General theory of fuzzy connectedness segmentations: reconciliation of two tracks of FC theory

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Based on a joint work with Gabor T. Herman and T. Yung Kong

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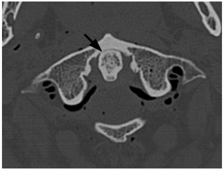
Preamble: What this talk is about

- The subject of this talk is theoretical, no experimental part
- Concerns an image segmentation theory that unifies two Fuzzy Connectedness, FC, tracks:
  - (I)RFC, on RFC and IRFC segmentations, favored by MIPG
  - MOFS, favored by Herman, Kong, and others
- Optimized algorithm for finding MOFS and IRFC objects which is the most efficient among existing algorithms finding MOFS or IRFC (that have no tie-zones ambiguity)
- All presented results concern "hard" segmentations, as opposed to fuzzy segmentations
  (Though, they could be easily reformulated to fuzzy sets framework)

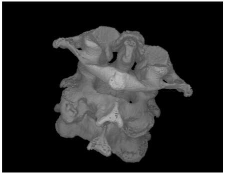
Outline

Outline

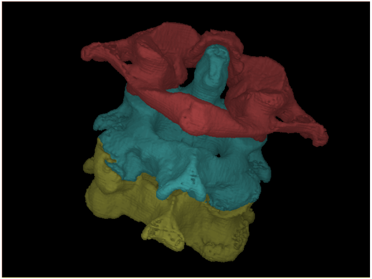
Image segmentation example: CT, cervical spine



A slice of an original 3D image



Surface rendition of segmented three vertebrae, together



Color surface rendition of the segmented three vertebra

Image segmentation — formal setting

- An *image* is a map f from a set V (of spels) into $\mathbb{R}^k$

  The value $f(c)$ represents image intensity at c, a k-dimensional vector each component of which indicates a measure of some aspect of the signal, like color.

- *Segmentation problem:* Given an image $f\colon V \to \mathbb{R}^k$,

  find a "desired" family $\{O_1, \ldots, O_M\}$ of subsets of V.

- We will assume the objects are indicated by disjoint sets S_i of seeds, imposing that $S_i \subset O_i \subset V \setminus \bigcup_{j \neq i} S_j$.

- We impose neither that O_i's are disjoint nor that $V = \bigcup_i O_i$.

Outline

Affinities: concise information coding of the image

In FC framework, all information on the image and (beside the seed sets) the objects is coded via affinity function(s):

maps $\psi \colon V \times V \to [0,1]$ such that $\psi(v,v) = 1$ for all $v \in V$.

The larger $\psi(c,d)$, the stronger c is (locally ψ-) connected to d.

No adjacency: for non-adjacent c and d we put $\psi(c,d) = 0$.

(Though adjacency/edge reappears in efficient algorithms.)

In MOFS track, each ith object has its own affinity ψ_i.

In (I)RFC track, there is single affinity ψ for all objects.

Even for (I)RFC results, we do not assume ψ is symmetric.
(Symmetric means $\psi(c,d) = \psi(d,c)$ for all $c,d \in V$.)

Path strength and connectivity via $W \subset V$

Given affinity $\psi\colon V \times V \to [0,1]$ and $A, B, W \subseteq V$,

- a *W-path* from A to B: any sequence $p = \langle w_0, \ldots, w_\ell \rangle$ of points in W such that $w_0 \in A$ and $w_\ell \in B$;
- the *ψ-strength* of $p = \langle v_0, \ldots, v_\ell \rangle$ is: $\psi(p) = 1$ if $\ell = 0$ and $\psi(p) = \min_{1 \le j \le \ell} \psi(v_{j-1}, v_j)$ if $\ell > 0$;

$$\psi^W(A,B) = \max\{\psi(p) \mid p \text{ is a } (W \cup A \cup B)\text{-path from } A \text{ to } B\}.$$

- Seed sets $S_1, \ldots, S_M$ are *consistent* with the affinities $\psi_1, \ldots, \psi_M$ if $\psi_i^V(S_i, S_j) < 1$ for all $i \neq j$.

Outline

The rationale behind the following three algorithms

- to emphasize great similarities among (I)RFC and MOFS segmentations
- to use their outputs as unambiguous definitions of RFC, IRFC and MOFS objects (there were some small ambiguities in definitions of IRFC, especially in IFT setting)
- to facilitate parallel development of the mathematics of (I)RFC and MOFS segmentations
- these thee algorithms are not optimized for computation of either of the FC segmentations

RFC objects

Algorithm 1: RFC Segmentation into M Objects O_i^{RFC}

Data: Seed sets $S_1, \ldots, S_M \subset V$; an affinity ψ on V

Result: The RFC segmentation $\langle O_1^{\mathrm{RFC}}, \ldots, O_M^{\mathrm{RFC}} \rangle$ of V

for $i \leftarrow 1$ **to** M **do** $T_i \leftarrow S_i$

sort $A = \psi[V \times V] \setminus \{0\}$ **into** $1 = \alpha_1 > \cdots > \alpha_{|A|}$

for $n \leftarrow 1$ **to** $|A|$ **do** /* the main loop */

  for $i \leftarrow 1$ **to** M **do**

  $newT_i \leftarrow T_i \cup \{v \in V \setminus \bigcup_j T_j \mid \psi^V(T_i, v) \geq \alpha_n\}$

  for $i \leftarrow 1$ **to** M **do** $T_i \leftarrow newT_i$

for $i \leftarrow 1$ **to** M **do** $O_i^{\mathrm{RFC}} \leftarrow T_i \setminus \bigcup_{j \neq i} T_j$

IRFC objects

Algorithm 2: IRFC Segmentation into M Objects O_i^{IRFC}

Data: Seed sets $S_1, \ldots, S_M \subset V$; an affinity ψ on V

Result: The IRFC segmentation $\langle O_1^{\mathrm{IRFC}}, \ldots, O_M^{\mathrm{IRFC}} \rangle$ of V

for $i \leftarrow 1$ **to** M **do** $T_i \leftarrow S_i$

sort $A = \psi[V \times V] \setminus \{0\}$ **into** $1 = \alpha_1 > \cdots > \alpha_{|A|}$

for $n \leftarrow 1$ **to** $|A|$ **do** /* the main loop */

  for $i \leftarrow 1$ **to** M **do**

  $newT_i \leftarrow T_i \cup \{v \in V \setminus \bigcup_j T_j \mid \psi^{V \setminus \bigcup_j T_j}(T_i, v) \geq \alpha_n\}$

  for $i \leftarrow 1$ **to** M **do** $T_i \leftarrow newT_i$

for $i \leftarrow 1$ **to** M **do** $O_i^{\mathrm{IRFC}} \leftarrow T_i \setminus \bigcup_{j \neq i} T_j$

MOFS objects

Algorithm 3: MOFS Segmentation into M Objects O_i^{MOFS}

Data: Seed sets $S_1, \ldots, S_M \subset V$; M affinities $\psi_1, \ldots, \psi_M$ on V
Result: The MOFS segmentation $\langle O_1^{\mathrm{MOFS}}, \cdots, O_M^{\mathrm{MOFS}} \rangle$ of V

1 **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow S_i$
2 **sort** $A = \bigcup_j \psi_j[V \times V] \setminus \{0\}$ **into** $1 = \alpha_1 > \cdots > \alpha_{|A|}$
3 **for** $n \leftarrow 1$ **to** $|A|$ **do** /* the main loop */
4 **for** $i \leftarrow 1$ **to** M **do**
$newT_i \leftarrow T_i \cup \{v \in V \setminus \bigcup_j T_j \mid \psi_i^{V \setminus \bigcup_j T_j}(T_i, v) \geq \alpha_n\}$
5 **for** $i \leftarrow 1$ **to** M **do** $T_i \leftarrow newT_i$
6 **for** $i \leftarrow 1$ **to** M **do** $O_i^{\mathrm{MOFS}} \leftarrow T_i$

Corollary and Correctness Theorem

Corollary (IRFC from MOFS)

If $\psi_1 = \cdots = \psi_M = \psi$, then $O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$ for all i.

The equation is true even when ψ is not symmetric.

Any algorithm finding MOFS objects finds also IRFC objects!

(This is important, as our optimized MOFS finding algorithm is more efficient than currently existing IRFC finding algorithms.)

Theorem (Correctness of the algorithms)

If the seed sets $S_1, \ldots, S_M$ are consistent with the affinities $\psi_1, \ldots, \psi_M$, then Algorithm 3 returns the same MOFS objects that were found by older algorithms.
If, in addition, $\psi_1 = \cdots = \psi_M = \psi$ and ψ is symmetric, then Algorithms 1 and 2 return the same RFC and IRFC objects that were found by older algorithms.

Outline

IRFC from MOFS, Round I

- If $\psi_1 = \cdots = \psi_M = \psi$ and ψ is symmetric, then $O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$
- If $\psi_1 = \cdots = \psi_M = \psi$ and ψ is non-symmetric, then we can still define $O_i^{\mathrm{IRFC}} \stackrel{\mathrm{def}}{=} O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$
  These new IRFC objects will still have good properties.
- If ψ_i's are distinct, even asymmetric, then objects $O_i = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$ will not have good properties.
  (Can be disconnected from the seeds – example latter.)
- With ψ_i's distinct,
  can we ensure that the objects O_i^{MOFS} are disjoint?
  Then such objects would have good properties and could be used as new IRFC objects.

IRFC from MOFS, Round II

Theorem

If $S_1, \ldots, S_M$ are consistent with the affinities $\psi_1, \ldots, \psi_M$, then objects O_i^{MOFS} are disjoint as long as

(D) *for any $c \neq d$, $u \neq v$, and $i \neq j$,*
$\psi_i(c, d) = \psi_j(u, v)$ *can happen only when* $\psi_i(c, d) = 0$.

In such case, O_i^{MOFS} can be treated as IRFC objects.

Property (D) can be insured by perturbing ψ_i's as follows:

if $M = 4$, $\psi_i(u, v) < 1$ for $u \neq v$, and values $\psi_i(u, v)$ are floating point binary values, then for any $u \neq v$ with $\psi_i(u, v) > 0$ set the two least significant bits of $\psi_i(u, v)$ to 00, 01, 10, or 11 for $i = 1, 2, 3, 4$, respectively.

Outline

Mathematical Characterizations of RFC and IRFC

Theorem (RFC, new only for non-symmetric ψ)

Assuming $S_1, \ldots, S_M$ are consistent with ψ, the RFC object O_i^{RFC} given by $S_1, \ldots, S_M$ and ψ satisfies

$$O_i^{\mathrm{RFC}} = \{v \in V \mid \max_{j \neq i} \psi^V(S_j, v) < \psi^V(S_i, v)\}, \quad (1)$$

$$O_i^{\mathrm{RFC}} = \{v \in V \mid \max_{j \neq i} \psi^V(S_j, v) < \psi^{O_i^{\mathrm{RFC}}}(S_i, v)\}. \quad (2)$$

Theorem (IRFC, new only uniqueness & non-symmetric case)

Assuming $S_1, \ldots, S_M$ are consistent with ψ, the IRFC object O_i^{IRFC} given by $S_1, \ldots, S_M$ and ψ is the unique set O such that

$$O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^V(S_i, v)\}. \quad (3)$$

Moreover, O_i^{IRFC} is also the unique set O that

$$O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^O(S_i, v)\}. \quad (4)$$

Corollaries on RFC and IRFC

Since $\psi^V(S_j, v) \geq \psi^{V \setminus O}(S_j, v)$, (1) and (3) immediately imply

Corollary (containment, new only for non-symmetric ψ)

$O_i^{\mathrm{RFC}} \subseteq O_i^{\mathrm{IRFC}}$ *for all* i.

Also (2) and (4) immediately imply

Corollary (connectedness, new only for non-symmetric ψ)

Any v *belonging to* $O_i \in \{O_i^{\mathrm{RFC}}, O_i^{\mathrm{IRFC}}\}$ *is connected to the object's seed set via an internal path of strength* $\psi^{O_i}(S_i, v) > 0$.

Mathematical Characterizations of IRFC and MOFS

Theorem (IRFC, repetition)

The IRFC object O_i^{IRFC} given by $S_1, \ldots, S_M$ and ψ is the unique set O such that

$$O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^{V}(S_i, v)\} \text{ and} \quad (5)$$

$$O = \{v \in V \mid \max_{j \neq i} \psi^{V \setminus O}(S_j, v) < \psi^{O}(S_i, v)\}. \quad (6)$$

Theorem (MOFS, new only part (8))

Assuming S_i's are consistent with ψ_i's, the sequence $\langle O_1^{\mathrm{MOFS}}, \ldots, O_M^{\mathrm{MOFS}} \rangle$ given by S_i's and ψ_i's is the unique sequence of sets $\langle O_1, \ldots, O_M \rangle$ such that

$$O_i = \{v \in V \mid \max_{j \neq i} \psi_j^{O_j}(S_j, v) \leq \psi_i^{O_i}(S_i, v) \neq 0\} \text{ for all } i. \quad (7)$$

Moreover, if $\psi_1 = \cdots = \psi_M = \psi$, then

$$O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}} \quad \text{for all } i. \quad (8)$$

Corollaries on IRFC and MOFS

(8) immediately implies

Corollary (containment, new only for non-symmetric ψ)

If $\psi_1 = \cdots = \psi_M = \psi$, then $O_i^{\mathrm{IRFC}} \subseteq O_i^{\mathrm{MOFS}}$ for all i.

Also (7) immediately implies

Corollary (connectedness, not new at all)

Any v belonging to O_i^{MOFS} is connected to the object's seed set via an internal path of strength $\psi^{O_i^{\mathrm{MOFS}}}(S_i, v) > 0$.

Outline

Robustness of MOFS on increasing sets of seeds

Let $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$ be a sequence of affinities;

$\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ — a sequence of seeds consistent with Ψ;

$O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ for the *i*th MOFS object given by Ψ and $\mathcal{S}$;

$\mathrm{Core}_i^{\Psi,\mathcal{S}} \subseteq O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ — a "big" set to be described latter.

Theorem (Robustness of MOFS on increasing sets of seeds)

Let $\mathcal{R} = \langle R_1, \ldots, R_M \rangle$ *be such that* $S_i \subseteq R_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$ *for* $1 \le i \le M$. *Then the sequence* $\mathcal{R}$ *is consistent with the affinities and* $O_i^{\mathrm{MOFS}}(\Psi, \mathcal{R}) = O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ *for* $1 \le i \le M$.

Robustness of MOFS objects on seeds choice

$\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ — a sequence of seeds consistent with Ψ;

$O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ for the ith MOFS object given by Ψ and $\mathcal{S}$;

Theorem (Robustness on increasing sets of seeds, repetition)

Let $\mathcal{R} = \langle R_1, \ldots, R_M \rangle$ be such that $S_i \subseteq R_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$ for $1 \le i \le M$. Then the sequence $\mathcal{R}$ is consistent with the affinities and $O_i^{\mathrm{MOFS}}(\Psi, \mathcal{R}) = O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ for $1 \le i \le M$.

Corollary (Robustness of MOFS objects on seeds choice)

Let $\mathcal{S}^* = \langle S_1^*, \ldots, S_M^* \rangle$ be any sequence of nonempty sets such that $S_i^* \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}}$ and $S_i \subseteq \mathrm{Core}_i^{\Psi,\mathcal{S}^*}$ for $1 \le i \le M$. Then we have that $O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S}^*) = O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$ for $1 \le i \le M$.

How big $O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})$ is?

Theorem (On the size of $O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})$)

The set $Q_i^{\Psi,\mathcal{S}}$ of all $v \in O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S}) \setminus \bigcup_{j\neq i} O_j^{\mathrm{MOFS}}(\Psi,\mathcal{S})$ such that $\psi_i^{O_i^{\mathrm{MOFS}}(\Psi,\mathcal{S})}(S_i, v) \geq \psi_i^V(v, \bigcup_{j\neq i} O_j^{\mathrm{MOFS}}(\Psi,\mathcal{S}))$ is contained in $\mathrm{Core}_i^{\Psi,\mathcal{S}}$.

For symmetric ψ_i, $\mathrm{Core}_i^{\Psi,\mathcal{S}} = Q_i^{\Psi,\mathcal{S}}$.

If all the affinities ψ_i are equal to the same symmetric affinity ψ, then $\mathrm{Core}_i^{\Psi,\mathcal{S}} = O_i^{\mathrm{IRFC}}(\psi,\mathcal{S})$.

This fact and previous theorem imply immediately

robustness results for IRFC segmentation!

What is missing in the robustness results

- We do not have any experimental studies on the size of $O_i^{\mathrm{MOFS}}(\Psi, \mathcal{S})$.
- This would be of interest even only in the case of a single non-symmetric affinity.

Outline

Efficient algorithm: Initialization

Algorithm 4: Finds O_i^{MOFS} and $\psi_i^{O_i^{\text{MOFS}}}(S_i, v)$ for each $v \in O_i^{\text{MOFS}}$

Data: seeds $\mathcal{S} = \langle S_1, \ldots, S_M \rangle$ and affinities $\Psi = \langle \psi_1, \ldots, \psi_M \rangle$

Result: an array $\sigma[\,]$, Boolean arrays $\chi^1[\,], \ldots, \chi^M[\,]$ such that,

$\sigma[v] =$ the approximation of $\psi_i^{O_i^{\text{MOFS}}}(S_i, v)$

$\chi^i[v] =$ **true** if v can belong to O_i^{MOFS}; **false** otherwise

```
foreach v ∈ V do                      /* initialization loop 1 */
    σ[v] ← 0
    for i ← 1 to M do χ^i[v] ← false
for i ← 1 to M do                     /* initialization loop 2 */
    foreach s ∈ S_i do
        σ[s] ← 1
        χ^i[s] ← true
H ← V
```

Efficient algorithm: Main Loop

while $\mathsf{H} \neq \emptyset$ **do** /* the main loop */
  remove an element w of $\arg\max_{u \in \mathsf{H}} \sigma[u]$ **from** H
  foreach x *such that* (w, x) *is a* Ψ*-edge* **do**
    foreach $\mathsf{i} \in \{1, \ldots, M\}$ *such that* $\chi^{\mathsf{i}}[\mathsf{w}] =$ **true do**
      $\sigma' \leftarrow \min(\sigma[\mathsf{w}], \psi_{\mathsf{i}}(\mathsf{w}, \mathsf{x}))$
      if $\sigma' > \sigma[\mathsf{x}]$ **then**
        $\sigma[\mathsf{x}] \leftarrow \sigma'$
        for $\mathsf{j} \leftarrow 1$ **to** M **do** $\chi^{\mathsf{j}}[\mathsf{x}] \leftarrow$ **false**
        $\chi^{\mathsf{i}}[\mathsf{x}] \leftarrow$ **true**
      else if $\sigma' = \sigma[\mathsf{x}]$ **and** $\sigma' > 0$ **and** $\chi^{\mathsf{i}}[\mathsf{x}] =$ **false then**
        $\chi^{\mathsf{i}}[\mathsf{x}] \leftarrow$ **true**
        if $\mathsf{x} \notin \mathsf{H}$ **then** $\mathsf{H} \leftarrow \mathsf{H} \cup \{\mathsf{x}\}$

Correctness and Efficiency of Algorithm 4

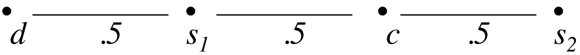
Theorem

- *Algorithm 4 indeed finds* O_i^{MOFS} *and* $\psi_i^{O_i^{\mathrm{MOFS}}}(S_i, v)$ *for each* i *and* $v \in O_i^{\mathrm{MOFS}}$.
- *In general, its running time* $O(|V| \log |V|)$.
- *Moreover, if all values of* ψ_i*'s are multiples of* $1/N$ *for an integer* N *that is* $O(|V|)$*, then we can use an array of doubly-linked lists instead of a heap to represent* H*, in which case the running time of Algorithm 4 will be* $O(|V|)$ *for any given value of* M.

Notice, that this last estimate is better than for the best IRFC algorithm existing up to this point, as the previously best algorithm required $O(M|V|)$ (or $O(M|V| \log |V|)$) operations for comparable task.

Outline

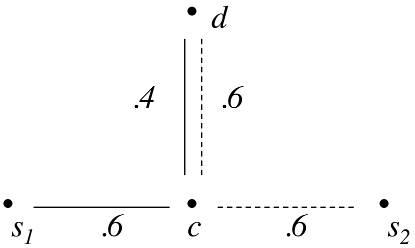
$O_i^{\mathrm{RFC}} \subsetneq O_i^{\mathrm{IRFC}} \subsetneq O_i^{\mathrm{MOFS}}$, single symmetric ψ



Example

With V and the symmetric affinity ψ as in Figure and $M = 2$, $O_1^{\mathrm{RFC}} = \{s_1\} \subsetneq O_1^{\mathrm{IRFC}} = \{s_1, d\} \subsetneq O_1^{\mathrm{MOFS}} = \{s_1, c, d\}$ and $O_2^{\mathrm{RFC}} = O_2^{\mathrm{IRFC}} = \{s_2\} \subsetneq O_2^{\mathrm{MOFS}} = \{s_2, c\}$.

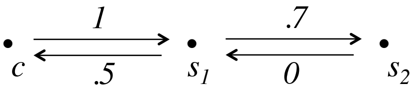
$O_i^{\mathrm{IRFC}} = O_i^{\mathrm{MOFS}} \setminus \bigcup_{j \neq i} O_j^{\mathrm{MOFS}}$ makes no sense for different symmetric affinities ψ_i



Example

With V and the symmetric affinities ψ_i as in Figure, $M = 2$, we get $O_1^{\mathrm{MOFS}} = \{s_1, c\}$ and $O_2^{\mathrm{MOFS}} = \{s_2, c, d\}$. So, object $O_2^{\mathrm{IRFC}} = O_2^{\mathrm{MOFS}} \setminus O_1^{\mathrm{MOFS}} = \{s_2, d\}$ is "disconnected."

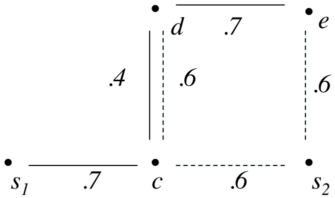
$\mathrm{Core}_i^{\Psi,\mathcal{S}} \neq Q_i^{\Psi,\mathcal{S}}$ for single non-symmetric affinity



Example

With V and a single non-symmetric affinity as in Figure, $M = 2$. Then $\mathrm{Core}_1^{\Psi,\mathcal{S}} = \{s_1, c\} = O_1^{\mathrm{MOFS}}$. But $c \notin Q_1^{\Psi,\mathcal{S}}$ because the ψ-strength of a ψ-strongest V-path from c to S_2 is 0.7, which exceeds the ψ-strength of a ψ-strongest V-path from S_1 to c.

$Q_i^{\Psi,\mathcal{S}} \neq O_i^{\rm MOFS}(\Psi,\mathcal{S}) \setminus \bigcup_{j\neq i} O_j^{\rm MOFS}(\Psi,\mathcal{S})$ for different symmetric affinities ψ_i



Example

With V and the symmetric affinities ψ_i as in Figure, $M=2$.
Then $O_1^{\rm MOFS}(\Psi,\mathcal{S})=\{s_1,c,d\}$ and $O_2^{\rm MOFS}(\Psi,\mathcal{S})=\{s_2,e\}$, so
$O_1^{\rm MOFS}(\Psi,\mathcal{S}) \setminus \bigcup_{j\neq 1} O_j^{\rm MOFS}(\Psi,\mathcal{S}) = O_1^{\rm MOFS}(\Psi,\mathcal{S})=\{s_1,c,d\}$.
But the set $\mathrm{Core}_1^{\Psi,\mathcal{S}} = Q_1^{\Psi,\mathcal{S}} = \{s_1,c\}$ is smaller.
For $R_1 = O_1^{\rm MOFS}(\Psi,\mathcal{S}) \not\subset \mathrm{Core}_1^{\Psi,\mathcal{S}}$ and $R_2=S_2$ no robustness:
$O_1^{\rm MOFS}(\Psi,\langle R_1,R_2\rangle)=\{s_1,c,d,e\}\neq O_1^{\rm MOFS}(\Psi,\mathcal{S})$.

Outline

Summary

- We described a general theory of FC segmentations that elegantly encompasses RFC, IRFC, and MOFC tracks.
- We showed how to use MOFS segmentations to immediately find IRFC segmentations in case where we have single symmetric affinity.
- We proposed extension of IRFC segmentation theory to the cases when we work with
  - a single non-symmetric affinity;
  - different affinities with essentially disjoint ranges $\psi_i[V \times V]$.
- We presented an algorithm for finding MOSF (so, IRFC) segmentations more efficient than older IRFC algorithms.
- We extended seeds placement robustness results for IRFC general case of MOFS segmentations.

Thank you for your attention!