Locally contractive maps and fixed point theorems

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The talk is based on the following papers:

- K.C. Ciesielski and J. Jasinski, *An auto-homeomorphism of a Cantor set with derivative zero everywhere*, J. Math. Anal. Appl. **434** (2016), 1267–1280.
- K.C. Ciesielski and J. Jasinski, *On fixed points of locally and pointwise contracting maps*, Topology Appl. **204** (2016), 70–78.
- K.C. Ciesielski and J. Jasinski, *Fixed point theorems of locally and pointwise contracting maps*, Canadian J. of Math. (2017) accepted.

Abstract

Definition (#6)

A selfmap $f : X \to X$ on a metric space $\langle X, d \rangle$ is *Pointwise Contractive,* (PC) if for any $x \in X$ there exist an $\varepsilon_x > 0$ and a $\lambda_x \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda_x d(x, y) \text{ for every } y \in B(x, \varepsilon_x).$$

The class PC is also known as Local Radial Contractions.

We discuss the following:

Theorem (KC & JJ, 2016)

If X is a compact rectifiably path connected space and $f : X \to X$ is a pointwise contractive map then f has a unique fixed point, that is, there exists a unique point $\xi \in X$ such that $f(\xi) = \xi$.

We will present it in the context of classical fixed point results for other classes of local and pointwise contractive maps.

Definition (#1)

A function $f : X \to X$ is called *Contractive*, (C), if there exists a constant $0 \leq \lambda < 1$ such that for any two elements $x, y \in X$ we have $d(f(x), f(y)) \leq \lambda d(x, y)$.

Theorem (Banach, 1922)

If (X, d) is a complete metric space and $f : X \to X$ is (C), then f has a unique fixed point.

Definition (#2)

A function $f : X \to X$ is called Shrinking, (S), if for any two elements $x, y \in X, x \neq y$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

If $\langle X, d\rangle$ is compact and $f : X \to X$ is (S), then f has a unique fixed point.

Local Classics

Definition (#3)

A function $f : X \to X$ is called *Locally Shrinking,* (LS), if for any element $z \in X$ there exists an $\varepsilon_z > 0$ such that $f \restriction B(z, \varepsilon_z)$ is shrinking, i.e. for any two $x \neq y \in B(z, \varepsilon_z)$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

Let $\langle X, d\rangle$ be compact and let $f : X \to X$.

(i) *If f is (LS), then f has a periodic point.*

(ii) *If f is (LS) and X is connected, then f has a unique fixed point.*

Definition (#4)

A function $f : X \to X$ is called *uniformly Pointwise Contracting,* (uPC), (a.k.a. uniform Local Radial Contractions) if there exists a $\lambda \in [0, 1)$ such that for every $z \in X$ there exists an $\varepsilon_z > 0$ such that for any element $x \in B(z, \varepsilon_z)$ we have $d(f(x), f(z)) \leq \lambda d(x, z)$.

Theorem (Hu and Kirk, 1978; proof corrected by Jungck, 1982)

If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f : X \to X$ is (uPC), then f has a unique fixed point.

Definition (#5)

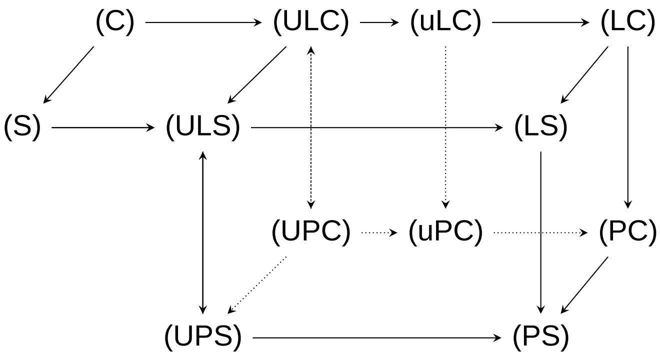
A function $f : X \to X$ is called *Uniformly Locally Contracting,* (ULC), if there exist a $\lambda \in [0, 1)$ and an $\varepsilon > 0$ such that for every $z \in X$ the restriction $f \restriction B(z, \varepsilon)$ is contractive with that λ.

Theorem (Edelstein, 1961)

Assume that $\langle X, d \rangle$ is complete and that $f : X \to X$ is (ULC)
If X is connected, then f has a unique fixed point.

Local Properties

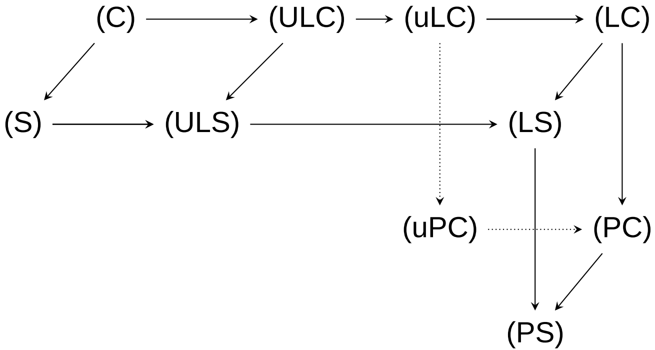
Classes of *Locally*, (L) (two variables) OR *Pointwise*, (P) (one variable)
AND *Contractive*, (C) (with λ) OR *Shrinking*, (S) (no λ)
make the following diagram:



Remark: (ULS)=(UPS) and (ULC)=(UPC).

Local Properties

Therefore,

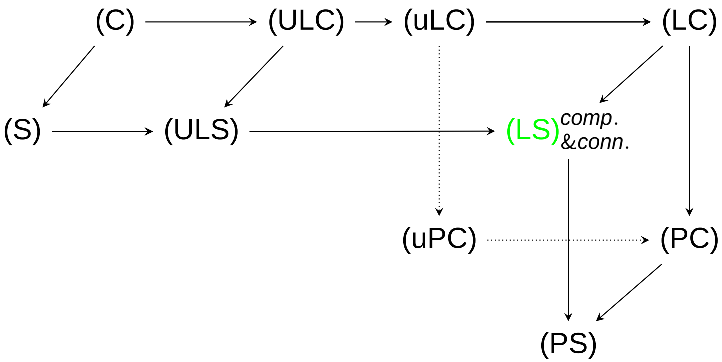


is the real picture.
So where are the fixed point theorems?
Of course it depends on the space X.

Summary of Local Properties with Fixed Points

All spaces X are assumed to be complete.

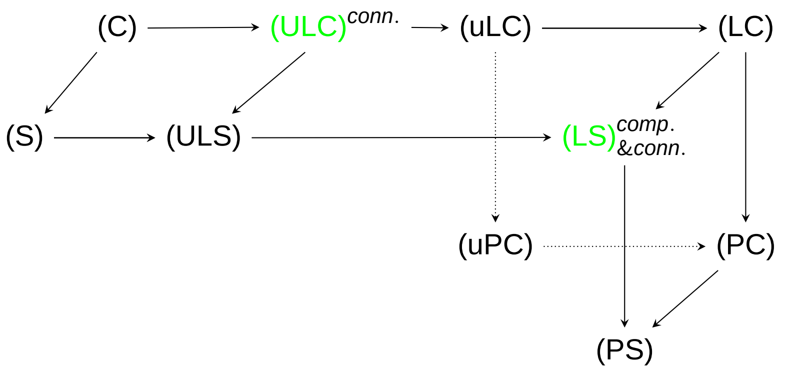
Edelstein (LS) + X compact and connected



Summary of Local Properties with Fixed Points

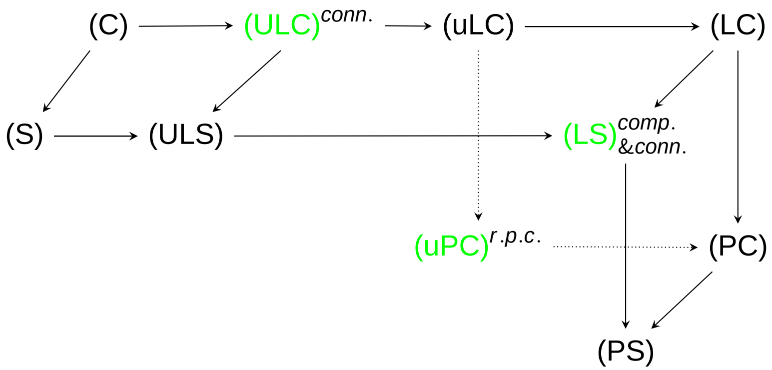
Edelstein (LS) + X compact and connected

Edelstein (ULC) + X connected



Summary of Local Properties with Fixed Points

Edelstein (LS) + X compact and connected
Edelstein (ULC) + X connected
Hu & Kirk (uPC) & X rectifiably path connected (r.p.c.)



Recall,

Definition (#6)

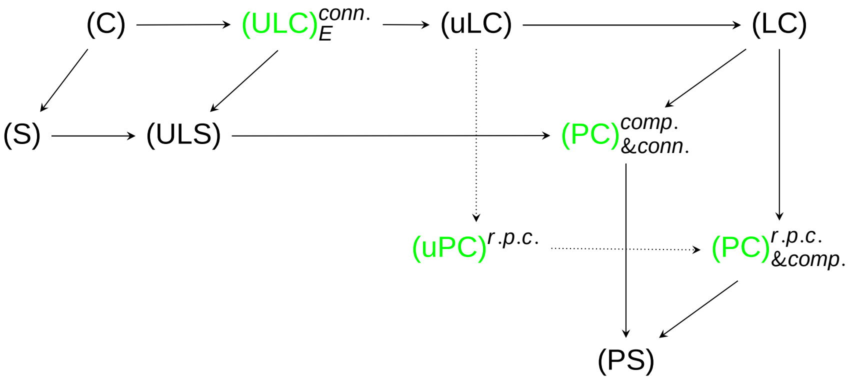
A function $f : X \to X$ is called *Pointwise Contractive,* (PC), if for every $z \in X$ there exist $\lambda_z \in [0, 1)$ and an $\varepsilon_z > 0$ such that $d(f(x), f(z)) \leq \lambda d(x, z)$ whenever $x \in B(z, \varepsilon)$.

Theorem (C & J, Top. and its App. 204 2016 70-78)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If $f : X \to X$ is (PC)*, then f has a unique fixed point.*

Local Properties with Fixed Points

Edelstein (LS) and X compact and connected
Edelstein (ULC) and X connected
Hu & Kirk (uPC) and X rectifiably path connected.
KC & JJ (PC) and X rectifiably path connected and compact



Necessity of compactness of X

Recall,

Theorem (KC & JJ, 2016)

If X is a compact rectifiably path connected space and $f : X \to X$ is a (PC) map then f has a unique fixed point, that is, there exists a unique point $\xi \in X$ such that $f(\xi) = \xi$.

Example (Munkres, 1974)

The map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right)$ has no periodic points because $f(x) > x$ for all $x \in X$. It is (PC), in fact, f is from the class $(\mathrm{S}) \cap (\mathrm{LC})$. This follows from the MVT because $f'(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$ hence, for any $a \in \mathbb{R}$, $f'[(-\infty, a]] = (0, c]$ for some $c \in (0, 1)$.

Necessity of compactness X

Definition (#7)

A map $f : X \to X$ is *uniformly Locally Contractive*, (uLC) if there exists a $\lambda \in [0,1)$ such for every $z \in X$ there is an $\varepsilon_x > 0$ so that $d(f(x), f(y)) \leq \lambda d(x,y)$ for any two $x, y \in B(z, \varepsilon_x)$.

Example (Rakotch, 1962)

There exists a closed, connected subset $X \subset \mathbb{R}^2$ and a map $f : X \to X$ which is (uLC) but every forward-orbit of f is divergent. So f has no periodic points.

Necessity of connectedness of X

For a selfmap f on $\langle X, d\rangle$ and a limit point $x \in X$, let

$$D^*f(x) = \limsup_{y\to x} \frac{d(f(x),f(y))}{d(x,y)},$$

and for isolated point x we set $D^*f(x) = 0$. D^* is called as *absolute derivative* by Charatonik and Insall.

Remark

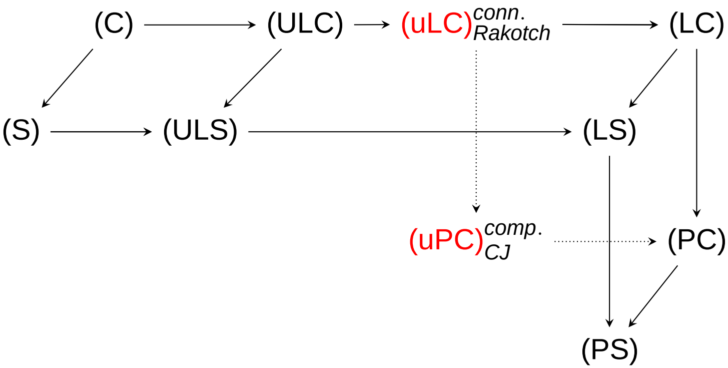
For $f : X \to X$,

- f is (PC) iff $D^*f(x) < 1$ for all $x \in X$.
- f is (uPC) iff $sup\{D^*f(x) : x| \in X\} < 1$

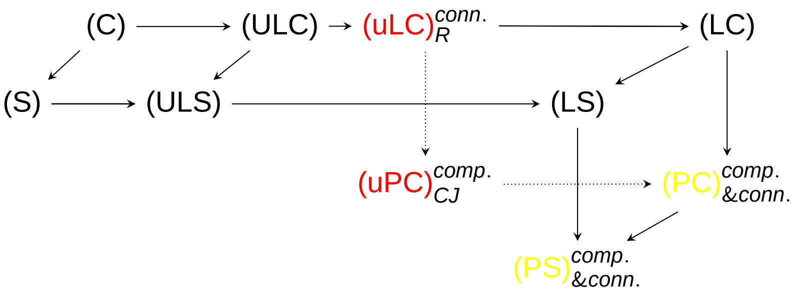
Example (KC & JJ, 2016)

There exists a compact perfect (Cantor-like) set $\mathfrak{X} \subseteq \mathbb{R}$ and an auto-homeomorphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ with $D^*\mathfrak{f}' \equiv 0$ (so $\mathfrak{f}$ is (uPC) with any $\lambda \in (0, 1)$) and without periodic points.

No periodic points Examples



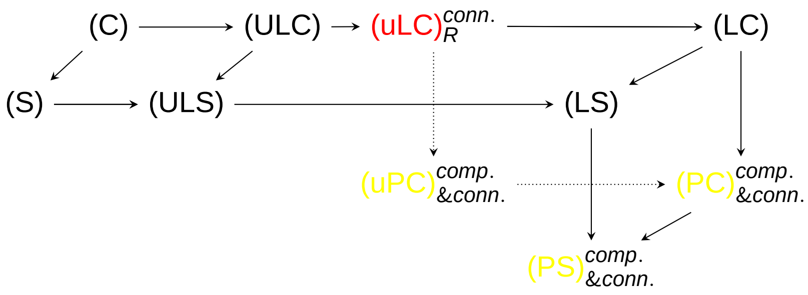
Open Problems



Problem (1)

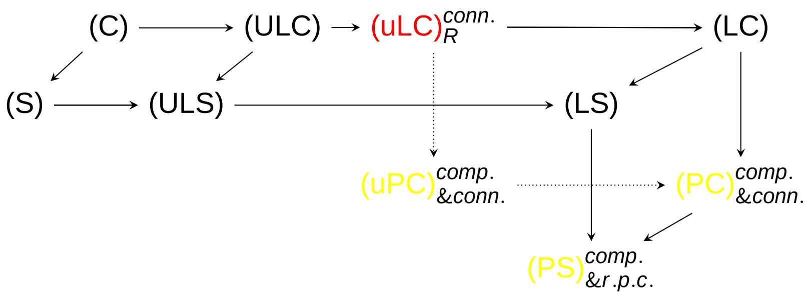
Assume that $\langle X, d\rangle$ is compact and either connected or path connected. If the map $f : \langle X, d\rangle \to \langle X, d\rangle$ is (PC), must f have either fix or periodic point? What if f is (PS)?

Open Problem



Problem (2)

Assume that $\langle X, d\rangle$ is compact and either connected or path connected. If the map $f : \langle X, d\rangle \to \langle X, d\rangle$ is (uPC)*, must f have either fix or periodic point?*



Problem (3)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If the map $f : \langle X, d\rangle \to \langle X, d\rangle$ is (PS), must f have either fix or periodic point?

Main Theorem Proof Outline

Recall,

Theorem (C & J, 2016)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If $f : X \to X$ is (PC), then f has a unique fixed point.

PROOF (outline). For $x, y \in X$ and a rectifiable path $p : [a, b] \to X, p(a) = x, p(b) = y$ let $\ell(p) = sup\{\sum_{i<n} d(t_i, t_{i+1}) : n < \omega$ and $a = t_0 < t_1 < ... < t_n = b\}$.
Define $D_0 : X^2 \to [0, \infty)$,
$D_0(x, y) = \inf\{\ell(p) : p$ is a rectifiable path from x to $y\}$.
We need to show:

(1) D_0 is a metric on X;
(2) $\langle X, D_0\rangle$ is complete;
(3) There exists $\bar{x} \in X$ such that $D_0(\bar{x}, f(\bar{x})) = L = \inf\{D_0(x, f(x)) : x \in X\}$;
(4) $L = 0$.

Even when $\langle X, d\rangle$ is compact, $\langle X, D_0\rangle$ does not need to be. Let X be the Topologist's Sine Curve with arc. Then $\langle X, d\rangle$ with standard metric from $\mathbb{R}^2$, is compact but $\langle X, D_0\rangle$ is not. It's actually homeomorphic with $[0, \infty)$. So (3) is not obvious.

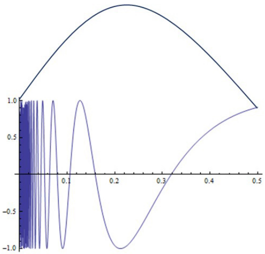


Figure: Topologist's Sine Curve with arc.

Proof of (3)

(3) There exists $\bar{x} \in X$ such that
$D_0(\bar{x}, f(\bar{x})) = L = \inf\{D_0(x, f(x)) : x \in X\}$.

Let $\langle x_n \in X : n < \omega \rangle$ be a sequence with
$\lim_{n\to\infty} D_0(x_n, f(x_n)) = L$. We have:

Theorem (Menger 1930)

In a metric space X, if there is a rectifiable path in X from x to y, then there is a geodesic, i.e. a path with minimal length ℓ, in X from x to y.

so for every $n < \omega$ there exists a path $p_n : [0,1] \to X$ from x_n to $f(x_n)$ with range $P_n \subseteq X$ and $\ell(p_n) = D_0(x_n, f(x_n))$.

We have the following:

Theorem (Myers 1945)

Let $\langle X, d\rangle$ be a compact metric space and, for any $n < \omega$, let $p_n\colon [0,1] \to X$ be a rectifiable path such that $\ell(p_n \restriction [0,t]) = t\ell(p_n)$ for any $t \in [0,1]$. If $L = \liminf_{n\to\infty} \ell(p_n) < \infty$, then there exists a subsequence $\langle p_{n_k} : k < \omega\rangle$ converging uniformly to a rectifiable path $p\colon [0,1] \to X$ with $\ell(p) \le L$.

WLOG, by reparametrizing our p_n, we can assume that for any $t \in [0,1]$, $\ell(p_n \restriction [0,t]) = t\ell(p_n)$.
So by the Myers' Theorem there exists a subsequence $\langle p_{n_k} : k < \omega\rangle$ converging uniformly to a rectifiable path $p\colon [0,1] \to X$ with $\ell(p) \le L$.
Take $\bar{x} = p(0) = \lim_{k\to\infty} p_{n_k}(0) = \lim_{k\to\infty} x_{n_k}$, then p is from $\bar{x}$ to $p(1) = \lim_{k\to\infty} p_{n_k}(1) = \lim_{k\to\infty} f(x_{n_k}) = f(\bar{x})$.
So, $D_0(\bar{x}, f(\bar{x})) \le \ell(p) \le L$, that is, $\bar{x}$ satisfies (3).

Thank you
for your attention.