Fixed point theorems for maps with various local contraction properties

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Twelfth Symposium on General Topology
Prague, July 25-29, 2016

Abstract

Let $\langle X, d \rangle$ be a metric space. We compare ten classes of continuous self-maps $f : X \to X$. All of these self-maps are proved to have fixed or periodic points for spaces X with certain topological properties. We will assume X to be

- complete
- complete and connected
- complete and rectifiably path connected
- complete and d-convex
- compact
- compact and connected
- compact and rectifiably path connected
- compact and d-convex

The Classics

Definition (#1)

A function $f : X \to X$ is called Contractive, (C), if there exists a constant $0 \leq \lambda < 1$ such that for any two elements $x, y \in X$ we have $d(f(x), f(y)) \leq \lambda d(x, y)$.

Theorem (Banach, 1922)

If (X, d) is a complete metric space and $f : X \to X$ is (C), then f has a unique fixed point, that is, there exists a unique $\xi \in X$ such that $f(\xi) = \xi$.

The Classics

Definition (#2)

A function $f : X \to X$ is called Shrinking, (S), if for any two elements $x, y \in X, x \neq y$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

If $\langle X, d\rangle$ is compact and $f : X \to X$ is (S), then it has a unique fixed point.

The Classics

Definition (#3)

A function $f : X \to X$ is called Locally Shrinking, (LS), if for any element $z \in X$ there exists an $\varepsilon_z > 0$ such that $f \restriction B(z, \varepsilon)$ is shrinking, i.e. for any two $x \neq y \in B(z, \varepsilon_z)$ we have $d(f(x), f(y)) < d(x, y)$.

Theorem (Edelstein, 1962)

Let $\langle X, d\rangle$ be compact and let $f : X \to X$.

(i) *If f is (LS), then f has a periodic point.* ♠

(ii) *If f is (LS) and X is connected, then f has a unique fixed point.*

The Classics

Definition (#4)

A function $f : X \to X$ is called Pointwise Contracting, (PC), if for every $z \in X$ there exists a $\lambda_z \in [0,1)$ and an $\varepsilon_z > 0$ such that for any element $x \in B(z, \varepsilon_z)$ we have $d(f(x), f(z)) \le \lambda_z d(x, z)$.

Definition (#5)

A function $f : X \to X$ is called uniformly Pointwise Contracting, (uPC), if there exists a $\lambda \in [0,1)$ such that for every $z \in X$ there exists an $\varepsilon_z > 0$ such that for any element $x \in B(z, \varepsilon_z)$ we have $d(f(x), f(z)) \le \lambda d(x, z)$.

Theorem (Hu and Kirk, 1978; proof corrected by Jungck, 1982)

If $\langle X, d\rangle$ is a rectifiably path connected complete metric space and a map $f : X \to X$ is (uPC), then f has a unique fixed point.

Classics/Recent

Definition (#6)

A function $f : X \to X$ is called Uniformly Locally Contracting, (ULC), if there exist a $\lambda \in [0, 1)$ and an $\varepsilon > 0$ such that for every $z \in X$ the restriction $f \restriction B(z, \varepsilon)$ is contractive with the same $\lambda_z = \lambda$.

Theorem

Assume that $\langle X, d\rangle$ is complete and that $f : X \to X$ is (ULC)

(i) (Edelstein, 1961) *If X is connected, then f has a unique fixed point.*

(ii) (C & J, 2016) *If X has a finite number of components , then f has a periodic point.*

Recent

Definition (#7)

A function $f : X \to X$ is called Pointwise Contractive, (PC), if for every $z \in X$ there exist $\lambda_z \in [0,1)$ and an $\varepsilon_z > 0$ such that $d(f(x), f(z)) \leq \lambda_z d(x,z)$ whenever $x \in B(z, \varepsilon_z)$.

Theorem (C & J, Top. and its App. 204 2016 70-78)

Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If $f : X \to X$ is (PC), then f has a unique fixed point.

Example (C & J, J. Math. Anal. Appl. 434 2016 1267 - 1280)

There exists a Cantor set $\mathfrak{X} \subset \mathbb{R}$ and a (PC) self-map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ without periodic points.

The Ten Contracting/Shrinking Properties

Global Properties. $f : X \to X$ is

(C) contractive if

$$\exists \lambda \in [0,1) \forall x, y \in X \, (d(f(x), f(y)) \leq \lambda d(x,y)),$$

(S) shrinking if

$$\forall x \neq y \in X \, (d(f(x), f(y)) < d(x,y)).$$

Clearly (C) $\Longrightarrow$ (S).

Each global property gives rise to two kinds of local properties, named local and pointwise, as follows:

The Ten Contracting/Shrinking Properties

Local Properties:

(LC) f *is locally contractive if* $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z) (d(f(x), f(y)) \leq \lambda_z d(x, y))$,

(LS) f *is locally shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x \neq y \in B(z, \varepsilon_z) (d(f(x), f(y)) < d(x, y))$,

Pointwise Properties (we fix y=z):

(PC) f *is pointwise contractive if* $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z) (d(f(x), f(z)) \leq \lambda_z d(x, z))$,

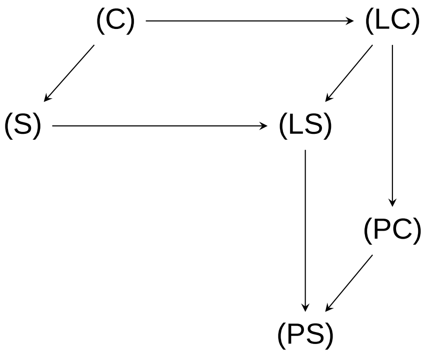
(PS) f *is pointwise shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z) (d(f(x), f(z)) < d(x, z))$,

Pointwise properties are also known as radial.

Clearly $(\text{Locally}) \implies (\text{Pointwise})$.

The Ten Contracting/Shrinking Properties

The following implications follow from the definitions:



The Ten Contracting/Shrinking Properties

Local properties can be made stronger by requiring uniformity, i.e. that the same $\lambda \in [0,1)$ and/or the same $\varepsilon > 0$ work for all $z \in X$.

Local Properties:

(LC) f is *locally contractive* if $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\,(d(f(x), f(y)) \leq \lambda_z d(x,y))$,

(uLC) f *is (weakly) uniformly locally contractive if* $\exists \lambda \in [0,1) \forall z \in X \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\,(d(f(x), f(y)) \leq \lambda d(x,y))$,

(ULC) f *is (strongly) Uniformly locally contractive if* $\exists \lambda \in [0,1) \exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\,(d(f(x), f(y)) \leq \lambda d(x,y))$,

(LS) f *is locally shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x, y \in B(z, \varepsilon_z)\,(d(f(x), f(y)) < d(x,y))$,

(ULS) f *is Uniformly locally shrinking if* $\exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\,(d(f(x), f(y)) < d(x,y))$.

The Ten Contracting/Shrinking Properties

Similarly, pointwise properties can be made stronger by requiring uniformity, i.e. that the same $\lambda \in [0,1)$ and/or the same $\varepsilon > 0$ works for all $z \in X$.

Pointwise Properties:

(PC) f is *pointwise contractive* if $\forall z \in X \exists \lambda_z \in [0,1) \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\,(d(f(x), f(z)) \leq \lambda_z d(x,z))$,

(uPC) f *is (weakly) uniformly pointwise contractive if* $\exists \lambda \in [0,1) \forall z \in X \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\,(d(f(x), f(z)) \leq \lambda d(x,z))$,

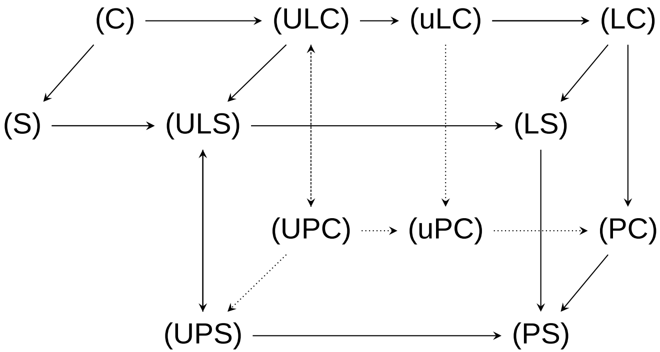
(UPC) f *is Uniformly pointwise contractive if* $\exists \lambda \in [0,1) \exists \varepsilon > 0 \forall z \in X \forall x \in B(z, \varepsilon)\,(d(f(x), f(z)) \leq \lambda d(x,z))$,

(PS) f *is pointwise shrinking if* $\forall z \in X \exists \varepsilon_z > 0 \forall x \in B(z, \varepsilon_z)\,(d(f(x), f(z)) < d(x,z))$,

(UPS) f *is Uniformly pointwise shrinking if* $\exists \varepsilon > 0 \forall z \in X \forall x, y \in B(z, \varepsilon)\,(d(f(x), f(y)) < d(x,y))$.

The Ten Contracting/Shrinking Properties or is it 12?

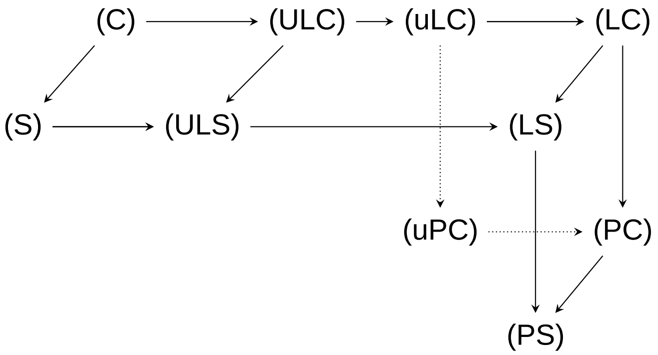
The following implications follow from the definitions:



Remark: (ULS)=(UPS) and (ULC)=(UPC). Any (λ, ε)-(UPC) function is $(\lambda, \frac{\varepsilon}{2})$-(ULC) and (ε)-(UPS) is $\left(\frac{\varepsilon}{2}\right)$-(ULS).

The Ten Contracting/Shrinking Properties

The following diagram

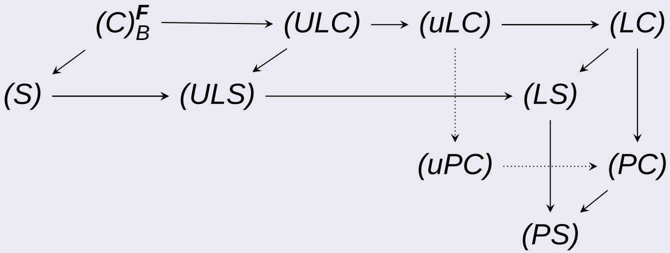


shows the essential classes and implications.

Fixed and Periodic Points

Theorem (Complete Spaces)

Assume X is complete. No combination of any of the properties shown imply any other property, unless the graph forces such implication. Neither does any combination of them imply the existence of a periodic point unless it contains (C).



Fixed and Periodic Points

Theorem (Complete Spaces cont.)

Specifically, there exist 9 complete spaces X with self-maps $f : X \to X$ without periodic points witnessing the following:

(PC): $(\mathrm{PC}) \not\Leftarrow (\mathrm{S})$

(uPC): $(\mathrm{uPC}) \not\Leftarrow (\mathrm{S})\&(\mathrm{LC})$

(LS): $(\mathrm{LS}) \not\Leftarrow (\mathrm{uPC})$

(ULS): $(\mathrm{ULS}) \not\Leftarrow (\mathrm{uLC})$

(S): $(\mathrm{S}) \not\Leftarrow (\mathrm{ULC})$

(LC): $(\mathrm{LC}) \not\Leftarrow (\mathrm{S})\&(\mathrm{uPC})$

(uLC): $(\mathrm{uLC}) \not\Leftarrow (\mathrm{S})\&(\mathrm{LC})\&(\mathrm{uPC})$

(ULC): $(\mathrm{ULC}) \not\Leftarrow (\mathrm{S})\&(\mathrm{uLC})$

(C): $(\mathrm{C}) \not\Leftarrow (\mathrm{S})\&(\mathrm{ULC})$

Fixed and Periodic Points, Blue does not imply yellow

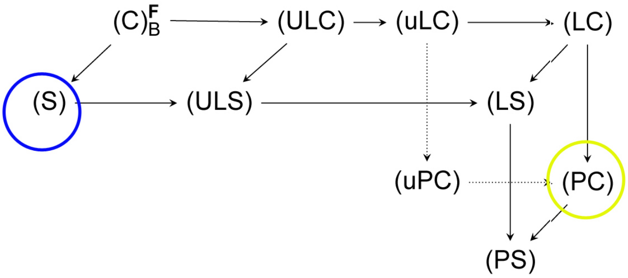


Figure: (PC) $\not\Rightarrow$ (S).

Remark: f is (PC) iff $limsup_{x \to z} \frac{d(f(x),f(z))}{d(x,z)} < 1$ for all $z \in X$.

Take $X = [0, \infty)$ and $f(x) = x + e^{-x^2}$ so $f'(x) = 1 - 2xe^{-x^2}$.
We have $f'(0) = 1$ so not-(PC) at $z = 0$. Also $f'[(0, \infty)] \subseteq (0, 1)$ so f is (S) by the MVT. For all $x \in [0, \infty)$,$f(x) > x$ so no periodic points.

Fixed and Periodic Points, Blue does not imply yellow

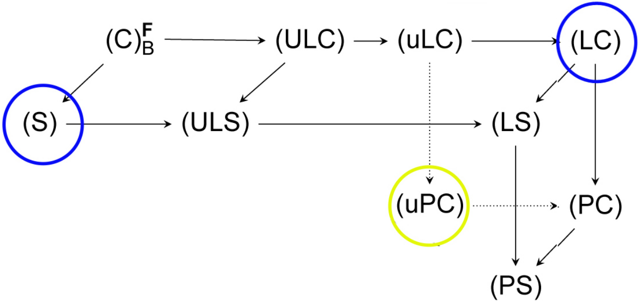


Figure: $(\mathrm{uPC}) \not\Leftarrow (\mathrm{S})\&(\mathrm{LC})$. Take $X = \mathbb{R}$ and $f(x) = \frac{1}{2}\left(x + \sqrt{x^2+1}\right)$. Then $f'(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$ so for any $a \in \mathbb{R}$, $f'[(-\infty, a]] = (0, c]$ for some $c < 1$ so MVT gives $(\mathrm{S})\&(\mathrm{LC})$. $\lim_{x\to\infty} f'(x) = 1$ so $\neg(\mathrm{uPC})$.

Fixed and Periodic Points, Blue does not imply yellow

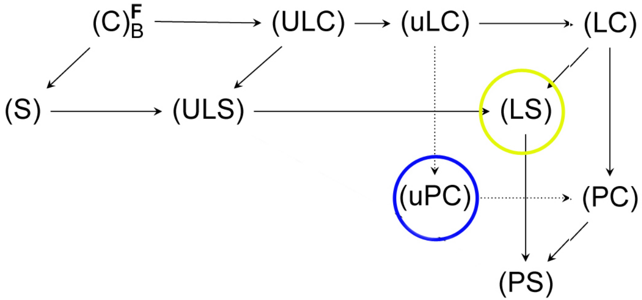


Figure: $(\mathrm{LS}) \not\Leftarrow (\mathrm{uPC})$ There exists a compact perfect set $\mathfrak{X} \subseteq \mathbb{R}$ and an autohomeomorphism $\mathfrak{f} : \mathfrak{X} \to \mathfrak{X}$ with $\mathfrak{f}' \equiv 0$. So $\mathfrak{f}$ is (uPC) with any $\lambda \in (0, 1)$ and $\mathfrak{f}$ has no periodic points, [C & J, 2015] so it is not (LS) by the Edelstein's Theorem ♠.

Fixed and Periodic Points, Blue does not imply yellow

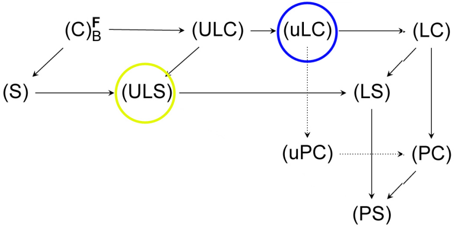


Figure: (ULS) $\not\Leftarrow$ (uLC) Take two increasing sequences: $0 < \beta_n \nearrow 1$ and $0 = a_0 < a_1 < \ldots \nearrow \infty$, $I_n = [a_n, a_{n+1}]$, such that $|I_{2n}| = |I_{2n+1}| = \frac{1}{n+1}$. Define metrics $\rho_n(x,y) = |I_n| \left(\frac{|x-y|}{|I_n|}\right)^{\beta_n}$ on I_n and *"make"* a metric ρ on $X = \bigcup_{n<\omega} I_n$ so that $f : X \to X$, mapping linearly and increasingly I_n onto I_{n+1} has needed properties. For $x \leq y$, $n < m$

$$\rho(x,y) = \begin{cases} \rho_n(x,y) & \text{if } x,y \in I_n \\ \rho_n(x,a_{n+1}) + |a_m - a_{n+1}| + \rho_m(a_m,y) & \text{if } x \in I_n, y \in I_m \end{cases}$$

Fixed and Periodic Points, Blue does not imply yellow

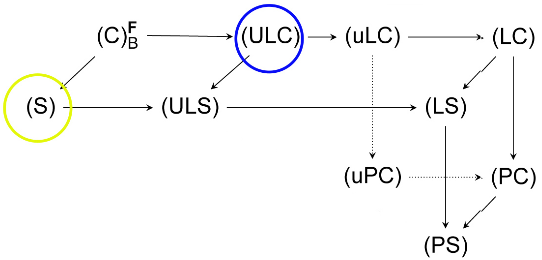


Figure: $(S) \not\Leftarrow (ULC)$ Remetrization.

Fixed and Periodic Points, Blue does not imply yellow

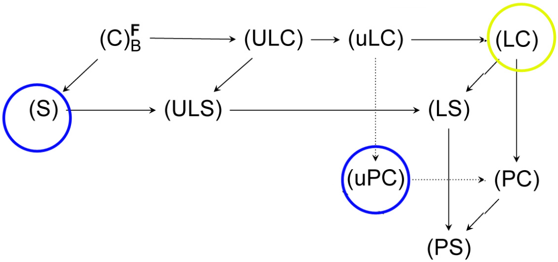


Figure: (LC) $\not\Leftarrow$ (S)&(uPC) Remetrization.

Fixed and Periodic Points, Blue does not imply yellow

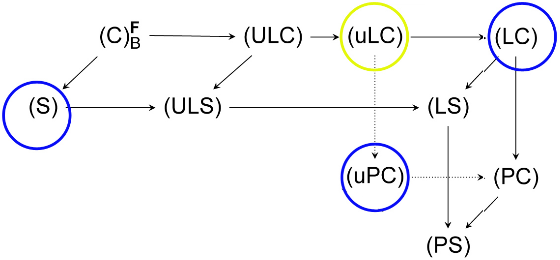


Figure: (uLC) $\not\Leftarrow$ (S)&(LC)&(uPC) Remetrization.

Fixed and Periodic Points, Blue does not imply yellow

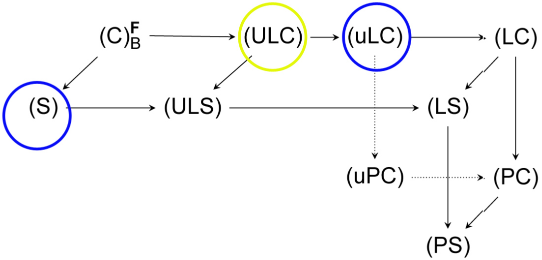


Figure: $(ULC) \not\Leftarrow (S)\&(uLC)$ Remetrization.

Fixed and Periodic Points, Blue does not imply yellow

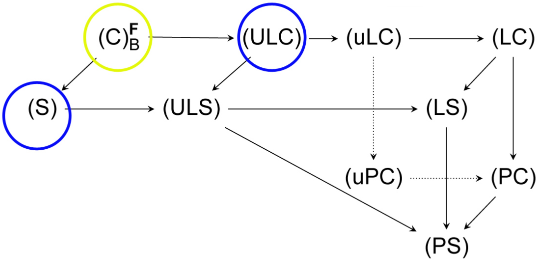
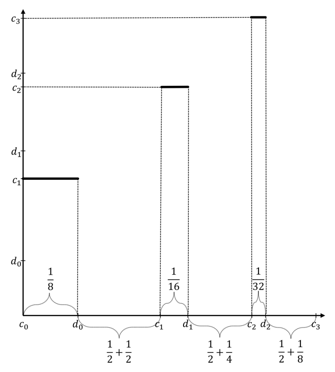


Figure: (C) $\not\Leftarrow$ (S)&(ULC) We have the following ...

Fixed and Periodic Points, Blue does not imply yellow

Example (A (S)&(ULC)&not(C) map f without periodic points)

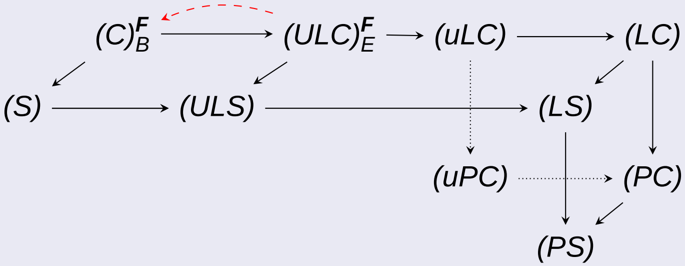
Define sequences $\langle c_n \rangle$ and $\langle d_n \rangle$: $c_0 = 0$, $d_n = c_n + 2^{-(n+3)}$ and $c_{n+1} = d_n + \frac{1}{2} + 2^{-(n+1)}$. Set $X = \bigcup_{n<\omega}[c_n, d_n]$ and let $f : X \to X$, $f(x) = c_{n+1}$ for $x \in [c_n, d_n]$. We have



Fixed and Periodic Points - Connected Spaces

Theorem (Connected Spaces)

Assume X is complete and connected. No combination of any of the properties shown imply any other property, unless the graph forces such implication. Neither does any combination imply the exitance of a periodic point unless it contains (C) *or* (ULC).



Fixed and Periodic Points - Connected Spaces

A sequence $s = \langle x_0, x_1, ..., x_n \rangle \in X^{n+1}$ is an ε-chain between x_0 and x_n if $d(x_i, x_{i+1}) \leq \varepsilon$. Let $\mathfrak{l}(s) = \sum_{i<n} d(x_i, x_{i+1})$. Define

$\hat{D}\colon X^2 \to [0, \infty), \hat{D}(x, y) = \inf\{\mathfrak{l}(s)\colon s \text{ is an } \varepsilon\text{-chain between } x \text{ and } y\}.$

Theorem (<- - - - - - - -)

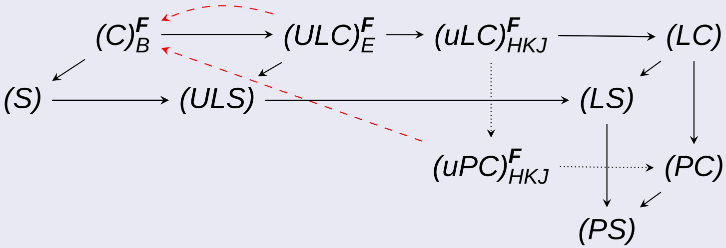
Assume $\langle X, d \rangle$ is connected.

- *For any $\varepsilon > 0$ there is an ε-chain between any two points.*
- *$\hat{D}$ is a metric topologically equivalent to d.*
- *If $\langle X, d \rangle$ is complete, than so is $\langle X, \hat{D} \rangle$.*
- *If $f\colon \langle X, d \rangle \to \langle X, d \rangle$ is (ULC), then $f\colon \langle X, \hat{D} \rangle \to \langle X, \hat{D} \rangle$ is (C).*
- *If $\langle X, d \rangle$ is also compact and $f\colon \langle X, d \rangle \to \langle X, d \rangle$ is (ULS), then $f\colon \langle X, \hat{D} \rangle \to \langle X, \hat{D} \rangle$ is (S).*

Fixed and Periodic Points - Connected Spaces

Theorem (Rectifiably Path Connected Spaces)

Assume X is complete and rectifiably path connected. No combination of any of the properties shown imply any other property, unless the graph forces such implication. Neither does any combination imply the exitance of a periodic point unless it contains (C), (ULC), (uLC) *or* (uPC).



Fixed and Periodic Points - Connected Spaces

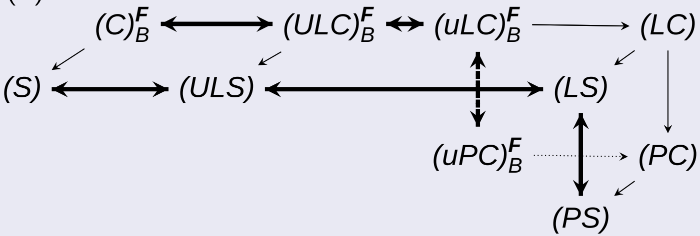
Definition

A metric space $\langle X, d\rangle$ is *d-convex* provided for any distinct points $x, y \in X$ there exists a path $p\colon [0,1] \to X$ from x to y such that $d(p(t_1), p(t_3)) = d(p(t_1), p(t_2)) + d(p(t_2), p(t_3))$ whenever $0 \leq t_1 < t_2 < t_3 \leq 1$.

Fixed and Periodic Points - Connected Spaces

Theorem (d-convex Spaces)

Assume X is complete and d-convex. Jungck (1982) showed $(\mathrm{uPC}) \Rightarrow (\mathrm{C})$ *with the same* λ*. A modified argument shows that* $(\mathrm{PS}) \Rightarrow (\mathrm{S})$*.*

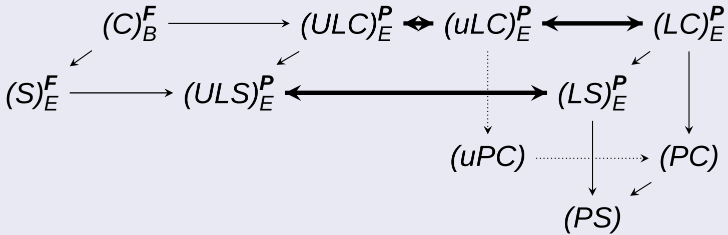


No combination of any of the properties shown imply any other property, unless the graph forces such implication. Neither does any combination imply the existence of a periodic point unless it contains (C)=(ULC)=(uLC)=(uPC).

Fixed and Periodic Points - Compact Spaces

Theorem (Compact Spaces)

Assume $\langle X, d\rangle$ is compact. Ding and Nadler (2002) and C&J 2015 showed $(\mathrm{LC}) \Rightarrow (\mathrm{ULC})$ *and* $(\mathrm{LS}) \Rightarrow (\mathrm{ULS})$.

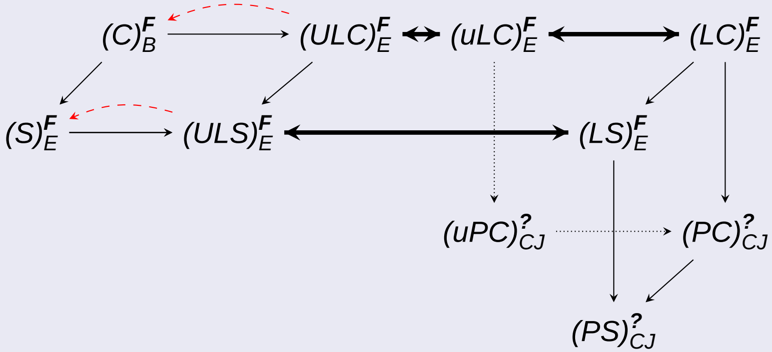


No combination of any of the properties shown imply any other property, unless the diagram forces such implication. Neither does any combination imply the existence of a fixed or periodic unless indicated on the diagram.

Fixed and Periodic Points - Compact Spaces

Theorem (Compact Connected Spaces)

Assume X is compact and connected.

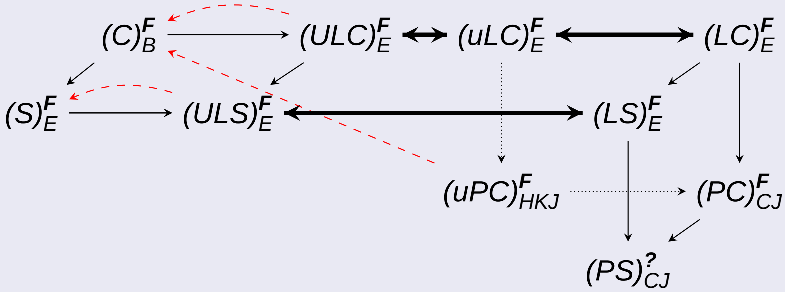


No combination of any of the properties shown imply any other property, unless the diagram forces such implication.

Fixed and Periodic Points - Compact Spaces

Theorem (Compact Rectifiably Path Connected Spaces)

Assume X is compact and rectifiably path connected.

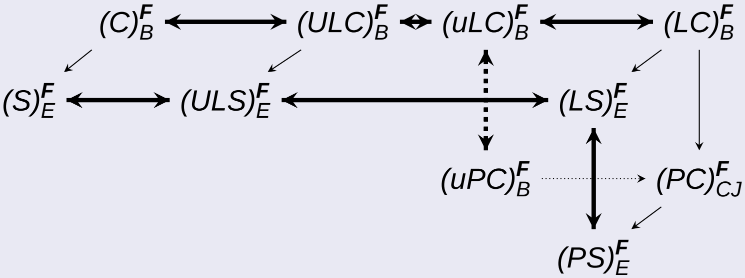


No combination of any of the properties shown imply any other property, unless the diagram forces such implication.

Fixed and Periodic Points - Compact Spaces

Theorem (Compact d-Convex Spaces)

Assume X is compact and d-convex.



No combination of any of the properties shown imply any other property, unless the diagram forces such implication.

Open Problems

1. Assume that $\langle X, d\rangle$ is compact and either connected or path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), must f have either fix or periodic point? What if f is (PC)? or (uPC)?
2. Assume that $\langle X, d\rangle$ is compact and rectifiably path connected. If the map $f\colon \langle X, d\rangle \to \langle X, d\rangle$ is (PS), does it imply that f has a fixed or periodic point?
3. Let $X \subset \mathbb{R}$ be compact perfect and let g be a function from X onto X^2. Can g be differentiable?

If a differentiable $g = \langle f, h\rangle$ as in Problem 3 existed then $f : X \to X$ would be a surjection with $f'(x) = 0$ except for a meager subset of X, [C&J, 2014].

Open Problems

Theorem (C & J, 2015)

There exists a perfect compact set $\mathfrak{X} \subseteq \mathbb{R}$ *and autohomeomorphism* $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ *with* $\mathfrak{f}'(x) = 0$ *for all* $x \in X$*.It follows that* $\mathfrak{f}$ *is* $\lambda - (\mathrm{uPC})$ *with any* $\lambda \in [0, 1)$*. Moreover,* $\langle \mathfrak{X}, \mathfrak{f} \rangle$ *is a minimal dynamical system so* $\mathfrak{f}$ *has no periodic points.*

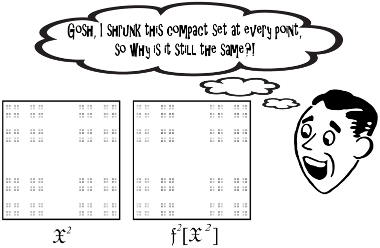
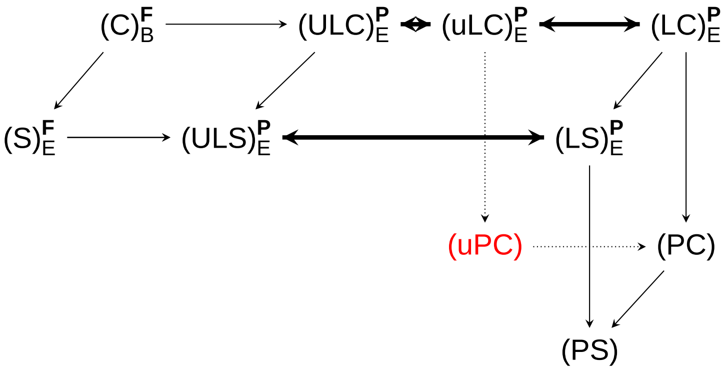


Figure: Action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2$.


(C)^F_B
(ULC)^P_E
(uLC)^P_E
(LC)^P_E
(S)^F_E
(ULS)^P_E
(LS)^P_E
(uPC)
(PC)
(PS)

Thank you
for your attention.