Differentiable pointwise contractive minimal dynamical systems

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Based on a joint work with Jakub Jasinski

see http://www.math.wvu.edu/~kcies/publications.html

Dynamical Systems session of the 51st Spring Topology and Dynamical Systems Conference, March 10, 2017

Credits: This presentation is based on the papers

- K.C. Ciesielski and J. Jasinski, *On closed subsets of $\mathbb{R}$ and of $\mathbb{R}^2$ admitting Peano functions*, Real Anal. Exchange 40(2) (2015), 309–317.
- K.C. Ciesielski and J. Jasinski, *An auto-homeomorphism of a Cantor set with zero derivative everywhere*, J. Math. Anal. Appl. 434(2) (2016), 1267–1280.
- K.C. Ciesielski and J. Jasinski, *On fixed points of locally and pointwise contracting maps*, Topology Appl. 204 (2016), 70–78;
- K.C. Ciesielski and J. Jasinski, *Fixed point theorems for maps with local and pointwise contraction properties*, 57 pages, Canad. J. Math., in print.

Non-standard term *"differential dynamical systems"*

Warning!

In this lecture *differential dynamical systems* means

a study of any self-map f of a metric space $\langle X, d\rangle$ s.t.

$D^*f(x) = \lim_{y\to x} \frac{d(f(y),f(x))}{d(y,x)} \in \mathbb{R}$ exists for every $x \in X$.

- $\langle X, d\rangle$ need not be a manifold!
- The "derivative" D^*f need not be continuous.

Outline

Outline

1. Background: from analysis to dynamical systems
2. Pointwise shrinking maps and minimal dynamical systems
3. Auto-homeomorphism $\mathfrak{f}$ of a Cantor set with $\mathfrak{f}' \equiv 0$
4. Open problems: Dynamical Systems or Fix Point Thms?
5. Summary

Can Peano-like functions be differentiable?

For perfect $P \subset \mathbb{R}$,

(Q1) Can surjective continuous map $f\colon P \to P^2$ be differentiable?

- No for P of positive Lebesgue measure, e.g., for $P = [0,1]$.
- [KC & JJ 2014]: Yes, if we allow unbounded sets P. Such an f can even have a C^∞ extension $F\colon \mathbb{R} \to \mathbb{R}^2$.
- [KC & JJ 2014]: No, if P is compact and f is extendable to a C^1 map $F\colon \mathbb{R} \to \mathbb{R}^2$.

Still Open Problem

Pr1: Question **(Q1) when P is compact of measure 0.**

From Peano problem Pr1 to dynamical systems

Theorem (KC & JJ 2014)

If $\langle f, g\rangle \colon P \to P^2$ is a differentiable surjection, then $f[K] = P$, where $K = \{x \in P \colon f'(x) = 0\}$.

Proof.

f is countable-to-one on the F_σ set $P \setminus K$. □

K need not be compact. But can it be?

(Q2) Does there exist $f \colon K \to \mathbb{R}$, with $K \subset \mathbb{R}$ compact perfect, such that $f' \equiv 0$ and $K \subseteq f[K]$?

Fact (corollaries from the theorems we will discuss)

- *For every f as in* **(Q2)** *there is a perfect $P \subset K$ s.t. $f \restriction P$ is a minimal dynamical system (i.e., the orbit of every $x \in P$ is dense in $P = f[P]$).*
- *There exist a minimal system $f \colon P \to P$ with $f' \equiv 0$.*

Outline

From shrinking maps to minimal dynamics

For a metric space $\langle X, d\rangle$ and a map $f\colon X \to X$

- *f is pointwise shrinking, PS*, if for every $x \in X$ there is open $U \ni x$ such that $d(f(x), f(y)) < d(x, y)$ for all $y \in U$, $y \neq x$.
- If $X \subset \mathbb{R}$ and $|f'| < 1$ everywhere, then f is *PS*.

Theorem (KC & JJ 2014)

If $f\colon X \to X$ is onto, PS, and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamical system.

Theorem (Edelstein 1962, almost contradicting above thm)

If $f\colon X \to X$ is LS and X is compact, then f has a periodic point,

- *f is locally shrinking, LS*, provided for every $y \in X$ there is open $U \ni y$ s.t. $f \restriction U$ is *shrinking*, that is, $d(f(x), f(x')) < d(x, x')$ for every distinct $x, x' \in U$.

Sketch of proof

$\langle X, d\rangle$ is infinite compact, $f\colon X \to X$ is pointwise shrinking

Thm: There is perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamics.

This is proved by showing the following facts:

1. $T \subseteq X$ infinite compact & $T \subset f[T]$, imply T is uncountable. ($T \subset f[T]$ for no countable T of Cantor-Bendixon rank $\alpha < \omega_1$.)
2. $F_m = \{x \in P\colon f^{(m)}(x) = x\}$ is finite for every $m \in \mathbb{N}$.
3. For every orbit $O(x)$ of $x \in F = \bigcup_{m\in\mathbb{N}} F_m$, $f[B(O(x), \varepsilon)] \subseteq B(O(x), \varepsilon)$ for every small enough $\varepsilon > 0$.
4. There is open $U \supset F$ s.t. $T = X \setminus U$ is infinite & $T \subset f[T]$.
5. Find minimal P in $\{P \subset T\colon$ compact $\neq \emptyset$ s.t. $P \subset f[P]\}$. (Exists by Zorn's Lemma—Birkhoff's argument.) Such P is as needed.

Outline

Back to **(Q2)**

(Q2) Does there exist $f\colon K \to \mathbb{R}$, with $K \subset \mathbb{R}$ compact perfect, such that $f' \equiv 0$ and $K \subseteq f[K]$?

Yes to **(Q2)** implies that

there is minimal dynamics $\mathfrak{f}$ on a Cantor set $\mathfrak{X} \subset \mathbb{R}$ with $\mathfrak{f}' \equiv 0$.

Would you believe, that such $\mathfrak{f}$ could exist?

We did not:

???

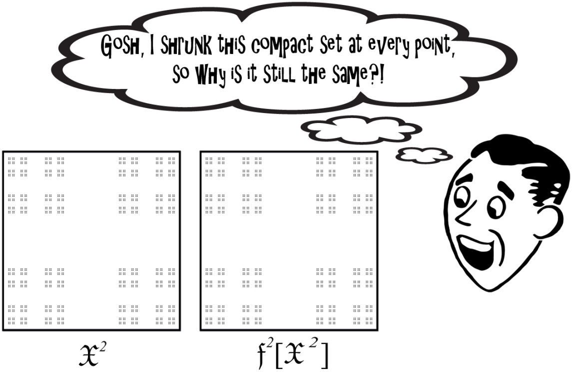

Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

. . . nevertheless, we proved:

Theorem (KC & JJ 2016)

There exists a compact perfect set $\mathfrak{X} \subset \mathbb{R}$ and a differentiable bijection $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ such that $\mathfrak{f}' \equiv 0$ on $\mathfrak{X}$. Moreover,

(i) *$\mathfrak{f}$ is a minimal dynamical system (i.e., the $\mathfrak{f}$-orbit $O(x) = \{\mathfrak{f}^{(n)}(x)\colon n \in \omega\}$ of every $x \in \mathfrak{X}$ is dense in $\mathfrak{X}$);*

(ii) *$\mathfrak{f}$ can be extended to a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$.*

Format of $\mathfrak{f}$: For some continuous injection $h\colon 2^\omega \to \mathbb{R}$,

$\mathfrak{X} = h[2^\omega]$ and $\mathfrak{f} = h \circ \sigma \circ h^{-1}\colon \mathfrak{X} \to \mathfrak{X}$, where

$\sigma\colon 2^\omega \to 2^\omega$ is an "add one and carry," odometer-like action:

for $s = \langle s_0, s_1, s_2, \ldots\rangle \in 2^\omega$, $\sigma(s) = s + \langle 1, 0, 0, ...\rangle$, i.e.

$$\begin{aligned} \sigma(1,1,1,\ldots) &= \langle 0,0,0,\ldots\rangle \\ \sigma(1,\ldots,1,0,s_{k+1},s_{k+2},\ldots) &= \langle 0,\ldots,0,1,s_{k+1},s_{k+2},\ldots\rangle. \end{aligned}$$

Properties of $\mathfrak{f} = h \circ \sigma \circ h^{-1} \colon \mathfrak{X} \to \mathfrak{X}$.

(i) $\mathfrak{f}$ is minimal since $\mathfrak{f}^{(n)} = h \circ \sigma^{(n)} \circ h^{-1}$:
density of the orbits of σ implies the same for $\mathfrak{f}$.

(ii) $\mathfrak{f}$ can be extended to a differentiable function $F \colon \mathbb{R} \to \mathbb{R}$:
follows immediately from a theorem of Jarník.

Delicate part: to choose h which ensures that $\mathfrak{f}' \equiv 0$.

For appropriately chosen $c_\tau \in \mathbb{R}$, $\tau \in 2^{<\omega}$, it is of the form

$$h(s) = \sum_{n<\omega} s_n c_{s\restriction n} \quad \text{for every } s \in 2^\omega.$$

Choice: tricky, based on 3 series with different convergence rates.

Checking $\mathfrak{f}' \equiv 0$: a bit tedious, but elementary.

Outline

Can $\mathfrak{X}$ from main example be (path) connected?

Open Problem (Pr2)

Let $\langle X, d\rangle$ be compact & either **connected** or **path connected**.
If $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is *PS*, must f have fix/periodic point?
What if f is *PC* or *uPC*, where

f is *pointwise contractive, PC*, if for every $x \in X$ there are open $U \ni x$ and $\lambda \in [0, 1)$ s.t. $d(f(x), f(y)) \le \lambda d(x, y)$ for all $y \in U$;

f is *uPC*, if there is $\lambda \in [0, 1)$ s.t. for every $x \in X$ there is open $U \ni x$ for which $d(f(x), f(y)) \le \lambda d(x, y)$ for all $y \in U$.

What is known on Problem Pr2

Pr2: For X compact & either **connected** or **path connected**, if $f\colon X\to X$ is *PS/PC/uPC*, must f have fix/periodic point?

- $\mathfrak{f}\colon \mathfrak{X}\to\mathfrak{X}$ shows that connectedness is essential;
- True, when X is rectifiably path connected and f is *PC*:

Theorem (KC & JJ 2016)

Assume that $\langle X, d\rangle$ is compact rectifiably path connected metric space. If $f\colon X\to X$ is PC, then f has a unique fixed point.

This is variant of 1978 theorem of Hu and Kirk 1978 (corrected by Jungck in 1982) proved without compactness of X, but with a stronger assumption that f is *uniformly PC, UPC*.

Compactness is essential: Hu and Kirk 1978 gave an example of path connected X and *uPC* map $f\colon X\to X$ with no periodic point.

One more problem

Open Problem (Pr3)

Let $\langle X, d\rangle$ be compact and **rectifiably path connected**.
If $f \colon \langle X, d\rangle \to \langle X, d\rangle$ is *PS*, must f have fix/periodic point?

Pr2 and Pr3 are the only open problems in our comprehensive study of ten classes of self-maps on metric spaces $\langle X, d\rangle$ with the local and pointwise (a.k.a. local radial) contraction properties.

The relations among the classes, assuming different topological properties of X, are represented as graphs, a sample of which is shown below.

Graph sample

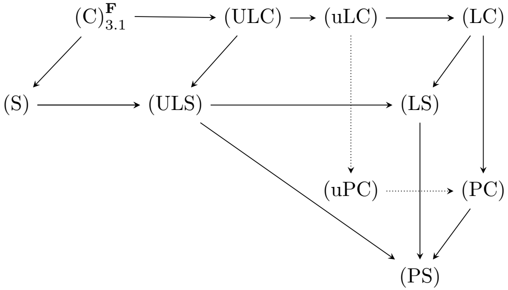

Figure: The relations between the local contractive and shrinking properties for the maps $f\colon X \to X$, with X being an arbitrary complete metric space.

Outline

Summary: new results on self maps

- If surjection $f\colon X \to X$ is *PS* and X is infinite compact, then there is a perfect $P \subset X$ s.t. $f \restriction P$ is a minimal dynamical system.
- There exist compact perfect $\mathfrak{X} \subset \mathbb{R}$ and bijection $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ s.t. $\mathfrak{f}' \equiv 0$ on $\mathfrak{X}$ and $\mathfrak{f}$ is a minimal dynamical system.
- If $\langle X, d \rangle$ is compact rectifiably path connected and $f\colon X \to X$ is *PC*, then f has a unique fixed point.

Summary: open problems on self maps

1. Let $\langle X, d \rangle$ be compact and **rectifiably path connected**.
If $f \colon \langle X, d \rangle \to \langle X, d \rangle$ is *PS*, must f have fix/periodic point?.

2. For X compact & either **connected** or **path connected**,
if $f \colon X \to X$ is *PS/PC/uPC*, must f have fix/periodic point?

We do not even know, what happens in the problems when X is a (topologically) manifold!

(Though, the maps must have fix points when the metric on X is convex.)

That is all!

Thank you for your attention!