Differentiability versus continuity: monstrous examples and extension theorems

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Based on papers/notes with links 121, 129, S21, S22
see also http://www.math.wvu.edu/~kcies/publications.html.
These notes are here.

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Preamble: new, 2017, results that are behind this talk

Example (New simple construction of a classic example)

There exists a differentiable nowhere monotone map $f \colon \mathbb{R} \to \mathbb{R}$.

Example (Greatly simplified construction of 2016 example)

There exists a differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect $\mathfrak{X} \subset \mathbb{R}$ with $\mathfrak{f}' \equiv 0$.

Theorem (C^1 interpolation thm, no Lebesgue measure needed)

For every continuous $f \colon \mathbb{R} \to \mathbb{R}$:

- *there is perfect $P \subset \mathbb{R}$ s.t. $f \restriction P$ is Lipschitz;*
- *there is C^1 map $g \colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.*

Theorem (Simple proof of Whitney and Jarník extension thms)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f \colon Q \to \mathbb{R}$ has differentiable extension $F \colon \mathbb{R} \to \mathbb{R}$.This F is C^1 iff such extension exists iff a simple (new) condition for f holds.

No familiarity
with Lebesgue measure

is needed to follow the proofs
sketched in this talk

Outline

1. Continuity from differentiability: classical results
2. Continuity from differentiability: Differentiable Monster
3. Differentiability from continuity: differentiable restrictions
4. Properties of differentiable maps on perfect $P \subset \mathbb{R}$
5. Differentiable extensions: Jarník and Whitney theorems

Outline

Positive, well known results

Clearly any differentiable function $F\colon \mathbb{R} \to \mathbb{R}$ is continuous.

True question: *To what extend $f = F'$ must be continuous?*

Theorem (Classic)

The set C_f of points of continuity of $f = F'$ is a dense G_δ-set.

Pr. F' is Baire 1: $F'(x) = \lim_n F_n(x)$, $F_n(x) = \frac{f(x+1/n)-f(x)}{1/n}$ cont.

Density of C_f: C_f contains a residual set

$$\bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \mathrm{int}(\{x \in \mathbb{R} \colon |F_k(x) - F_m(x)| \le 1/n \text{ for all } m, k \ge N\}).$$

C_f is a G_δ-set:

$$C_f = \bigcap_{n=1}^{\infty} \bigcup_{\delta>0} \{x \colon |f(s) - f(t)| < 1/n \text{ for all } s, t \in (x-\delta, x+\delta)\}.$$

This is it:

Theorem (Classic)

For any dense G_δ-set G there is a derivative f with $C_f = G$.

Pr. Enough to prove for maps on $(0,1)$.

Let $(0,1) \setminus G = \bigcup_{n=1}^{\infty} P_n$, each P_n closed nowhere dense.

Define $f_n \colon (0,1) \to \mathbb{R}$ as 0 on P_n and as

$$f_n(x) = \frac{(x-a)^2(x-b)^2}{(b-a)^2}\left[\sin\left(\frac{1}{x-a}\right) + \sin\left(\frac{1}{x-b}\right)\right]$$

on any component (a,b) of $(0,1) \setminus P_n$.

Then $f = \sum_{n=1}^{\infty} 3^{-n} f_n$ is as needed. □

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Differentiable monster

Example (Köpcke 1887-1890; Denjoy 1915; Katznelson & Stromberg 1974; Weil 1976; and many others)

There is differentiable $f\colon \mathbb{R} \to \mathbb{R}$ which is nowhere monotone.

Note that

- Differentiable f is a monster iff f' attains on every interval both positive and negative values.
- So, the derivative f' of a differentiable monster is discontinuous on the dense set $Z^c = \{x\colon f'(x) \neq 0\}$.

Simple construction of a differentiable monster follows.

A variant of Pompeiu function, of 1907

Fix $r \in (0,1)$ and $\mathbb{Q} = \{q_i \colon i \in \mathbb{N}\}$ such that $|q_i| \le i$ for all $i \in \mathbb{N}$.

Lemma (KC; small variation of Pompeiu's result)

(i) $g(x) = \sum_{i=1}^{\infty} r^i (x - q_i)^{1/3}$ *is continuous, "differentiable," strictly increasing, onto* $\mathbb{R}$, *with* $g'(q) = \infty$ *for all* $q \in \mathbb{Q}$.

(ii) $h = g^{-1} \colon \mathbb{R} \to \mathbb{R}$ *is everywhere differentiable with* $h' \ge 0$ *and* $Z = \{x \in \mathbb{R} \colon h'(x) = 0\}$ *being a dense* G_δ*-set.*

(iii) $Z^c = \mathbb{R} \setminus Z$ *is also dense in* $\mathbb{R}$.

Pr. (i) Continuity follows from $|g(x)| \le \sum_{i=1}^{\infty} r^i (|x| + i + 1)$.

Differentiability requires $g'(x) = \sum_{i=1}^{\infty} r^i \frac{1}{3} \frac{1}{(x-q_i)^{2/3}}$. Easy when series $= \infty$. Other case follows from $0 < \frac{\psi_i(y) - \psi_i(x)}{y - x} \le 6\psi_i'(x)$.

(ii) and (iii) easily follow from (i).

New simple construction of a differentiable monster

Lemma There is a strictly increasing differentiable $h\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{x \in \mathbb{R}\colon h'(x) = 0\}$ being a dense G_δ-set.

Theorem (KC 2017)

If h is as in Lemma, then $f(x) = h(x - t) - h(x)$ is a differentiable monster for any typical $t \in \mathbb{R}$.

Pr. Let $D \subset \mathbb{R} \setminus Z$ be countable dense. $h' > 0$ on D.

Any t in residual $G = \bigcap_{d \in D}((-d + Z) \cap (d - Z))$ works.

Clearly f is differentiable with $f'(x) = h'(x - t) - h'(x)$.

$f' > 0$ on $t + D$: $f'(t + d) = h'(d) - h'(t + d) = h'(d) > 0$, as $t + d \in Z$.

$f' < 0$ on D: $f'(d) = h'(d - t) - h'(d) = -h'(d) < 0$, as $d - t \in Z$. □

Outline

How much differentiability continuous map must have

None?

Example (Weierstrass Monster)

There exists continuous $F\colon \mathbb{R} \to \mathbb{R}$ differentiable at no point.

Proof.

Put $F(x) = \sum_{n=0}^{\infty} 4^n min\{|x - \frac{k}{8^n}|\colon k \in \mathbb{Z}\}$.

Continuous at each $x \in \mathbb{R}$, since for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$
$|f(y) - f(x)| \le \left|\sum_{i=0}^{n} 4^i f_i(y) - \sum_{i=0}^{n} 4^i f_i(x)\right| + \frac{1}{2^n}$.

Not differentiable at $x \in \mathbb{R}$, since for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $x \in \left[\frac{k}{8^n}, \frac{k+1}{8^n}\right]$ there exists a $y_n \in \left\{\frac{k}{8^n}, \frac{k+1}{8^n}\right\} \setminus \{x\}$ such that
$\left|\frac{f(x)-f(y_n)}{x-y_n}\right| \ge \left|\frac{f(\frac{k+1}{8^n})-f(\frac{k}{8^n})}{\frac{k+1}{8^n}-\frac{k}{8^n}}\right| = \left|\sum_{i=0}^{n-1} \frac{f_i(\frac{k+1}{8^n})-f_i(\frac{k}{8^n})}{\frac{k+1}{8^n}-\frac{k}{8^n}}\, 4^i\right| =$
$\left|\sum_{i=0}^{n-1} \pm 4^i\right| \ge \frac{2}{3} 4^{n-1}$. □

Differentiable restriction theorem

Some differentiability after all!

Theorem (Laczkovich 1984)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is perfect $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable.

Remark

There are continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f \restriction Q$ can be differentiable only when Q is both first category and meager.

Proof.

Let $f = (f_1, f_2)\colon [0,1] \to [0,1]^2$ be the classical (ternary-like) Peano curve. Ciesielski and Larson proved in 1991 that f_1 is nowhere approximately and $\mathcal{I}$-approximately differentiable. So it is as in the remark. □

New proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Theorem (With new simple proof, by KC)

For every continuous increasing $f\colon [a,b] \to \mathbb{R}$ there is perfect P such that $f \restriction P$ is Lipschitz.

Proof based on the following results, due to Riesz:

Lemma (Rising sun lemma, proof is an easy exercise)

If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \le g(d)$ for every component (c,d) of $\{x \in [a,b)\colon g(x) < g(y) \text{ for some } y \in (x,b]\}$.

Fact (Proved by induction)

Let $a < b$ and $\mathcal{J}$ be a family of open intervals with $\bigcup \mathcal{J} \subset (a,b)$.

(i) *If $[\alpha,\beta] \subset \bigcup \mathcal{J}$, then $\sum_{I\in\mathcal{J}} \ell(I) > \beta - \alpha$.*

(ii) *If $I \in \mathcal{J}$ are pairwise disjoint, then $\sum_{I\in\mathcal{J}} \ell(I) \le b - a$.*

Proof of Lipschitz part

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Have: If $g\colon [a,b] \to \mathbb{R}$ is cont, then $g(c) \le g(d)$ for every comp. (c,d) of $\{x \in [a,b)\colon g(x) < g(y) \text{ for some } y \in (x,b]\}$.

Sketch of proof. Fix $L > \frac{f(b)-f(a)}{b-a}$, put $g(t) = f(t) - Lt$, and

$$U = \{x \in [a,b)\colon g(y) > g(x) \text{ for some } y \in (x,b]\}.$$

f is Lipschitz on $P = [\bar{a}, b] \setminus U$ with constant L, where

$\bar{a} = \sup\{x\colon [a,x) \subset U)\}$. Fix $X = \{x_n\colon n \in \mathbb{N}\}$. Need $P \setminus X \neq \emptyset$.

If $\mathcal{J} =$ open components of U, then $\ell(f[J]) \ge L\ell(J)$ for $J \in \mathcal{J}$.

By Fact (ii), $\sum_{J\in\mathcal{J}} \ell(f[J]) \le f(b) - f(\bar{a})$. So,

$\sum_{J\in\mathcal{J}} \ell(J) \le \frac{1}{L} \sum_{J\in\mathcal{J}} \ell(f[J]) \le \frac{f(b)-f(\bar{a})}{L} < b - \bar{a}$, and by Fact (i),

$P \neq \emptyset$. To get $P \setminus X \neq \emptyset$ increase slightly $\mathcal{J}$.

End of proof of differentiable restriction theorem

Goal: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont, then $f \restriction Q$ is diff. for some perfect Q.

Have: If $f\colon \mathbb{R} \to \mathbb{R}$ is cont $\nearrow$, then $f \restriction P$ is Lipschitz for a perfect P.

Proof of differentiable restriction theorem.

f is Lipschitz on some perfect P: proved above for somewhere monotone f; otherwise f is constant on some perfect set.

For function $f \restriction P$ use Morayne theorem to find perfect $Q \subset P$ such that the quotient map for $f \restriction Q$ is uniformly continuous. Then Q is as needed. □

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Are differentiable $f\colon P \to \mathbb{R}$, $P \subset \mathbb{R}$ perfect, good?

Not at all!

Example (Ciesielski & Jasinski 2016 example simplified)

There exists differentiable auto-homeomorphism $\mathfrak{f}$ of a compact perfect subset $\mathfrak{X}$ of the Cantor ternary set $\mathfrak{C}$ such that $\mathfrak{f}' \equiv 0$.

Counterintuitive, as $\mathfrak{f}$ is shrinking at every $x \in \mathfrak{X}$
($|\mathfrak{f}(x) - \mathfrak{f}(y)| < |x - y|$ for every $y \in \mathfrak{X}$ with small $|x - y| > 0$)
but it maps compact $\mathfrak{X}$ **onto** itself.

Actually, $\mathrm{Lip}\, \mathfrak{f}(x) < \varepsilon$ for all $\varepsilon > 0$ and $x \in \mathfrak{X}$.

$\mathfrak{f} = h \circ \sigma \circ h^{-1}$, where $h\colon 2^\omega \to \mathbb{R}$ is embedding and
$\sigma\colon 2^\omega \to 2^\omega$ is the “add one and carry” adding machine:

$$\sigma(s) = \begin{cases} \langle 0,0,0,\ldots \rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0,0,\ldots,0,1,s_{k+1},s_{k+2},\ldots \rangle & \text{if } s_k = 0 \text{ and } s_i = 1 \text{ for } i < k. \end{cases}$$

Definition of $h\colon 2^\omega \to \mathbb{R}$ with $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

$h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$, where $N(s \restriction 0) = 1$ and
$N(s \restriction n) = \sum_{i<n-1} s_i 2^i + (1 - s_{n-1})2^{n-1} + 2^n$ for $n > 0$.

Fact: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega\colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \overset{\text{(i)}}{\leq} |h(s) - h(t)| \overset{\text{(ii)}}{\leq} 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Proof. For $h_n(s) = \sum_{k<n} 2s_k 3^{-(k+1)N(s\restriction k)}$ we have

$$h_n(s) + 2s_n 3^{-(n+1)N(s\restriction n)} \leq h(s) \leq h_n(s) + (2s_n + 1)3^{-(n+1)N(s\restriction n)},$$

as $h(s) = h_n(s) + 2s_n 3^{-(n+1)N(s\restriction n)} + 2\sum_{k>n} 3^{-(k+1)N(s\restriction k)}$ and

$$0 \leq 2\sum_{k>n} 3^{-(k+1)N(s\restriction k)} \leq 2\sum_{i=1}^{\infty} 3^{-[(n+1)N(s\restriction n)+i]} = 3^{-(n+1)N(s\restriction n)}$$

So, (i): $|h(s) - h(t)| \geq 3^{-(n+1)N(s\restriction n)}$;
and (ii): $|h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}$.

Proof of $\mathfrak{f}' \equiv 0$ for $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

Def: $h(s) = \sum_{n=0}^{\infty} 2s_n 3^{-(n+1)N(s\restriction n)}$,

Have: If $s \neq t \in 2^\omega$ and $n = \min\{i < \omega \colon s_i \neq t_i\}$, then

$$3^{-(n+1)N(s\restriction n)} \leq |h(s) - h(t)| \leq 3 \cdot 3^{-(n+1)N(s\restriction n)}.$$

Also (a): $\forall s \in 2^\omega\ \exists k < \omega\ N(\sigma(s) \restriction n) = N(s \restriction n) + 1$ for all $n > k$

as it fails only for $s = \langle s_0, \ldots, s_{n-2}, s_{n-1}, \ldots \rangle = \langle 1, \ldots, 1, 0, \ldots \rangle$.

Proof of $\mathfrak{f}' \equiv 0$.

To see $\mathfrak{f}'(h(s)) = 0$: pick $k < \omega$ from (a) and $\delta > 0$ s.t. $0 < |h(s) - h(t)| < \delta$ implies $n = \min\{i < \omega \colon s_i \neq t_i\} > k$. Then,

$$\frac{|\mathfrak{f}(h(s)) - \mathfrak{f}(h(t))|}{|h(s) - h(t)|} \leq \frac{3 \cdot 3^{-(n+1)N(\sigma(s)\restriction n)}}{3^{-(n+1)N(s\restriction n)}} = 3 \cdot 3^{-(n+1)}.$$

So $\mathfrak{f}'(h(s)) = 0$, as $3 \cdot 3^{-(n+1)}$ is arbitrarily small for small δ. □

Must every orbit of $\mathfrak{f}$ be dense in $\mathfrak{X}$?

For a metric space $\langle X, d\rangle$ and a map $f\colon X \to X$

- *f is pointwise contractive, PC*, if for every $x \in X$ there is open $U \ni x$ and $L \in [0,1)$ s.t. $\mathrm{Lip}\, f(x) \le L$ for all $x \in U$.

Theorem (KC & JJ 2014: YES, essentially)

If $f\colon X \to X$ is onto, PC, and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamical system, i.e., every orbit of $f \restriction P$ is dense in P.

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LC and X is compact, then f has a periodic point,

- *f is locally contractive, LC*, provided for every $x \in X$ there is open $U \ni x$ s.t. $f \restriction U$ is Lipschitz with constant < 1.

Related open problem

Open Problem

Let $\langle X, d\rangle$ be compact & either **connected** or **path connected**.
If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is *PC*, must f have fix/periodic point?

- $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ shows that connectedness is essential;
- True, when X is rectifiably path connected and f is *PC*:

Theorem (KC & JJ 2016)

Assume that $\langle X, d\rangle$ is compact rectifiably path connected metric space. If $f\colon X \to X$ is PC, then f has a unique fixed point.

(Variant of 1978 Hu and Kirk thm, corrected by Jungck in 1982.)

Compactness is essential, Hu and Kirk 1978: there is path connected X and *PC* map $f\colon X \to X$ with no periodic point.

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Notation

For $J = (a, a + h)$ let $I_J = [a + h/3, a + 2h/3]$, middle third of J.

For closed $Q \subset \mathbb{R}$ and $f \colon Q \to \mathbb{R}$ let

$\hat{Q} = Q \cup \bigcup \{I_J \colon J \text{ is a bounded connected component of } \mathbb{R} \setminus Q\}$,

$\bar{f} \colon \mathbb{R} \to \mathbb{R}$ — "the" linear interpolation of f, $\hat{f} = \bar{f} \restriction \hat{Q}$.

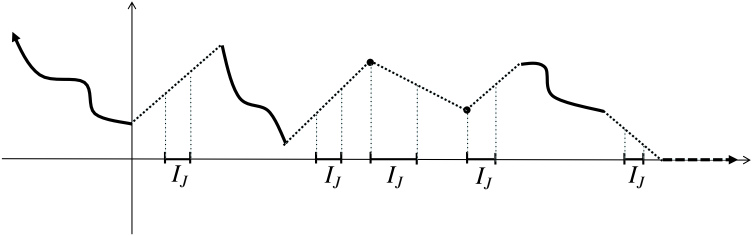


Figure: The linear interpolation $\bar{f}$ of f, represented by thick curves.

Jarník's differentiable extension theorems

Theorem (Jarník 1923)

If $Q \subset \mathbb{R}$ is perfect, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proved in:

V. Jarník, *O rozšíření definičního oboru funkcí jedné proměnné, přičemž zůstává zachována derivabilita funkce* (in Czech) Rozpravy Čes. akademie, II. tř., XXXII (1923), No. 15, 15 p.

Sketched in: V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dé rivabilité de la fonction* (in French), Bull. Internat. de l'Académie des Sciences de Bohême (1923), 1–5.

Independently proved in 1974 by Petruska and Laczkovich.

Jarník and Whitney differentiable extension theorems

Theorem (Jarník and Whitney thms, version of MC&KC 2017)

If $Q \subset \mathbb{R}$ is closed, than any differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$. This F is C^1 iff such extension exists iff $\hat{f}$ is continuously differentiable.

Corollary (Agronsky, Bruckner, Laczkovich, Preiss 1985: C^1 interpolation theorem)

For every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ with $f \cap g$ uncountable.

Proof of Corollary: We proved that there is perfect $Q \subset \mathbb{R}$ s.t. the quotient map of $h = f \restriction Q$ is uniformly continuous.

It is easy to see that $\hat{h}$ is continuously differentiable for such h.

Our proof of Jarník and Whitney thms (for perfect Q)

Differentiable $f\colon Q \to \mathbb{R}$ has differentiable extension $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition (Linear interpolation almost works)

If $f\colon Q \to \mathbb{R}$ is differentiable, then the unilateral derivatives of $\bar{f}$ exist and are finite everywhere.
So, $\bar{f}$ is differentiable at any $x \in \mathbb{R}$ which is not in the set E of the end-points of connected components of $\mathbb{R} \setminus \tilde{Q}$.

Proof.

Clear, unless $x \in \tilde{Q} \setminus E$.
Estimate is clear, unless $y \in (a, b)$ for a component of $\tilde{Q}^c$, when

$$\frac{|\bar{f}(y)-\bar{f}(x)|}{y-x} \text{ is between } \frac{|f(a)-f(x)|}{a-x} \text{ and } \frac{|f(b)-f(x)|}{b-x}, \text{ so}$$

$$\left|f'(x) - \frac{|\bar{f}(y)-\bar{f}(x)|}{y-x}\right| \le \max\left\{\left|f'(x) - \frac{|f(a)-f(x)|}{a-x}\right|, \left|f'(x) - \frac{|f(b)-f(x)|}{b-x}\right|\right\}$$

□

From linear interpolation to differentiable extension

Goal: Differentiable $f\colon Q \to \mathbb{R}$ has differ. extension $F\colon \mathbb{R} \to \mathbb{R}$.

Have: Linear interpolation almost works.

Solution: Small modification of linear interpolation, $F = \bar{f} + g$:

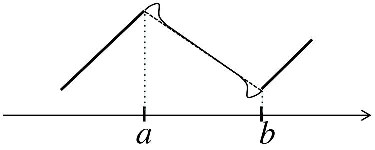


Figure: A format of the graph (thin continuous curve) of $F = \bar{f} + g$ on a component (a, b) of $\mathbb{R} \setminus Q$. Thick segments: parts of the graph of f

g must satisfy: $g \restriction [a, b]$ is C^1,

$(\tilde{*})$ $D^+ g(a) = \tilde{f}'(a) - \frac{\tilde{f}(b)-\tilde{f}(a)}{b-a}$ and $D^- g(b) = \tilde{f}'(b) - \frac{\tilde{f}(b)-\tilde{f}(a)}{b-a}$.

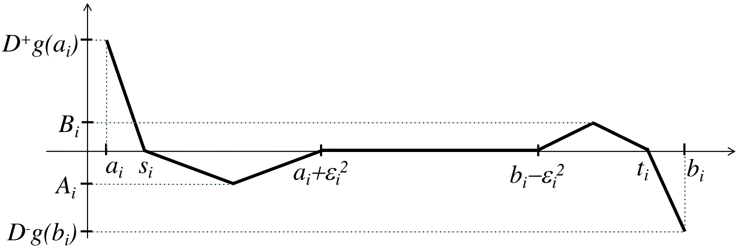
Format of a modifier g

$\{(a_i, b_i)\colon 1 \le i \le \kappa\}$ enumeration of components of Q^c; need

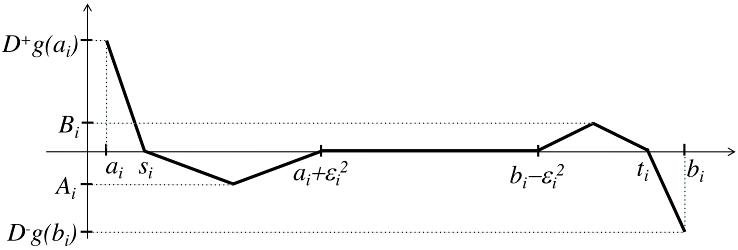
$$(*) \quad D^+g(a_i) = f'(a_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i},\ D^-g(b_i) = f'(b_i) - \frac{f(b_i)-f(a_i)}{b_i-a_i}$$

and g is small enough to preserve differentiability at Q.

$g \restriction [a_i, b_i]$ defined as $g(x) = \int_{a_i}^{x} h_i(r)\, dr$, where h_i is as in figure



How to ensure that g works?



For $\ell_i = \min\{1, b_i - a_i\}$ let $\varepsilon_i \in (0, 3^{-i}\ell_i)$ be such that

(a) $\left| f'(a_i) - \frac{f(x)-f(a_i)}{x-a_i} \right| < 3^{-i}$ for every $x \in Q \cap [a_i - \varepsilon_i, a_i)$;

(b) $\left| f'(b_i) - \frac{f(x)-f(b_i)}{x-b_i} \right| < 3^{-i}$ for every $x \in Q \cap (b_i, b_i + \varepsilon_i]$.

Pick $s_i \in (a_i, a_i + \varepsilon_i^2)$ s.t. $\int_{a_i}^{s_i} |h_i(r)|\, dr = \frac{1}{2}|h_i(a_i)|(s_i - a_i) < \varepsilon_i^2$; A_i so that $\int_{a_i}^{a_i+\varepsilon_i^2} h_i(r)\, dr = 0$; similarly for t_i and B_i. Prove it works.

Summary

- $f\colon \mathbb{R} \to \mathbb{R}$ differentiable implies f' is continuous on big set.
- This is it: there is a Differentiable Monster.
- $f\colon \mathbb{R} \to \mathbb{R}$ continuous implies $f \restriction P$ differentiable for a perfect $P \subset \mathbb{R}$.
- $f \restriction P$ can be extended to differentiable $F\colon \mathbb{R} \to \mathbb{R}$.
- Differentiable $f \restriction P$ can be **very weird**!

That is all!

Thank you for your attention!