Path-value functions for which Dijkstra's algorithm returns optimal mapping

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Dijkstra Algorithm, DA: Why should you care?

- It is one of the fastest algorithms used in image precessing, including image segmentation:

  (essentially) linear time with respect to image size

- It is the power engine behind

  - Fuzzy Connectedness, FC, segmentation software

- Can be used to find Watershed transform

- Usable in boundary tracking, morphological reconstructions, fast binary morphology, shape description, clustering, and classification

Q: In what other situations DA can be used?

- Q was investigated in the paper

  [FSL] Falcão, Stolfi, and Lotufo, *IFT*, TPAMI, 2004
- They found “sufficient” conditions for **DA** to be usable
- I started search for *necessary and sufficient* conditions
- Indeed, I found such conditions
- In the process, I found also that

  “sufficient” conditions in [FSL] are not sufficient!

  (Practical conclusions from [FSL] seem to be intact.)

What's ahead: Talk's outline

1. The algorithm
2. Characterization Theorem for **DA**
3. **DA***: a slight modification of **DA**
4. What is in [FSL] paper
5. Final Remarks
6. Summary

Outline

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Definitions and notation needed for DA

- $G = \langle V, E \rangle$ – finite directed graph

  (Applications and our examples use simple grids.)

- *Path (in G)*: $p = \langle v_0, \ldots, v_\ell \rangle$, $\langle v_j, v_{j+1} \rangle \in E$ for $j < \ell$;

  from $S \subset V$ to $v \in V$ when $v_0 \in S$ and $v_\ell = v$;

  $p\hat{\ }w = \langle v_0, \ldots, v_\ell, w \rangle$; Π_G – all paths in G.

- Path cost function: a map ψ from Π_G to $\langle [-\infty, \infty], \preceq \rangle$,

  $\preceq$ is either $\leq$ or $\geq$.

- DA for ψ tries to find, for every $v \in V$, the ψ-minimizer:

$$\psi(v) = \min_{\preceq}\{\psi(p) \colon p \text{ is a path to } v\}$$

Examples of path cost functions ψ

$G = \langle V, E \rangle$ and non-empty $S \subset V$ are fixed

- Fuzzy connectedness: given *affinity* map $\psi\colon E \to [0,1]$,

  seeks for maximizers (i.e., $\preceq$-minimizers with $\preceq$ being $\geq$):

  $\psi_{\min}(\langle v_0, \ldots, v_\ell \rangle) = \min_{1 \le j \le \ell} \psi(v_{j-1}, v_j)$ for $\ell > 0$

  $\psi_{\min}(\langle v_0 \rangle) = 1$ if $v_0 \in S$, $\psi_{\min}(\langle v_0 \rangle) = 0$ if $v_0 \notin S$

- Shortest path (classic DA): given *distance* $\omega_E\colon E \to [0,\infty)$,

  $\psi_{\mathrm{sum}}(\langle v_0, \ldots, v_\ell \rangle) = \sum_{1 \le j \le \ell} \omega_E(v_{j-1}, v_j)$ for $\ell > 0$

  $\psi_{\mathrm{sum}}(\langle v_0 \rangle) = 0$ if $v_0 \in S$, $\psi_{\mathrm{sum}}(\langle v_0 \rangle) = \infty$ if $v_0 \notin S$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\leq$)

More examples of path cost functions ψ

- Watershed transform: given *altitude* map $\omega_V \colon V \to [0,\infty)$,

  $\psi_{\rm peak}(\langle v_0,\ldots,v_\ell\rangle) = \max_{1\le j\le \ell}\{h(v_0),\omega_V(v_j)\}$ for $\ell>0$

  $\psi_{\rm peak}(\langle v_0\rangle) = h(v_0)$ for some h, $h(v_0)\ge \omega_V(v_0)$ for $v_0\in V$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\le$)

- Barrier Distance transform: given map $\omega_V \colon V \to [0,\infty)$,

  $\psi_{\rm dif}(\langle v_0,\ldots,v_\ell\rangle) = \max_{0\le j\le \ell}\omega_V(v_j) - \min_{0\le j\le \ell}\omega_V(v_j)$ for $\ell>0$

  $\psi_{\rm dif}(\langle v_0\rangle) = 0$ if $v_0\in S$, $\psi_{\rm dif}(\langle v_0\rangle) = \infty$ if $v_0\notin S$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\le$)

Yet another example of a path cost function ψ

- The last value: given a map $\omega_V \colon V \to [0, \infty)$,

  $\psi_{\mathrm{last}}(\langle v_0, \ldots, v_\ell \rangle) = \omega_V(v_\ell)$ for $\ell > 0$

  $\psi_{\mathrm{last}}(\langle v_0 \rangle) = \omega_V(v_0)$ if $v_0 \in S$, $\psi_{\mathrm{last}}(\langle v_0 \rangle) = \infty$ if $v_0 \notin S$

  seeks for minimizers (i.e., $\preceq$-minimizers with $\preceq$ being $\leq$)

Its applications are concerned with a particular case of the riverbed boundary tracking and can be used to support connectivity constraints in region-based image segmentation.

Dijkstra Algorithm, DA, aiming to find ψ-optimal map

Data: $G = \langle V, E \rangle$ and ψ from Π_G to $\langle [-\infty, \infty], \preceq \rangle$

Result: an array $\sigma[\,]$, aiming for being ψ-optimal map

Additional: an array $\pi[\,]$ of paths, such that, at any time, for any $v \in V$, $\pi[v]$ is a path to v with $\sigma[v] = \psi(\pi[v])$

foreach $\mathrm{v} \in V$ **do** $\pi[\mathrm{v}] \leftarrow \langle \mathrm{v} \rangle$; $\sigma[\mathrm{v}] \leftarrow \psi(\pi[\mathrm{v}])$ /* init. */

$\mathrm{H} \leftarrow V$

while $\mathrm{H} \neq \emptyset$ **do** /* the main loop */

  remove an element w of arg $\preceq$-$\min_{u \in \mathrm{H}} \sigma[u]$ **from** H

  foreach x *such that* $\langle \mathrm{w}, \mathrm{x} \rangle \in E$ **do**

    $\sigma' \leftarrow \psi(\pi[\mathrm{w}]\hat{\ }\mathrm{x})$

    if $\sigma[\mathrm{x}] \succ \sigma'$ **then** $\sigma[\mathrm{x}] \leftarrow \sigma'$; $\pi[\mathrm{x}] \leftarrow \pi[\mathrm{w}]\hat{\ }\mathrm{x}$

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Special paths

For fixed $\psi\colon \Pi_G \to \mathbb{R}$ and a path $p = \langle v_0, \ldots, v_\ell \rangle \in \Pi_G$ to v, let

$$\Psi(\langle v_0, \ldots, v_\ell \rangle) = \max_{\preceq}\{\psi(\langle v_0, \ldots, v_i \rangle)\colon i = 0, 1, \ldots, \ell\}.$$

We say that p

- is *ψ-optimal* if it is $\preceq$-minimal, that is, provided $\psi(p) \preceq \psi(q)$ for any other path $q \in \Pi_G$ to v;
- is *hereditarily ψ-optimal* provided every initial segment $\langle v_0, \ldots, v_k \rangle$, $k \le \ell$, of p is ψ-optimal;
- is *hereditarily optimal, HO,* provided it is hereditarily ψ-optimal and $\Psi(\langle v_0, \ldots, v_k \rangle) \preceq \Psi(p)$ for every $k \le \ell$ and hereditarily ψ-optimal p to v_k;
- is *Ψ-minimal* (in a strong sense) provided $\Psi(p_v) \prec \Psi(q\hat{\ }v)$ for every $q\hat{\ }v \in \Pi_G$ such that $\psi(p_v) \prec \psi(q\hat{\ }v)$ and q is either empty or HO.

More terminology

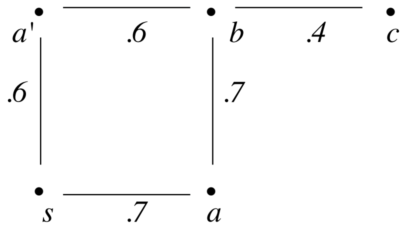
A path $p = \langle v_0, \ldots, v_\ell \rangle \in \Pi_G$ to v

- *has the replacement property (R)* provided $\psi(\langle v_0, \ldots, v_i \rangle) = \psi(q\hat{\ } v_i)$ for every $i \in \{1, \ldots, \ell\}$ and every HO path $q \in \Pi_G$ to v_{i-1};
- is *monotone* provided $\psi(\langle v_0, \ldots, v_i \rangle) \preceq \psi(\langle v_0, \ldots, v_j \rangle)$ whenever $0 \leq i \leq j \leq \ell$;
- is *hereditarily ψ-optimal monotone, HOM,* provided it is both hereditarily ψ-optimal and monotone.

Remark

Every HOM path $p_v = \langle v_0, \ldots, v_\ell \rangle \in \Pi_G$ is a Ψ-minimal HO path.

Examples: for FC cost $\psi_{\min}$ with $S = \{s\}$



- $\langle s, a, b \rangle$ is hereditarily $\psi_{\min}$-optimal
- $\langle s, a', b \rangle$ is not $\psi_{\min}$-optimal
- $\langle s, a, b, c \rangle$ is hereditarily $\psi_{\min}$-optimal
- $\langle s, a', b, c \rangle$ is $\psi_{\min}$-optimal but not hereditarily

Facts related to special paths

For costs $\psi_{\min}$, ψ_{sum}, and ψ_{peak} there is a map f s.t.

(I) $\psi(p\hat{\ }v) = f(\psi(p), a, v)$ for any path p to a and edge $\langle a, v\rangle$.

Any ψ-optimal path has replacement property if ψ satisfies (I).

$\psi_{\min}$, ψ_{sum}, and ψ_{peak} have strong replacement property:

(R*) $\psi(\langle v_0, \ldots, v_\ell\rangle) \preceq \psi(q\hat{\ }v_\ell)$ all paths $\langle v_0, \ldots, v_\ell\rangle$ and q to $v_{\ell-1}$ with $\psi(\langle v_0, \ldots, v_{\ell-1}\rangle) \preceq \psi(q)$.

For $\psi_{\min}$, ψ_{sum}, ψ_{peak}, and ψ_{dif}: (M) any path is monotone

(M) and (R*) imply that every v admits HOM path

So, for $\psi_{\min}$, ψ_{sum}, and ψ_{peak}, every v admits HOM path

The theorem for **DA**

Theorem

Let $\psi\colon \Pi_G \to [-\infty, \infty]$ be a path cost function. If

(E) *for every $v \in V$ there exists a Ψ-minimal HO path to v with the replacement property,*

then $\sigma[\,]$ returned by **DA is guaranteed** *to be ψ-optimal;*

$\pi[\,]$ returned by **DA**: *$\pi[v] = \langle v_0, \ldots, v_\ell \rangle$ is HO path to v; $\pi[v_i] = \langle v_0, \ldots, v_i \rangle$ for every $i \in \{0, \ldots, \ell\}$.*

Conversely, if

(M) *$\psi(q) \preceq \psi(p)$ for every path p and its initial segment q,*

then $\sigma[\,]$ returned by **DA cannot be** *ψ-optimal,*

unless for every v there is a hereditarily ψ-optimal path to v.

ψ_{last} satisfies (E) but is not monotone!

Corollary: Characterization Theorem

Corollary

If $\psi\colon \Pi_G \to \mathbb{R}$ satisfies (M) and

(R) $\psi(p) = \psi(q\hat{\ }v)$ *for every HOM* $p = \langle v_0, \ldots, v_\ell \rangle$ *& q to* $v_{\ell-1}$,

then $\sigma[\,]$ *returned by* **DA** *is the* ψ*-optimal map if, and only if,*

for every $v \in V$ *there exists a hereditarily* ψ*-optimal path to* v.

PROOF. (E) follows from (M) and (R).

The rest follows from Theorem. □

Practical consequences

Corollary

ψ_{sum}, ψ_{min}, *and* ψ_{peak} *satisfy (E).*

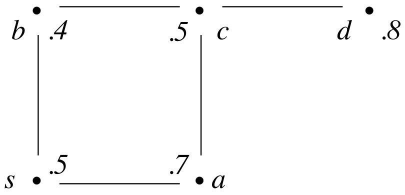
DA *works correctly for these functions.*

PROOF. (R*) implies:

- $\psi(\langle v_0, \ldots, v_\ell \rangle) = \psi(q\hat{\ } v_\ell)$ for all optimal paths $\langle v_0, \ldots, v_\ell \rangle$ and q to $v_{\ell-1}$ with $\psi(\langle v_0, \ldots, v_{\ell-1} \rangle) \preceq \psi(q)$.

So, (E) holds. □

Another consequence



Corollary

***DA** need not return optimal map for Barrier Distance ψ_{dif}.*

PROOF. No hereditarily ψ_{dif}-optimal path from $S = \{s\}$ to d.

As ψ_{dif} satisfies (M), the result follows from the Theorem. □

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Problems with **DA** for general path costs

Consider graph $s \longleftrightarrow a$

Put $\psi(\langle s\rangle) = .2$, $\psi(p) = 0$ for any other path from s, and

$\psi(p) = 0$ for p from a. For minimization, we get

There is no hereditary ψ-optimal path for any $v \in V$, since $\langle v\rangle$ is suboptimal.

ψ satisfies (R), in void, since there are no HO paths.

DA returns a non-trivial circular path: **DA** terminates with $\pi[a] = \langle s, a\rangle$ and the cycle $\pi[s] = \langle s, a, s\rangle$.

This contradicts Lemma 2 from [FSL]

DA returns optimal $\sigma[\,]$

DA*, which cannot return cycles for any ψ

Algorithm 1: DA*, aiming to find the ψ-optimal map

Data: $G = \langle V, E \rangle$ and ψ from Π_G to $\langle [-\infty, \infty], \preceq \rangle$
Result: an array $\sigma[\,]$, aiming for being ψ-optimal map
Additional: an array $\pi[\,]$ of paths, such that, at any time, for any $v \in V$, $\pi[v]$ is a path to v with $\sigma[v] = \psi(\pi[v])$

```
foreach v ∈ V do π[v] ← ⟨v⟩; σ[v] ← ψ(π[v])      /* init.        */
H ← V
while H ≠ ∅ do                                   /* the main loop */
    remove an element w of  arg ⪯-min_{u∈H} σ[u]  from H
    foreach x such that ⟨w, x⟩ ∈ E and x ∈ H do
        σ′ ← ψ(π[w]^x)
        if σ[x] ≻ σ′ then σ[x] ← σ′; π[x] ← π[w]^x
```

Main Theorem for DA*: no cycles

Theorem

Let $\psi\colon \Pi_G \to [-\infty, \infty]$ *be a path cost function.*

- *If* $\pi[\,]$ *is returned by* **DA***, *then, for every* $v \in V$, $\pi[v] = \langle v_0 \ldots, v_\ell \rangle$ *is a path to* v *with no repetitions such that* $\pi[v_i] = \langle v_0 \ldots, v_i \rangle$ *for every* $i \in \{0, \ldots, \ell\}$.
- *If (E) holds, then* $\sigma[\,]$ *returned by* **DA* is guaranteed** *to be the* ψ*-optimal map. Moreover, the returned map* $\pi[\,]$ *consists of hereditary* ψ*-optimal paths.*
- *Conversely,* $\sigma[\,]$ *returned by* **DA* cannot be** ψ*-optimal, unless for every* $v \in V$ *there exists a hereditary* ψ*-optimal path to* v.

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Smooth functions from [FSL]

A path cost map ψ is a smooth function provided

for any v there exists ψ-optimal p to v s.t. either $p = \langle v \rangle$, or

$p = q\hat{}v$, where q is a path to w, $\langle w, v \rangle$ is an edge, and

C1. $\psi(q) \preceq \psi(p)$,

C2. q is ψ-optimal,

C3. for any ψ-optimal path r to w, $\psi(r\hat{}v) = \psi(p)$.

It is claimed (incorrectly) in [FSL] that for smooth ψ **DA** must return ψ-optimal map $\sigma[\,]$.

There is no proof of this in [FSL]. Instead, authors claim (without proof) that C1-C3 imply stronger properties C1*-C3* and proceed to prove that they imply **DA**'s good behavior.

Properties C1*-C3*: hereditary versions of C1-C3

For any v there exists a ψ-optimal path $p = \langle v_0, \ldots, v_\ell \rangle$ to v s.t. for any $k \in \{0, \ldots, \ell - 1\}$

C1*. $\psi(\langle v_0, \ldots, v_k \rangle) \preceq \psi(p)$,

C2*. $\langle v_0, \ldots, v_k \rangle$ is ψ-optimal,

C3*. for any ψ-optimal path q to v_k, $\psi(q\hat{}\langle v_{k+1}, \ldots, v_\ell \rangle) = \psi(p)$.

C1*&C2* means that p is an HOM path

C3* is close to our (R), demanding that

$$\psi(q\hat{}v_{k+1}) = \psi(\langle v_0, \ldots, v_{k+1} \rangle)$$

Q. Why did I bother, when [FSL] contains proof that C1*-C3* are sufficient?

A. The proof in [FSL], using C1*-C3*, is incorrect!

C1-C3 does not imply C1*-C3*

Example

Graph: $\{0,\dots,5\}\times\{0,\dots,5\}$ with 4-adjacency.

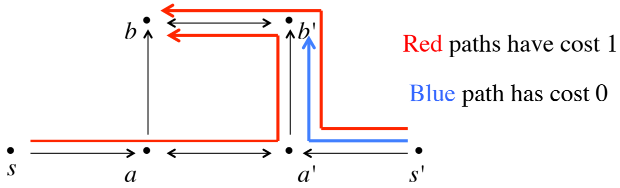
Seed: $s=\langle 0,0\rangle$. Problem: minimization, i.e., $\preceq$ is $\leq$.

If s appears in $p=\langle v_0,\dots,v_\ell\rangle$ only as v_0:
$\psi(p)=\ell$ when $\ell\leq 3$; $\psi(p)=0$ otherwise.

$\psi(p)=100$ for all other paths p.

- $\psi(v)=0$ for every v
- C1-C3 are satisfied (by any path of length ≥ 5)
- C1*-C2* are not satisfied (only s admits HOM path)
- for any v adjacent to s, **DA** returns a suboptimal value 1.

C1*-C3* do not imply good behavior of DA or DA*



$S = \{s, s'\}$; maximization problem (i.e., $\preceq$ is $\geq$)

$\psi(p) = 1$ for any p from S of the form ($\psi(p) = 0$ otherwise):

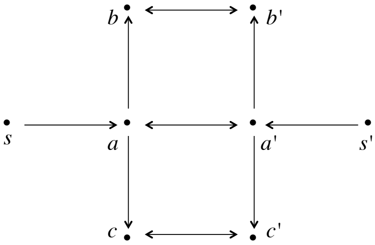
- to a $v \in \{s, s', a, a'\}$ or having repeated vertices;
- $\langle \ldots, a', b', b \rangle$, $\langle s, a, a', b' \rangle$, $\langle \ldots, a, b, b' \rangle$, or $\langle s', a', a, b \rangle$.

C1*-C3* satisfied: by $\langle s, a, a', b', b \rangle$ and $\langle s', a', a, b, b' \rangle$

May terminate with suboptimal σ: Starting with $\langle s, a \rangle$ and $\langle s', a' \rangle$

May terminate with optimal σ: Starting with $\langle s, a, a' \rangle$

Stronger example: σ cannot be optimal



$\psi(p) = 1$ for any p from $\{s, s'\}$ of the form ($\psi(p) = 0$ otherwise):

- to a $v \in \{s, s', a, a'\}$ or having repeated vertices;
- $\langle \ldots, a', b', b\rangle$, $\langle s, a, a', b'\rangle$, $\langle \ldots, a, b, b'\rangle$, or $\langle s', a', a, b\rangle$.
- $\langle \ldots, a', c', c\rangle$, $\langle s', a', c'\rangle$, $\langle \ldots, a, c, c'\rangle$, or $\langle s, a, c\rangle$.

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Final tune-ups

If ψ, like $\psi_{\min}$, ψ_{sum}, and ψ_{peak}, satisfies

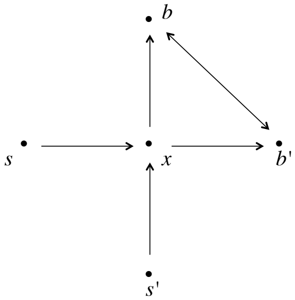
(I) $\psi(p\hat{\ }v) = f(\psi(p), a, b)$ for any path p to a and edge $\langle a, b\rangle$,

then, in **DA** and **DA***, there is no need to store paths in $\pi[\,]$.
The similar trick can be used for ψ_{dif}.

If ψ satisfies (M), "x $\in$ H" in line 5 of **DA*** is redundant.

For such ψ it makes sense to replace, both in **DA** and **DA***, the condition in line 5 with "x such that $\langle \mathrm{w}, \mathrm{x}\rangle \in E$ and x $\in$ H," to avoid unnecessary compution of $\psi(\pi[\mathrm{w}]\hat{\ }\mathrm{x})$.

Is the replacement requirement necessary?



$S = \{s, s'\}$; maximization problem (i.e., $\preceq$ is $\geq$)
$\psi(p) = 1$ for any p from S of the form ($\psi(p) = 0$ otherwise):

- $\langle s, x, b, b' \rangle$, $\langle s', x, b', b \rangle$, and their initial segments.

b and b' admits no optimal path with the replacement property.

DA and **DA*** return optimal maps:

with $\pi[b] = \langle s', x, b', b \rangle$ or $\pi[b'] = \langle s, x, b, b' \rangle$.

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Summary

- For some classes of path cost functions ψ, we found a necessary and sufficient conditions on ψ, for Dijkstra algorithm to return correct optimizer.
- We identified the errors in the [FSL] paper and shown how these errors can be patched.
- We showed how our characterization theorem can be used for some practically used path cost functions.
- The application of these characterization theorem to other path cost functions is currently investigated.

Thank you for your attention!