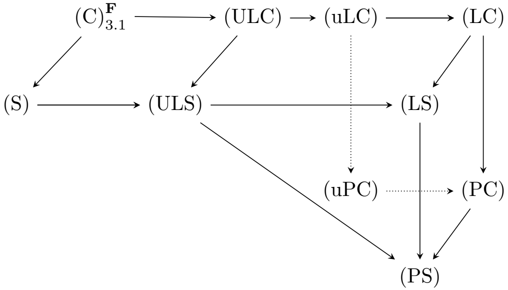

Figure: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being an arbitrary complete metric space.

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Can Peano-like functions be differentiable?

For perfect $P \subset \mathbb{R}$,

(Q1) Can surjective continuous map $f\colon P \to P^2$ be differentiable?

- No for P of positive Lebesgue measure, e.g., for $P = [0,1]$.
- [KC & JJ 2014]: Yes, if we allow unbounded sets P. Such an f can even have a C^∞ extension $F\colon \mathbb{R} \to \mathbb{R}^2$.
- [KC & JJ 2014]: No, if P is compact and f is extendable to a C^1 map $F\colon \mathbb{R} \to \mathbb{R}^2$.

Still Open Problem

Pr3: Question **(Q1) when P is compact of measure 0.**

From Peano problem Pr1 to dynamical systems

Theorem (KC & JJ 2014)

If $\langle f, g\rangle\colon P \to P^2$ is a differentiable surjection, then $f[K] = P$, where $K = \{x \in P\colon f'(x) = 0\}$.

Proof.

f is countable-to-one on the F_σ set $P \setminus K$. □

K need not be compact. But can it be?

(Q2) Does there exist $f\colon K \to \mathbb{R}$, with $K \subset \mathbb{R}$ compact perfect, such that $f' \equiv 0$ and $K \subseteq f[K]$?

Fact (corollaries from the theorems we discussed)

- *For every f as in* **(Q2)** *there is a perfect $P \subset K$ s.t. $f \restriction P$ is a minimal dynamical system (i.e., the orbit of every $x \in P$ is dense in $P = f[P]$).*
- *There exist a minimal system $f\colon P \to P$ with $f' \equiv 0$.*

Summary: open problems on self maps

1. Let $\langle X, d \rangle$ be compact and **rectifiably path connected**.
   If $f : \langle X, d \rangle \to \langle X, d \rangle$ is *PS*, must f have fix/periodic point?.

2. For X compact & either **connected** or **path connected**,
   if $f : X \to X$ is *PS/PC/uPC*, must f have fix/periodic point?

We do not even know, what happens in the problems when X is a (topologically) manifold!

(Though, the maps must have fix points when the metric on X is convex.)

That is all!

Thank you for your attention!