Lineability and additivity cardinals for real-valued functions: old results and new developments

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Outline

1. Lineability in terms of cardinal coefficients $\mathcal{L}$
2. Additivity number A vs lineability coefficients $\mathcal{L}$
3. Darboux-like functions
4. Different levels of surjectivity: the newest results
5. Some interesting open problems

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3. Darboux-like functions
4. Different levels of surjectivity: the newest results
5. Some interesting open problems

General lineability problem, studied in the last decade

Given a vector space W and $M \subset W$ let

$$\mathcal{V}(M) = \{V \subset M \cup \{0\} \colon V \text{ is a subspace of } W\}$$

How big $\dim(V)$ can be, when $V \in \mathcal{V}(M)$?

Inconvenience: $\lambda(M) \stackrel{\text{df}}{=} \max\{\dim(V) \colon V \in \mathcal{V}(M)\}$ may not exist.

Problem better expressed via lineability number

$$\mathcal{L}(M) = \min\{\kappa \colon \neg\exists V \in \mathcal{V}(M)(\kappa = \dim(V))\} \stackrel{\text{if } \lambda(M) \text{ exists}}{=} \lambda(M)^+$$

Clearly $0 < \mathcal{L}(M) \leq \dim(M)^+$ for any $M \subset W$.

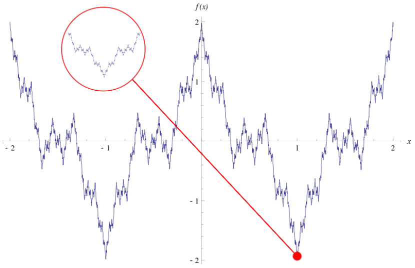
M is μ-lineable when $\mu < \mathcal{L}(M)$.

The systematic investigation of lineability started around 2004.

Warming up examples: # 1

For $W = C([0,1])$ and ND – the Weierstrass' monsters:

$$ND = \{f \in W\colon f \text{ is nowhere differentiable}\}$$



Jiménez-Rodríguez, Muñoz-Fernández, Seoane-Sepúlveda 2013: $\mathcal{L}(ND)$ has the maximal possible value of $\dim(W)^+$:

$$\mathcal{L}(ND) = \mathfrak{c}^+$$

Warming up examples: # 2

For $W = \mathbb{R}^{\mathbb{R}}$ and sSZ – surjective Sierpiński-Zygmund

(i.e., surjective with $f \restriction X$ discontinuous for every $X \in [\mathbb{R}]^{\mathfrak{c}}$)

K. Płotka 2015, implicitly: under GCH

sSZ is $2^{\mathfrak{c}}$-lineable: $\mathcal{L}(sSZ) = (2^{\mathfrak{c}})^+$

(Balcerzak, KC, Natkaniec 1997) it is consistent with ZFC that

$sSZ = \emptyset$: $\mathcal{L}(sSZ) = 1$

(KC, Pawlikowski 2004) under *Covering Property Axiom* **CPA**

$sSZ = \emptyset$: $\mathcal{L}(sSZ) = 1$

More $\mathcal{L}$ numbers: when W has topology τ

For $M \subset W$ let

$$\mathcal{V}_\tau(M) = \{V \subset M \cup \{0\} \colon V \text{ is a } \tau\text{-closed subspace of } W\}$$

$$\mathcal{L}_\tau(M) = \min\{\kappa \colon \neg\exists V \in \mathcal{V}_\tau(M)(\kappa = \dim(V))\}$$

M is μ-spacable when $\mu < \mathcal{L}_\tau(M)$.

For $W = \mathbb{R}^X$ with $X = \mathbb{R}^n$: τ_u and τ_p are topologies of uniform and pointwise convergence; $\mathcal{L}_u = \mathcal{L}_{\tau_u}$ and $\mathcal{L}_p = \mathcal{L}_{\tau_p}$

Clearly

$$\mathcal{L}_p(M) \le \mathcal{L}_u(M) \le \mathcal{L}(M)$$

Define also

$$\mathrm{m}\mathcal{L}(M) = \min\{\dim(V) \colon V \text{ is a maximal linear subspace of } M \cup \{0\}\}$$

Clearly $\quad \mathrm{m}\mathcal{L}(M) < \mathcal{L}(M)$

Outline

Additivity number $A(M)$, studied extensively in 1990s

For a vector space W over field K (usually $K = \mathbb{R}$) and $M \subset W$:

$$A(M) = \min\left(\{|F|\colon F \subset W \,\&\, (\forall w \in W)(w + F \not\subset M)\} \cup \{|W|^+\}\right)$$

$$\mathrm{st}(M) = \{w \in W\colon (K \setminus \{0\})w \subset M\}$$

Proposition

If $\emptyset \neq M \subsetneq W$, then

1. $2 \leq A(M) \leq |W|$ and $\mathrm{m}\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+$
2. if $\mathrm{st}(M) = M$ and $A(M) > |K|$, then

$$A(M) \leq \mathrm{m}\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+$$

Full comparison of A, m$\mathcal{L}$, and $\mathcal{L}$

Theorem (K.C, Gámez-Merino, Pellegrino, Seoane-Sepúlveda)

For a vector space W over K with $\dim(W) \geq \omega$

- *if* $\emptyset \neq \mathrm{st}(M) = M \subsetneq W$ *(commonly satisfied), then*

$$A(M) \leq \mathrm{m}\mathcal{L}(M) < \mathcal{L}(M) \leq \dim(W)^+$$

- *Conversely, for any cardinals* α, μ, *and* λ *with*

$$|K| < \alpha \leq \mu < \lambda \leq \dim(W)^+$$

  there exists $M \subsetneq W$ *with* $0 \in M = \mathrm{st}(M)$ *such that*

$$A(M) = \alpha,\ \mathrm{m}\mathcal{L}(M) = \mu, \text{ and } \mathcal{L}(M) = \lambda$$

Little else is known about m$\mathcal{L}$.

Outline

Darboux-like maps $f\colon X \to \mathbb{R}$, $X \subset \mathbb{R}^n$: definitions

These maps have some properties of continuous functions

D(X): f is *Darboux (has the Intermediate Value Property)* if $f[K]$ is connected for every connected $K \subseteq X$

Conn(X): f is a *connectivity* map if $f \restriction Z$ is connected in $Z \times \mathbb{R}$ for any connected $Z \subseteq X$

AC(X): f is *almost continuous* if for each open $U \subseteq X \times \mathbb{R}$ with $f \subset U$ there is a $g \in \mathcal{C}(X)$ with $g \subset U$

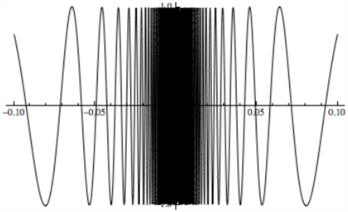
Ext(X): f is *extendable* provided there is an $F \in \text{Conn}(X \times [0,1])$ such that $f(x) = F(x,0)$ for every $x \in X$

PC(X): f is *peripherally continuous* if for every $x \in X$, open $U \ni x$, and open $V \ni f(x)$, there is open $W \subset U$ with $x \in W$ and $f[\text{bd}(W)] \subset V$

Example of discontinuous Darboux $f\colon \mathbb{R} \to \mathbb{R}$

For any $c \in [-1, 1]$,

$$f(x) = \begin{cases} \sin\left(x^{-1}\right) & \text{for } x \neq 0 \\ c & \text{for } x = 0. \end{cases}$$



Actually, this f belongs to all Darboux-like classes of functions

since it is Baire class one, $\mathcal{B}_1$, and (on $\mathbb{R}$)

Brown, Humke, Laczkovich, 1988:

$\text{Ext} \cap \mathcal{B}_1 = \text{AC} \cap \mathcal{B}_1 = \text{Conn} \cap \mathcal{B}_1 = \text{D} \cap \mathcal{B}_1 = \text{Ext} \cap \mathcal{B}_1 = \text{PC} \cap \mathcal{B}_1$

Darboux-like maps $f\colon \mathbb{R}^n \to \mathbb{R}$, $n > 1$: inclusions

(More important case of $n = 1$ we discuss latter.)

Theorem (Many authors, see KC 1997)

$$
\begin{array}{ccccccc}
 & & \mathrm{Conn}(\mathbb{R}^n) & & & & \mathrm{AC}(\mathbb{R}^n) \\
 & & \parallel & & & \nearrow & \\
\mathrm{C}(\mathbb{R}^n) & \longrightarrow & \mathrm{Ext}(\mathbb{R}^n) & \longrightarrow & \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) & & \\
 & & \parallel & & & \searrow & \\
 & & \mathrm{PC}(\mathbb{R}^n) & & & & \mathrm{D}(\mathbb{R}^n)
\end{array}
$$

Figure: Arrows indicate strict inclusions

A and $\mathcal{L}$ values for Darboux-like maps on $\mathbb{R}^n$, $n > 1$

$$\mathrm{Conn}(\mathbb{R}^n) = \mathrm{Ext}(\mathbb{R}^n) = \mathrm{PC}(\mathbb{R}^n)$$

$$\mathrm{C}(\mathbb{R}^n) \longrightarrow \mathrm{Ext}(\mathbb{R}^n) \longrightarrow \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n)$$

$$\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \longrightarrow \mathrm{AC}(\mathbb{R}^n), \qquad \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \longrightarrow \mathrm{D}(\mathbb{R}^n)$$

Theorem

- $A(\mathrm{Conn}(\mathbb{R}^n)) = A(\mathrm{Ext}(\mathbb{R}^n)) = A(\mathrm{PC}(\mathbb{R}^n)) = A(\mathrm{D}(\mathbb{R}^n)) = 1$
- $\mathfrak{c}^+ \leq A(\mathrm{AC}(\mathbb{R}^n)) \leq 2^{\mathfrak{c}}$ *is all that can be proved in ZFC*

Theorem (K.C, Gámez-Merino, Pellegrino, Seoane-Sepúlveda)

- $\mathcal{L}_u(\mathcal{F}) = \mathcal{L}_p(\mathcal{F}) = \mathcal{L}(\mathcal{F}) = \mathfrak{c}^+$ *for* $\mathcal{F} \in \{\mathrm{C}(\mathbb{R}^n), \mathrm{PC}(\mathbb{R}^n)\}$
- $\mathcal{L}_p(\mathcal{F}) = \mathcal{L}(\mathcal{F}) = (2^{\mathfrak{c}})^+$ *for* $\mathcal{F} \in \{\mathrm{AC}(\mathbb{R}^n), \mathrm{D}(\mathbb{R}^n)\}$

Problem: Find precise value of $\mathcal{L}(\mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n))$

More Darboux-like functions $f\colon \mathbb{R} \to \mathbb{R}$

CIVP f has *Cantor intermediate value property* if for every $x < y$ and perfect K between $f(x)$ and $f(y)$ there is a perfect set $C \subset (x, y)$ with $f[C] \subset K$

SCIVP f has *strong* CIVP if for every $x < y$ and perfect K between $f(x)$ and $f(y)$ there is a perfect set $C \subset (x, y)$ such that $f[C] \subset K$ and $f \restriction C$ is continuous

WCIVP f has *weak* CIVP if for every $x, y \in \mathbb{R}$ with $f(x) < f(y)$ there exists a perfect set C between x and y such that $f[C] \subset (f(x), f(y))$

PR f has *perfect road* if for every $x \in \mathbb{R}$ there is a perfect set $P \subset \mathbb{R}$ having x as a bilateral limit point for which $f \restriction P$ is continuous at x.

Darboux-like maps $f\colon \mathbb{R} \to \mathbb{R}$: inclusions

Theorem (Many authors, see KC 1997)

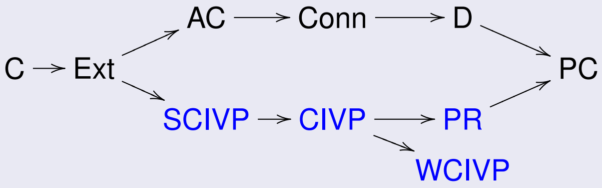
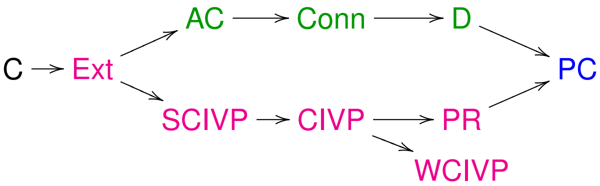


Figure: Arrows indicate strict inclusions

A and $\mathcal{L}$ values for Darboux-like maps $f\colon \mathbb{R} \to \mathbb{R}$



Theorem (K.C, Gámez-Merino, Pellegrino, Seoane-Sepúlveda)

- $\mathcal{L}_p(\mathcal{F}) = (2^{\mathfrak{c}})^+$ *for all Darboux-like classes* $\mathcal{F}$ *except* C.

Theorem

KC & Recław 1995*:* $A(PC) = 2^{\mathfrak{c}}$ *and*
$A(\mathcal{F}) = \mathfrak{c}^+$ *for* $\mathcal{F} \in \{\text{Ext}, \text{SCIVP}, \text{CIVP}, \text{WCIVP}, \text{PR}\}$
KC & A. Miller 1994*:* $\mathfrak{c}^+ \le A(\text{AC}) = A(\text{Conn}) = A(\text{D}) \le 2^{\mathfrak{c}}$
is all that can be proved in ZFC

A vs $\mathcal{L}$: connection deeper than just $A(M)^+ \leq \mathcal{L}(M)$

$A(\text{Ext})^+ = \mathfrak{c}^{++}$ needs not be equal $\mathcal{L}(\text{Ext}) = (2^{\mathfrak{c}})^+$.

Still, proof of $\mathcal{L}(\text{Ext}) = (2^{\mathfrak{c}})^+$ is based on proof of $A(\text{Ext}) = \mathfrak{c}^+$:

Proposition (Basis for proving $A(\text{Ext}) > \mathfrak{c}$)

There is a family $\mathcal{F} \in \mathbb{R}^{\mathbb{R}}$ of cardinality $\mathfrak{c}$ and
a family $\{M_f \colon f \in \mathcal{F}\}$ of pairwise disjoint subsets of $\mathbb{R}$ such that

- if $g \restriction M_f = f \restriction M_f$, for some $f \in \mathcal{F}$, then $g \in \text{Ext}$.

Proof of $\mathcal{L}(\text{Ext}) > 2^{\mathfrak{c}}$: Can assume $f(x) = 0$ for $f \in \mathcal{F}$ & $x \notin M_f$.

Then $V = \{\sum_{f\in\mathcal{F}} h(f)\cdot f \colon h \in \mathbb{R}^{\mathcal{F}}\}$ proves $2^{\mathfrak{c}}$-lineability of Ext.

Outline

Classes of surjective maps $f\colon \mathbb{R}\to\mathbb{R}$: definitions

$\mathcal{S}$: f is *surjective* if $f[\mathbb{R}]=\mathbb{R}$;

ES: f is *everywhere surjective* if $f[(a,b)]=\mathbb{R}$ for every $a<b$;

SES: f is *strongly everywhere surjective* if $|(f^{-1}(y)\cap(a,b)|=\mathfrak{c}$ for every $a<b$ and $y\in\mathbb{R}$;

$F_{<\mathfrak{c}}$: $f\in F_{<\mathfrak{c}}$ if $|(f^{-1}(y)|<\mathfrak{c}$ for every $y\in\mathbb{R}$;

SZ: f is *Sierpiński-Zygmund* if $f\restriction X\notin C(X)$ for every $X\in[\mathbb{R}]^{\mathfrak{c}}$;

Basic interrelations:

- SES $\subsetneq$ ES $\subsetneq$ $\mathcal{S}$, ES $\subsetneq$ D, SZ $\subsetneq$ $F_{<\mathfrak{c}}$;
- SES $\cap$ SZ $=\emptyset$, ES $\cap$ SZ $\subset$ ES $\setminus$ SES;
- It is independent of ZFC that ES $\cap$ SZ $=\mathcal{S}\cap$ SZ $=\emptyset$;
- ES $\cap F_{<\mathfrak{c}}\subsetneq$ ES $\setminus$ SES.

The class ES \ SES: the hard object to study

Problem (Gamez, Munoz, Sanchez, Seoane 2010)

Is $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES})=(2^{\mathfrak{c}})^+$? *(Still open in ZFC!)*

Proposition (Class SES is well understood)

- [GMSS 2010]: $\mathcal{L}(\mathrm{SES})=(2^{\mathfrak{c}})^+$. [E.g., our earlier example $\{\sum_{f\in\mathcal{F}} h(f)\cdot f\colon h\in\mathbb{R}^{\mathcal{F}}\}\subset \mathrm{Ext}$, of size $2^{\mathfrak{c}}$, justifies it.]
- [KC & Miller 1994]: $\mathfrak{c}^+\le A(\mathrm{SES})=A(\mathrm{D})=A(\mathrm{AC})\le 2^{\mathfrak{c}}$ is all that can be proved in ZFC

Results from [Bartoszewicz, Bienias, Głąb, Natkaniec, 2016?] and (implicitly) [Płotka 2015] imply that

$\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES})=(2^{\mathfrak{c}})^+$ is consistent with ZFC.

Our new results show considerably more!

$A(\mathrm{ES}\setminus\mathrm{SES})$ and more on $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES})$

Theorem (Ciesielski & Gamez & Seoane 2016)

$\mathrm{ES}\setminus\mathrm{SES}$ is $\mathfrak{c}^+$-lineable, that is, $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) > \mathfrak{c}^+$

So, $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) = (2^{\mathfrak{c}})^+$ follows from $2^{\mathfrak{c}} = \mathfrak{c}^+$

Theorem (Ciesielski & Gamez & Seoane 2016)

If $\mathfrak{c}$ is regular, then $A(\mathrm{ES}\setminus\mathrm{SES}) \leq \mathfrak{c}$. In particular,

$A(\mathrm{ES}\setminus\mathrm{SES})^+ < \mathcal{L}(\mathrm{ES}\setminus\mathrm{SES})$ in "almost all" models of ZFC.

Proof of $A(\text{ES} \setminus \text{SES}) \leq \mathfrak{c}$, assuming $\mathfrak{c}$ is regular

Put $\mathbb{R} = \{r_\xi : \xi < \mathfrak{c}\}$ and $A_\xi = \{r_\zeta : \zeta < \xi\}$.

Then $F = \{r\chi_{A_\xi} + y : r, y \in \mathbb{R} \ \& \ \xi < \mathfrak{c}\}$ justifies the result.

To see this, an fix $g \in \mathbb{R}^\mathbb{R}$. Need to show $g + F \not\subset \text{ES} \setminus \text{SES}$.

Indeed, $g = g + \chi_{A_0} \in g + F$. If $g \in \text{SES}$, we are done.

So, assume not. Fix a, b, y with $A = g^{-1}(y) \cap (a, b) \in [\mathbb{R}]^{<\mathfrak{c}}$.

Pick $\xi < \mathfrak{c}$ with $A \subset A_\xi$ and $0 \neq r \in \mathbb{R} \setminus (g - y)[A_\xi]$.

Then $g - y - r\chi_{A_\xi} \in g + F$.

But $(a, b) \cap (g - y - r\chi_{A_\xi})^{-1}(0) = \emptyset$, that is, $g - y - r\chi_{A_\xi} \notin \text{ES}$.

Proof of $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) > \mathfrak{c}^+$ is based on two results:

Fact (easy remark)

$\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) > \mathfrak{c}^\kappa$ *for every* $\kappa < \mathfrak{c}$.

Proof: Use $\{\sum_{f\in\mathcal{F}} h(f)\cdot f \colon h\in\mathbb{R}^{\mathcal{F}}\}$, for natural $\mathcal{F}$, $|\mathcal{F}| = \kappa$.

Lemma (seems easy and natural; it is natural, but . . .)

If $\mathfrak{c}$ *is regular, then* $\mathcal{L}(\mathrm{ES}\cap F_{<\mathfrak{c}}) > \mathfrak{c}^+$.

Proof of $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) > \mathfrak{c}^+$:

- If $\mathfrak{c}$ is regular, then $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) \geq \mathcal{L}(\mathrm{ES}\cap F_{<\mathfrak{c}}) > \mathfrak{c}^+$
- If $\mathfrak{c}$ is singular, then $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES}) > \mathfrak{c}^{\mathrm{cof}(\mathfrak{c})} \geq \mathfrak{c}^+$

Proof of $\mathcal{L}(ES \cap F_{<\mathfrak{c}}) > \mathfrak{c}^+$, assuming $\mathfrak{c}$ is regular

Enough to show that if $\mathcal{G} \subset (\mathrm{ES} \cap F_{<\mathfrak{c}}) \cup \{0\}$ is linear with $|\mathcal{G}| \le \mathfrak{c}$, then $\mathcal{G}$ can be further extended.

By induction we find $f \in \mathbb{R}^{\mathbb{R}}$ with $f - \mathcal{G} \subset \mathrm{ES} \cap F_{<\mathfrak{c}}$. (So, $f \notin \mathcal{G}$.)

Then $\mathbb{R}(f - \mathcal{G}) \subset (\mathrm{ES} \cap F_{<\mathfrak{c}}) \cup \{0\}$ is a desired extension of $\mathcal{G}$.

Finding f, an easy inductive argument? ... True. But wait!

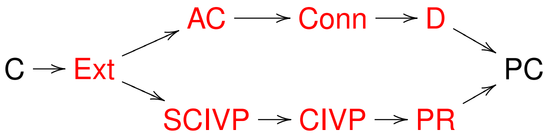
Doesn't this **contradict** $A(ES \cap F_{<\mathfrak{c}}) \le \mathcal{L}(\mathrm{ES} \setminus \mathrm{SES}) \le \mathfrak{c}$?

It seems: there is $G \in \left[\mathbb{R}^{\mathbb{R}}\right]^{\mathfrak{c}}$ with $f - G \not\subset \mathrm{ES} \cap F_{<\mathfrak{c}}$ for every $f \in \mathbb{R}^{\mathbb{R}}$

Luckily, our $\mathcal{G}$ is special: is contained in $(\mathrm{ES} \cap F_{<\mathfrak{c}}) \cup \{0\}$.

So, maybe construction of f is not that straightforward, after all?

News on $A(\mathcal{F} \cap \text{Darboux-like})$ for $\mathcal{F} \in \{\text{SES}, \text{ES}, \mathcal{S}\}$



Theorem

For every $\mathcal{F} \in \{\text{SES}, \text{ES}, \mathcal{S}\}$ *we have*

- $A(\mathcal{F} \cap \mathcal{G}) = A(\mathcal{G}) = \mathfrak{c}^+$ *for* $\mathcal{G} \in \{\text{Ext}, \text{SCIVP}, \text{CIVP}, \text{PR}\}$*;*
- $\mathfrak{c}^+ \leq A(\mathcal{F} \cap \mathcal{G}) = A(\mathcal{F}) = A(\text{AC}) = A(\text{Conn}) = A(\text{D}) \leq 2^{\mathfrak{c}}$ *for every* $\mathcal{G} \in \{\text{AC}, \text{Conn}, \text{D}, \text{PC}, \mathbb{R}^{\mathbb{R}}\}$*.*

Relatively new components: $A(\text{SES} \cap \text{Ext}) \geq \mathfrak{c}^+$

$A(\mathcal{S}) \leq A(\text{SES})$ & $A(\text{SES}) \leq A(\text{SES} \cap \text{AC})$

Outline

Some chosen open problems on $\mathcal{L}(F)$ and $A(F)$

1. Can $\mathcal{L}(\mathrm{ES}\setminus\mathrm{SES})=(2^{\mathfrak{c}})^+$ be proved in ZFC?
2. Can we prove $A(\mathrm{ES}\setminus\mathrm{SES})\leq\mathfrak{c}$ in ZFC?

   What else can be said about $A(\mathrm{ES}\setminus\mathrm{SES})$ or $A(\mathrm{ES}\cap F_{<\mathfrak{c}})$?
3. Are numbers $A(\mathrm{D}\cap\mathrm{SZ})$, $A(\mathrm{ES}\cap\mathrm{SZ})$, and $A(\mathcal{S}\cap\mathrm{SZ})$ provably (in ZFC) equal?

   What about $\mathcal{L}(\mathrm{D}\cap\mathrm{SZ})$, $\mathcal{L}(\mathrm{ES}\cap\mathrm{SZ})$, and $\mathcal{L}(\mathcal{S}\cap\mathrm{SZ})$?
4. Under what conditions $A(M)=\mathrm{m}\mathcal{L}(M)$?

That is all!

Thank you for your attention!

Some References

- K.C. Ciesielski, J.L. Gamez-Merino, D. Pellegrino, and J.B. Seoane-Sepulveda, Lineability, spaceability, and additivity cardinals for Darboux-like functions, Linear Algebra Appl. 440 (2014), 307-317.
- Survey: K.C. Ciesielski, Set Theoretic Real Analysis, J. Appl. Anal. 3(2) (1997), 143-190.
- K.C. Ciesielski, J.L. Gamez-Merino, and J.B. Seoane-Sepulveda, Darboux and Sierpiński-Zygmund functions and related lineability questions, draft.
- For the *existence* (non-constructive) of a $\mathfrak{c}$-dimensional linear uubspace of Weierstrass' monsters see also: Fonf, Gurariy, and Kadets (1999) or Rodriguez-Piazza (1999).