An auto-homeomorphism of a Cantor set with zero derivative everywhere

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Based on a joint work with Jakub Jasinski

see http://www.math.wvu.edu/˜kcies/publications.html

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Outline

1. The example and why do we care
2. The construction of the main example
3. Why the construction works? Sketch of a proof

Outline

The main result

Theorem ([KC & JJ])

There exists a compact perfect set $\mathfrak{X} \subset \mathbb{R}$ and a differentiable bijection $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{X}$ such that $\mathfrak{f}' \equiv 0$ on $\mathfrak{X}$. Moreover,

(i) *$\mathfrak{f}$ is a minimal dynamical system (i.e., the $\mathfrak{f}$-orbit $O(x) = \{f^{(n)}(x) \colon n \in \omega\}$ of every $x \in \mathfrak{X}$ is dense in $\mathfrak{X}$);*

(ii) *$\mathfrak{f}$ can be extended to a differentiable function $F \colon \mathbb{R} \to \mathbb{R}$.*

Fact: $\mathfrak{f}' \equiv 0$ implies that $\mathfrak{f}$ is *locally radially contractive*:

(LRC) for every $x \in \mathfrak{X}$ there are $\varepsilon_x > 0$ and $\lambda_x \in [0, 1)$ such that $|\mathfrak{f}(x) - \mathfrak{f}(y)| \le \lambda_x |x - y|$ for any $y \in \mathfrak{X}$ with $|x - y| < \varepsilon_x$.

Radially $\equiv$ only one variable, y, can vary near x.

Existence of $\mathfrak{f}$ seems paradoxical!

Fact: Assume that $X \subseteq \mathbb{R}$ and $f\colon X \to \mathbb{R}$.

(i) $X \not\subseteq f[X]$ when X is a bounded closed interval and $|f'| \le \lambda < 1$ on X since then, by the Mean Value Theorem, $|f(y) - f(z)| \le \lambda |y - z|$ for every $y, z \in X$, so that the diameter of $f[X]$ is strictly smaller than the diameter of X. If $\mathfrak{f}' \equiv 0$, then f is constant.

(ii) $X \not\subseteq f[X]$ when X has a positive finite Lebesgue measure $m(X)$ and $|f'| \le \lambda < 1$ on X since then $m(f[X]) \le \lambda m(X)$.

(iii) $X \not\subseteq f[X]$ when $|f'| < 1$ on X and f can be extended to a **continuously** differentiable function $F\colon \mathbb{R} \to \mathbb{R}$. This has been proved by the authors, *RAEx* **39**(1), 2014.

???

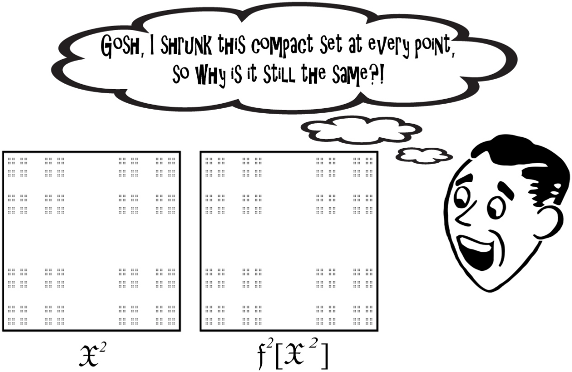


Figure: The result of the action of $\mathfrak{f}^2 = \langle \mathfrak{f}, \mathfrak{f} \rangle$ on $\mathfrak{X}^2 = \mathfrak{X} \times \mathfrak{X}$

The example vs Banach Fixed Point Theorem: where Banach Theorem meets Dynamical Systems

Convexity assumed?	$f\colon X \to X$ has fixed or periodic point when f is		
	contractive (C)	(LC)	(LRC)
Yes	fixed point, Banach 1922	fixed point, Edelstein 1962	fixed point, Hu & Kirk 1978
No	fixed point, Banach 1922	periodic point, Edelstein 1962	NEITHER KC & JJ 2015

Table: Fixed/periodic point properties of $f\colon X \to X$; X is compact and either arbitrary, or a convex subspace of a Banach space

(C) $\exists\, \lambda \in [0,1)$ s.t. $d(f(y), f(z)) \leq \lambda d(y,z)$ for every $y, z \in X$.

(LC) for every $x \in X$ there is $\varepsilon_x > 0$ s.t. $f \restriction B(x, \varepsilon_x)$ is (C), i.e. for every $x \in X$ there are $\varepsilon_x > 0$ and $\lambda_x \in [0,1)$ s.t. $|f(y) - f(z)| \leq \lambda_x |y - z|$ for any $y, z \in B(x, \varepsilon_x)$.

(LRC) for every $x \in X$ there are $\varepsilon_x > 0$ and $\lambda_x \in [0,1)$ s.t. $|f(x) - f(y)| \leq \lambda_x |x - y|$ for any $y \in B(x, \varepsilon_x)$.

(LRC) map which is not (LC)

$1 = b_1 > a_1 > b_2 > a_2 > \cdots > \lim_n a_n = 0$ and
$X = \{0\} \cup \bigcup_{n=1}^{\infty} [a_n, b_n]$.

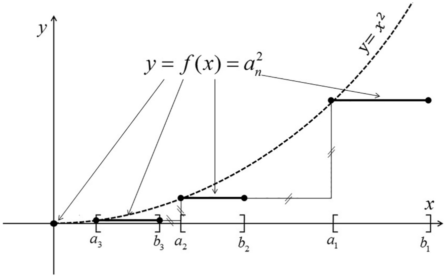


Figure: $f(0) = 0$; for any $n = 1, 2, 3, \ldots$,
$f(a_n) - f(b_{n+1}) = a_n - b_{n+1}$ and $f(x) = (a_n)^2$ for any $x \in [a_n, b_n]$.

$\mathfrak{f}$ is minimal, does it have to be?

Yes, our example must be based on a minimal dynamics:

Theorem (KC & JJ)

Let X be an infinite compact metric space and assume that $f: X \to X$ is an (LRC) surjection. Then there exists a perfect subset $Y \subseteq X$ such that $f \restriction Y$ is a minimal dynamical system.

Open problem

Question (KC & JJ)

Assume that $f\colon X \to X$ is (LRC) (or even that $f' \equiv 0$ on X).
If X is compact and connected (or even path connected),
must f has a fixed point?

What is known:

- True if assumption that f is (LRC) is strengthen of (LC) — Edelstein result.
- False if assumption that X is compact is weakened to complete — Hu & Kirk result requires that X is rectifiable path connected; without rectifiability the result is false, KH.
- False if assumption that X is connected is removed — our new example.

Outline

Adding machine: a minimal dynamics on Cantor set 2^ω

"Add one and carry," odometer-like action $\sigma\colon 2^\omega \to 2^\omega$:

for $s = \langle s_0, s_1, s_2, \ldots\rangle \in 2^\omega$, $\sigma(s) = s + \langle 1, 0, 0, ...\rangle$, i.e.

$$\sigma(s) = \begin{cases} \langle 0, 0, 0, \ldots\rangle & \text{if } s_i = 1 \text{ for all } i < \omega, \\ \langle 0, 0, \ldots, 0, 1, s_{k+1}, s_{k+2}, \ldots\rangle & \text{if } s_k = 0,\ s_i = 1 \text{ for all } i < k. \end{cases}$$

Alternatively

$$\begin{aligned} \sigma(1, 1, 1, \ldots) &= \langle 0, 0, 0, \ldots\rangle \\ \sigma(1, \ldots, 1, 0, s_{k+1}, s_{k+2}, \ldots) &= \langle 0, \ldots, 0, 1, s_{k+1}, s_{k+2}, \ldots\rangle. \end{aligned}$$

Fact: σ is bijective and minimal on 2^ω.

Format of the example

- We construct continuous injection $h\colon 2^\omega \to \mathbb{R}$.
- Put $\mathfrak{X} = h[2^\omega]$ and $\mathfrak{f} = h \circ \sigma \circ h^{-1}\colon \mathfrak{X} \to \mathfrak{X}$.

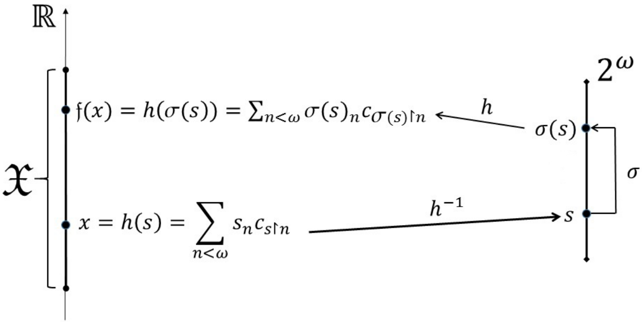


Figure: $\mathfrak{f} = h \circ \sigma \circ h^{-1}$

What can be said on $\mathfrak{f} = h \circ \sigma \circ h^{-1} \colon \mathfrak{X} \to \mathfrak{X}$.

Difficult part:

- to ensure that $\mathfrak{f}' \equiv 0$.

Easy consequences:

(i) $\mathfrak{f}$ is minimal since $\mathfrak{f}^{(n)} = h \circ \sigma^{(n)} \circ h^{-1}$: density of the orbits of σ implies the same for $\mathfrak{f}$.

(ii) $\mathfrak{f}$ can be extended to a differentiable function $F \colon \mathbb{R} \to \mathbb{R}$: follows immediately from a theorem of Jarník.

Format of the injection $h\colon 2^\omega \to \mathbb{R}$

$$h(s) = \sum_{n<\omega} s_n c_{s\restriction n} \text{ for every } s \in 2^\omega$$

for appropriately chosen numbers $c_\tau \in \mathbb{R}$ for $\tau \in 2^{<\omega}$.

To ensure that $\mathfrak{f}'(x) = 0$ for $x = h(s)$ with $s \in 2^\omega$, we need

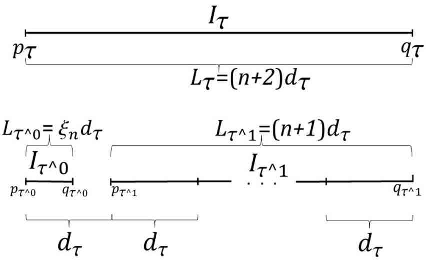
$$\Delta_{st} = \frac{|\mathfrak{f}(x) - \mathfrak{f}(y)|}{|x-y|} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|} \to_{\ell\to\infty} 0,$$

where $\ell = \min\{i < \omega \colon s_i \neq t_i\}$; that is, eventually,

$$|h(\sigma(s)) - h(\sigma(t))| << |h(s) - h(t)|.$$

d_τ's, related to $\sum_n \frac{1}{n^2}$, — first approximation of c_τ's

$d_{s\restriction n} = \frac{1}{n+2} L_{s\restriction n} = \frac{1}{n+2} |I_{s\restriction n}|$ from Cantor-like set construction:



$I_\emptyset = [0,1]$; $I_{\tau^\frown 1}$ — the terminal $\frac{n+1}{n+2}$-th part of I_τ;

$I_{\tau^\frown 0}$ — the initial $\frac{\xi_n}{n+2}$-th part of I_τ, with $\xi_n = \frac{1}{2}\frac{1}{(n+4)^{1/2}}$.

The fun begins: full definition of c_τ's

$$c_{s\restriction n} = a_{s\restriction n}\beta_n^{-b_{s\restriction n}} d_{s\restriction n},$$

where $\beta_n = \ln(n+3) > 1$,

$$a_{s\restriction n} = \begin{cases} -1 & \text{when } s \restriction n = \langle 1, 1, \ldots, 1\rangle, \\ 1 & \text{otherwise} \end{cases}$$

$b_{s\restriction n} = \sum_{i<\nu_n} s_i 2^i$ with $\nu_n = \max\left\{m < \omega \colon (\beta_n)^{2^m - 1} < \sqrt{n+2}\right\}$.

The definition is complicated to ensure an intricate comparison of different rates of convergence of the components.

Outline

$\frac{|h(\sigma(s))-h(\sigma(t))|}{|h(s)-h(t)|} \to_{\ell\to\infty} 0$ for $s = \langle 1, 1, 1, \dots \rangle$

For $\ell = \min\{i < \omega \colon s_i \neq t_i\}$ large enough, some work gives (using essentially $a_{s\restriction n}$ and $d_{s\restriction n}$ from $c_{s\restriction n} = a_{s\restriction n}\beta_n^{-b_{s\restriction n}} d_{s\restriction n}$)

$$|h(\sigma(s)) - h(\sigma(t))| \leq \frac{1}{\ell+1}\frac{1}{\ell} \qquad (1)$$

$$|h(s) - h(t)| \geq \sum_{n\geq\ell} |c_{s\restriction n}| \geq \sum_{n\geq\ell} \frac{1}{(n+2)^{1/2}}\frac{1}{n+2}\frac{1}{n+1}. \qquad (2)$$

Since $\sum_{n\geq\ell} \frac{1}{(n+2)^{1/2}}\frac{1}{n+2}\frac{1}{n+1} \geq \sum_{n\geq\ell} \frac{1}{(n+2)^{2.5}} \geq \int_{\ell+2}^{\infty} x^{-2.5}\,dx = \frac{1}{1.5}\frac{1}{(\ell+2)^{1.5}}$, (1) and (2) imply the required convergence:

$$\Delta_{st} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|} \leq \frac{\frac{1}{\ell(\ell+1)}}{\frac{1}{1.5}\frac{1}{(\ell+2)^{1.5}}} = 1.5\frac{(\ell+2)^{1.5}}{\ell(\ell+1)} \to_{\ell\to\infty} 0.$$

$\frac{|h(\sigma(s))-h(\sigma(t))|}{|h(s)-h(t)|} \to_{\ell\to\infty} 0$ for $s \neq \langle 1,1,1,\ldots\rangle$

For ℓ large enough and $u \in \{s,t\}$ with $u_\ell = 1$, some work gives (using $\beta_n^{-b_{s\restriction n}}$ and $d_{s\restriction n}$, but not $a_{s\restriction n}$ from $c_{s\restriction n} = a_{s\restriction n}\beta_n^{-b_{s\restriction n}} d_{s\restriction n}$)

$$\begin{aligned} |h(\sigma(s)) - h(\sigma(t))| &\leq \tfrac{3}{2}\sum_{n\geq\ell} u_n |c_{\sigma(u)\restriction n}| \\ |h(s) - h(t)| &\geq \tfrac{1}{2}\sum_{n\geq\ell} u_n |c_{u\restriction n}| > 0. \end{aligned} \tag{3}$$

Also there is a constant $E_k > 0$ depending only on k such that

$$\frac{|c_{\sigma(u)\restriction n}|}{|c_{u\restriction n}|} = \frac{|a_{\sigma(u)\restriction n}\beta_n^{-b_{\sigma(u)\restriction n}} d_{\sigma(u)\restriction n}|}{|a_{u\restriction n}\beta_n^{-b_{u\restriction n}} d_{u\restriction n}|} = E_k\beta_n^{-1} \leq E_k\beta_\ell^{-1} \text{ for } n \geq \ell. \tag{4}$$

This guarantees the desired convergence, as then

$$\Delta_{st} = \frac{|h(\sigma(s)) - h(\sigma(t))|}{|h(s) - h(t)|} \leq \frac{\frac{3}{2}\sum_{n\geq\ell} u_n |c_{\sigma(u)\restriction n}|}{\frac{1}{2}\sum_{n\geq\ell} u_n |c_{u\restriction n}|} \leq 3E_k\beta_\ell^{-1} \to_{\ell\to\infty} 0.$$

Which details of the proof were left?

- The proofs of estimates (1), (2), and (3).
  (Each takes a short paragraph of an argument.)
- A proof that $\frac{|d_{\sigma(u)\restriction n}|}{|d_{u\restriction n}|} = E_k$, k being the first 1 in u, part of (4).
  (A short paragraph of an argument.)
- A proof that h is actually an injection.
  (An argument is easy, but takes about a page of explanations.)

That is all!

Thank you for your attention!