Separate continuity and its generalizations history, recent progress, and open problems

Krzysztof Chris Ciesielski

Department of Mathematics, West Virginia University
and
MIPG, Departmentof Radiology, University of Pennsylvania

Dept. of Mathematics, Warsaw University, Poland
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Outline

1. Separate and linear continuity – prehistory
2. Discontinuity sets of separately/linearly continuous functions
3. Functions with continuous restrictions to k-flats
4. $\mathcal{F}$-continuity, allowing curvy surfaces in $\mathcal{F}$
5. When $\mathcal{F}$-continuity implies continuity?
6. Some proofs
7. Summary

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Basic definitions; separate and linear continuities

We consider mainly functions $f\colon \mathbb{R}^n \to \mathbb{R}$, $n = 2, 3, 4, \ldots$ fixed.

For a fixed collection $\mathcal{F}$ of subsets of $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$

- f is $\mathcal{F}$-continuous iff $f \restriction F$ is continuous for every $F \in \mathcal{F}$

For $k \le n$, $\mathcal{F}_{k,n}$: all k-dimensional flats (affine subspaces) of $\mathbb{R}^n$

$\mathcal{F}^+_{k,n}$: all $F \in \mathcal{F}_{k,n}$ parallel to spaces spanned by coordinate vectors

- f is separately continuous iff it is $\mathcal{F}^+_{1,n}$-continuous
- f is linearly continuous iff it is $\mathcal{F}_{1,n}$-continuous

Continuity vs $\mathcal{F}$-continuity: prehistory (for $n = 2$)

Cauchy, in 1821 book *Cours d'analyse*, incorrectly claimed:

separate continuity implies continuity!

Counterexamples:

- J. Thomae calculus text 1870 (and 1873), due to E. Heine:

  $F(x,y) = \sin\left(4\arctan\left(\frac{y}{x}\right)\right)$ for $\langle x,y\rangle \neq \langle 0,0\rangle$, $F(0,0) = 0$.

- 1884 treatise on calculus by Genocchi and Peano:

  $P(x,y) = \frac{xy^2}{x^2+y^4}$ for $\langle x,y\rangle \neq \langle 0,0\rangle$, $P(0,0) = 0$.

Baire classification of separate continuous functions

Theorem ([Baire 1899] for $n = 1$, [Lebesgue 1905] for all n)

Every separately continuous function on $\mathbb{R}^n$ is Baire class $n-1$, but need not be of lower Baire class, as

- *for every Baire class $n-1$ function $g\colon [0,1] \to \mathbb{R}$ there is a separately continuous function F on $\mathbb{R}^n$ such that*

$$F(x, \ldots, x) = g(x) \text{ for all } x \in [0,1].$$

Corollary

Every linearly continuous function on $\mathbb{R}^n$ is Baire class $n-1$

Question (I believe open and very interesting)

Is the Baire class the best in the Corollary above?

Nothing is known for $n \geq 3$.

Baire classification of linearly continuous functions?

Corollary

Every linearly continuous function on $\mathbb{R}^n$ is Baire class $n-1$

Question (I believe open and very interesting)

Is the Baire class the best in the Corollary above?

Theorem (KC, very partial answer, preliminary work)

For every Baire class 1 function $g\colon [0,1] \to \mathbb{R}$ there is a linearly continuous function F on $\mathbb{R}^2$ such that

$$F(x, x^2) = g(x) \text{ for all } x \in [0,1].$$

Outline

Sets of discontinuity points for $\mathcal{F}$-continuous functions

$D(f)$ denotes the set of points of discontinuity of f

$\mathcal{D}(\mathcal{F}) = \{D(f) \colon f \text{ is } \mathcal{F}\text{-continuous}\}$

Theorem (Kershner 1943, characterization of $\mathcal{D}(\mathcal{F}^{+}_{1,n})$)

For any set $D \subset \mathbb{R}^n$

- *$D = D(f)$ for some separately continuous f on $\mathbb{R}^n$ iff*
- *D is an F_σ set and every orthogonal projection of D onto a coordinate hyperplane has first category image.*

Question (Kronrod 1944, still not fully answered)

Find a characterization of $\mathcal{D}(\mathcal{F}_{1,n})$ (similar to that of Kershner) that is, of sets $D(f)$ for linearly continuous functions f.

On sets $D(f)$ for linearly continuous functions f

Theorem (Slobodnik 1976: upper bound for $\mathcal{D}(\mathcal{F}_{1,n})$)

If $D \subset \mathbb{R}^n$ is the set of discontinuity points of some linearly continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, then

$$D = \bigcup_{i<\omega} D_i,$$

where each D_i is isometric to the graph of a Lipschitz function $\phi_i\colon K_i \to \mathbb{R}$ with K_i being compact nowhere dense in $\mathbb{R}^{n-1}$.

In particular, such D must have Hausdorff dimension $\leq n-1$,

while there is a separately continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ having positive Lebesgue (so, n-Hausdorff) measure.

New results on sets $D(f)$ for linearly continuous f

Theorem (KC and T. Glatzer: lower bound for $\mathcal{D}(\mathcal{F}_{1,n})$)

If D is a restriction of a convex $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ to a compact nowhere dense subset of $\mathbb{R}^{n-1}$, then $D = D(f)$ for a linearly continuous $f\colon \mathbb{R}^n \to \mathbb{R}$.

For $n = 2$ the results remains true when ϕ is $\mathcal{C}^2$ (continuously twice differentiable).

In particular, D may have positive $(n-1)$-Hausdorff measure.

Note a gap between classes of convex and Lipschitz functions

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Can $\mathcal{F}_{k,n}$-continuity imply continuity?

Recall: $\mathcal{F}_{k,n}$ – all k-dimensional flats (affine subspaces) of $\mathbb{R}^n$

$P(x,y) = \frac{xy^2}{x^2+y^4}$ is discontinuous and $\mathcal{F}_{1,n}$-continuous.

Theorem (KC, submitted)

$$f_n(\vec{x}) = \frac{x_0(x_0)^{4^0}(x_1)^{4^1}\cdots(x_{n-1})^{4^{n-1}}}{(x_0)^{2^n}+(x_1)^{2^{n+1}}+\cdots+(x_{n-1})^{2^{n+(n-1)}}} = \frac{x_0\prod_{i=0}^{n-1}(x_i)^{2^{2i}}}{\sum_{i=0}^{n-1}(x_i)^{2^{n+i}}}$$

for $\vec{x} = \langle x_0, x_1, \ldots, x_{n-1}\rangle \neq \theta$, $f_n(\theta) = 0$, is $\mathcal{F}_{n-1,n}$-continuous but not continuous (on a path $\vec{p}(t) = \langle t^{2^n}, t^{2^{n-1}}, \ldots, t^{2^2}, t^{2^1}\rangle$).

- $f_2(x_0,x_1) = \frac{(x_0)(x_0)(x_1)^4}{(x_0)^4+(x_1)^8} = P((x_0)^2,(x_1)^2)$
- $f_3(x_0,x_1,x_2) = \frac{(x_0)(x_0)(x_1)^4(x_2)^{16}}{(x_0)^8+(x_1)^{16}+(x_2)^{32}}$, etc

Can $\mathcal{F}_{k,n}$-continuity imply continuity?

Here $\mathcal{F}_k$ denotes $\mathcal{F}_{k,n}$ and $\mathcal{F}_k^+$ denotes $\mathcal{F}_{k,n}^+$

$\mathcal{F}_n^+$- and $\mathcal{F}_n$-continuities are the standard continuity

Every function is $\mathcal{F}_0^+$- and $\mathcal{F}_0$-continuous

Theorem (KC and T. Glatzer, to appear)

For every $n \geq 2$,

$$\begin{array}{ccccccc}
\mathcal{F}_n\text{-cont} & \Longrightarrow & \mathcal{F}_{n-1}\text{-cont} & \Longrightarrow & \cdots & \Longrightarrow & \mathcal{F}_1\text{-cont} \\
\Updownarrow & & \Downarrow & & & & \Downarrow \\
\mathcal{F}_n^+\text{-cont} & \Longrightarrow & \mathcal{F}_{n-1}^+\text{-cont} & \Longrightarrow & \cdots & \Longrightarrow & \mathcal{F}_1^+\text{-cont}
\end{array}$$

None of the implications can be reversed.

On the families $\mathcal{D}^+_{k,n} = \mathcal{D}(\mathcal{F}^+_{k,n})$, $\mathcal{F}^+_{k,n}$ – right k-flats

Theorem (KC and T. Glatzer, to appear)

For any $k < n$, $D \in \mathcal{D}^+_{k,n}$ iff D is an F_σ-set whose orthogonal projection $\pi_F[D]$ on any $(n-k)$-flat $F \in \mathcal{F}^+_{n-k}$ is of first category.

Corollary (KC and T. Glatzer)

$P^k \times \mathbb{R}^{n-k} \in \mathcal{D}^+_{k-1,n} \setminus \mathcal{D}^+_{k,n}$ for any nowhere dense perfect $P \subset \mathbb{R}$. In particular, these sets can have positive n-dimensional Lebesgue measure.

$$\{\emptyset\} = \mathcal{D}^+_{n,n} \subsetneq \mathcal{D}^+_{n-1,n} \subsetneq \cdots \subsetneq \mathcal{D}^+_{1,n} \subsetneq \mathcal{D}^+_{0,n}$$

Corollary (KC and T. Glatzer)

If $D \in \mathcal{D}_{k,n}$, then $\pi_F[D]$ is of first category for any $F \in \mathcal{F}_{n-k}$.

On the families $\mathcal{D}_{k,n} = \mathcal{D}(\mathcal{F}_{k,n})$, $\mathcal{F}_{k,n}$ – all k-flats

Theorem (KC and T. Glatzer, to appear)

For any $0 < k < n$ and $D \in \mathcal{D}_{k,n}$ there exists a sequence $\langle f_i \rangle_{i<\omega}$ of Lipschitz functions f_i from $V_i \in \mathcal{F}_{n-k}$ into a perpendicular k-flat whose graphs cover D.
So, every $D \in \mathcal{D}_{k,n}$ has Hausdorff dimension $\leq n-k$.

Proposition (KC and T. Glatzer)

$\{0\}^k \times P \times \mathbb{R}^{n-k-1} \in \mathcal{D}_{k,n}$ for any compact nowhere dense $P \subset \mathbb{R}$. In particular,
$\mathcal{D}_{k,n}$ contains the sets of positive $(n-k)$-Hausdorff measure.

$$\begin{array}{ccccccccccc} \{\emptyset\} & = & \mathcal{D}_{n,n} & \subsetneq & \mathcal{D}_{n-1,n} & \subsetneq & \cdots & \subsetneq & \mathcal{D}_{1,n} & \subsetneq & \mathcal{D}_{0,n} \\ & & \| & & \cap & & & & \cap & & \| \\ & & \mathcal{D}^+_{n,n} & \subsetneq & \mathcal{D}^+_{n-1,n} & \subsetneq & \cdots & \subsetneq & \mathcal{D}^+_{1,n} & \subsetneq & \mathcal{D}^+_{0,n} \end{array}$$

Characterization of $\mathcal{D}_{k,n} = \mathcal{D}(\mathcal{F}_{k,n})$ for $k \geq n/2$

Definition (Topology on $\mathcal{F}_{k,n}$)

Generated by a subbase formed by the sets
$\mathcal{F}(U) = \{F \in \mathcal{F}_k \colon F \cap U \neq \emptyset\}$, where U is an open set in $\mathbb{R}^n$.

Definition (Ideal $\mathcal{J}_{k,n}$)

$\mathcal{J}_{k,n}$ – all bounded sets $S \subset \mathbb{R}^n$ s.t. there is an increasing sequence $\langle \mathcal{L}_i \colon i < \omega \rangle$ of closed subsets of $\mathcal{F}_k$ such that $\bigcup_{i<\omega} \mathcal{L}_i = \mathcal{F}_k$ and, for every $i < \omega$,
S is disjoint with the interior $\mathrm{int}(\bigcup \mathcal{L}_i)$ of the set $\bigcup \mathcal{L}_i \subset \mathbb{R}^n$.

Theorem (KC and T. Glatzer, to appear)

Let $0 < k < n$ be such that $k \geq \frac{n}{2}$. A set $D \subset \mathbb{R}^n$ is in $\mathcal{D}_{k,n}$ iff D is a countable union of compact sets from $\mathcal{J}_{k,n}$.

Outline

More history; $\mathcal{F}$ consisting of graphs of functions

Scheefer 1890, Lebesgue 1905: for $\mathcal{A}$ =analytic functions

$\mathcal{A}$-continuity (for $n = 2$) does not imply continuity.

Theorem ([Rosenthal 1955])

- *D^2-continuity (for $n = 2$) does not imply continuity; however*
- *$\mathcal{C}^1$-continuity is equivalent to continuity (for every n),*

where $\mathcal{C}^1$ and D^2 are, respectively, continuously and twice differentiable functions.

Here, functions are with respect of any of coordinate hyperplanes, e.g., from x to y and from y to x.

On sets $D(f)$ for D^2-continuous functions f

Remember (Rosenthal) that $\mathcal{C}^1$-continuity implies continuity.

Theorem (KC and T. Glatzer)

There exists a D^2-continuous $f: \mathbb{R}^2 \to \mathbb{R}$ for which $D(f)$ has positive one dimensional Hausdorff measure.

The example can be "lifted" to a D^2-continuous $f: \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ of positive $(n-1)$-Hausdorff measure.

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For which families $\mathcal{F} \subset \mathcal{P}(\mathbb{R}^2)$, $\mathcal{D}(\mathcal{F}) = \emptyset$?

- $\mathcal{D}(\text{all converging sequences}) = \emptyset$.

Luzin's 1948 text: If $f_h(x) = f(x, h(x))$ is continuous for every continuous h, then $f(x, y)$ is continuous. In particular,

- $\mathcal{D}(\mathcal{C}(\mathbb{R})) = \emptyset$ (only graphs from x to y!)

Theorem (KC and Joseph Rosenblatt, submitted)

- $\mathcal{D}(\text{“}C^1\text{”}) = \emptyset$ *(we allow infinite derivatives)*
- $\mathcal{D}(D^1) \neq \emptyset$ *(basically, example of KC and TG)*

$T(h)$-continuity, $T(h)$ translations of single h

Theorem (KC and Joseph Rosenblatt, submitted)

- $\mathcal{D}(T(h)) \neq \emptyset$ *for every continuous* $h\colon \mathbb{R}^n \to \mathbb{R}$
- $\mathcal{D}(T(h)) = \emptyset$ *for a Baire class 1 function* $h\colon \mathbb{R}^n \to \mathbb{R}$*;* *We can have* $D(h) = P^n$ *with* P *compact measure 0.*

Theorem (KC and Joseph Rosenblatt)

- $\mathcal{D}(T(X)) = \emptyset$ *for any Borel set* $X \subset \mathbb{R}^n$ *which is either of positive measure or of the second category*
- $\mathcal{D}(T(P^n)) = \emptyset$ *for a compact* $P \subset \mathbb{R}$ *of measure zero.*

$I(h)$-continuity, $I(h)$ all isometric copies of h

Theorem (KC and Joseph Rosenblatt, submitted)

- *$T(h)$-continuity does not imply $I(h)$-continuity*

  For $h\colon \mathbb{R} \to \mathbb{Q}$, $h(x) = 0$ for all $x \notin \mathbb{Q} \cap [0,1]$,

  $h \restriction \mathbb{Q} \cap [0,1]$ having a dense graph in $[0,1] \times \mathbb{R}$.

Question

- Does there exist a continuous $h\colon \mathbb{R} \to \mathbb{R}$ with $\mathcal{D}(I(h)) = \emptyset$?
- What can be said about the sets X with $\mathcal{D}(I(X)) = \emptyset$?

Outline

Separately continuous f on $\mathbb{R}^n$ is Baire class $\leq n-1$

Follows immediately, by induction on n, from the following result used with $X = \mathbb{R}^n$.

Theorem (From Z. Piotrowski's book, in prepatration)

Let X be Polish space and $f\colon X \times \mathbb{R} \to \mathbb{R}$ be such that

- *$f(x, \cdot)$ is continuous for every $x \in X$, and*
- *$f(\cdot, r)$ is Baire class $n-1$ for every $r \in \mathbb{R}$.*

For every $k = 0, 1, 2, \ldots$ let $f_k\colon X \times \mathbb{R} \to \mathbb{R}$ be the linear interpolation of $f \restriction X \times \{m/2^k\colon m \in \mathbb{Z}\}$,

i.e., $f_k(x, \cdot)$ is linear on $[m/2^k, (m+1)/2^k]$ for all x and m.

Then each f_k is of Baire class $n-1$ and $f_k \to_k f$ pointwise.

Separately continuous f on $\mathbb{R}^n$ is not Baire $n-2$

Theorem ([Lebesgue]; simplified proof of Maslyuchenko 1999)

For every Baire class $n-1$ function $g\colon [0,1] \to \mathbb{R}$ there is a separately continuous function f on $\mathbb{R}^n$ such that

$$f(x,\ldots,x) = g(x) \text{ for all } x \in [0,1].$$

By induction on n. Obvious for $n = 1$.

Notation, for $n \geq 1$:

$d_n\colon \mathbb{R} \to \mathbb{R}^n$, $d_n(x) = \langle x,\ldots,x\rangle$.

Notation, for $n > 1$ and $i \in \{1,\ldots,n\}$:

$q_i\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$, $q_i(x_1,\ldots,x_n) = \langle x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\rangle$.

Notation and inductive step

$d_n(x) = \langle x, \dots, x \rangle$, $q_i(x_1, \dots, x_n) = \langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \rangle$

Δ_n – diagonal in $\mathbb{R}^n$;

G_k's: open in $\mathbb{R}^n$ s.t. $\Delta_n = \bigcap_k G_k$ & $\mathrm{cl}(G_{k+1}) \subset G_k$

$\varphi_k \colon \mathbb{R}^n \to [0,1]$: $\{\varphi_k\}_k$ is a partition of unity of $\mathbb{R}^n \setminus \Delta_n$ with respect to the open cover $\{G_{k-1} \setminus \mathrm{cl}(G_{k+2})\}_k$:

$\varphi_k(x) = 0$ outside $G_{k-1} \setminus \mathrm{cl}(G_{k+2})$; $\sum_k \varphi_k(x) = 1$ outside Δ_n

$g_k \colon \mathbb{R} \to \mathbb{R}$ of Baire class $n-2$, $g_k \to_k g$.

By inductive assumption, for every k there exists a separately continuous function $f_k \colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that $g_k = f_k \circ d_{n-1}$.

Need separately cont. $f \colon \mathbb{R}^n \to \mathbb{R}$ s.t. $g = f \circ d_n$.

Inductive step

$d_n(x) = \langle x, \dots, x \rangle$, $q_i(x_1, \dots, x_n) = \langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \rangle$

$\varphi_k \colon \mathbb{R}^n \to [0,1]$: $\{\varphi_k\}_k$ is a partition of unity of $\mathbb{R}^n \setminus \Delta_n$

$g_k \colon \mathbb{R} \to \mathbb{R}$ of Baire class $n-2$, $g_k \to_k g$.

$f_k \colon \mathbb{R}^{n-1} \to \mathbb{R}$ separately continuous s.t. $g_k = f_k \circ d_{n-1}$.

For $i \in \{1, \dots, n\}$ define $h_i \colon \mathbb{R}^n \to \mathbb{R}$ as

$$h_i(x) = \begin{cases} \sum_k \varphi_k(x) f_k(q_i(x)) & \text{if } x \in \mathbb{R}^n \setminus \Delta_n \\ g(d_n^{-1}(x)) & \text{if } x \in \Delta_n. \end{cases}$$

Claim

h_i is separately continuous on $\mathbb{R}^n \setminus \Delta_n$;
h_i is continuous on Δ_n with respect to the ith coordinate

Proof modulo Claim, $n = 2$

$q_i(x_1, \dots, x_n) = \langle x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \rangle$

$$h_i(x) = \begin{cases} \sum_k \varphi_k(x) f_k(q_i(x)) & \text{if } x \in \mathbb{R}^n \setminus \Delta_n \\ g(d_n^{-1}(x)) & \text{if } x \in \Delta_n. \end{cases}$$

Claim

h_i is separately continuous on $\mathbb{R}^n \setminus \Delta_n$;
h_i is continuous on Δ_n with respect to the ith coordinate

Now, for $n = 2$, $f = h_1$ works, since
h_1 is also continuous with respect to the second coordinate, as

for every distinct $s, t \in \mathbb{R}$ we have
$h_1(s,t) = \sum_k \varphi_k(s,t) f_k(q_1(s,t)) = \sum_k \varphi_k(s,t) g_k(s)$
which, by Claim, converges to $g(s) = h_1(s,s)$ as $t \to s$.

Proof modulo Claim, $n > 2$

$$q_i(x_1, \ldots, x_n) = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle$$

Claim

h_i is separately continuous on $\mathbb{R}^n \setminus \Delta_n$;
h_i is continuous on Δ_n with respect to the ith coordinate

For $i \in \{1, \ldots, n\}$, let $D_i = q_i^{-1}(\Delta_{n-1})$.

Sets $D_i \setminus \Delta_n$ are pairwise disjoint and closed in $\mathbb{R}^n \setminus \Delta_n$.

By normality of $\mathbb{R}^n \setminus \Delta_n$, for every $i \in \{1, \ldots, n\}$ there is continuous $\psi_i \colon \mathbb{R}^n \setminus \Delta \to [0, 1]$ such that $\psi_i(x) = 1$ for $x \in D_i \setminus \Delta_n$, and $\psi_i(x) = 0$ for $x \in D_j \setminus \Delta_n$, $j \neq i$.

Proof modulo Claim, $n > 2$, continuation

$$q_i(x_1, \ldots, x_n) = \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle$$

Claim

h_i is separately continuous on $\mathbb{R}^n \setminus \Delta_n$;
h_i is continuous on Δ_n with respect to the ith coordinate

$D_i = q_i^{-1}(\Delta_{n-1})$; $\psi_i \colon \mathbb{R}^n \setminus \Delta \to [0, 1]$ s.t. $\psi_i(x) = 1$ for $x \in D_i \setminus \Delta_n$, and $\psi_i(x) = 0$ for $x \in D_j \setminus \Delta_n$, $j \neq i$.

Define $f \colon \mathbb{R}^n \to \mathbb{R}$ as

$$f(x) = \begin{cases} \sum_{i=1}^n \psi_i(x) h_i(x) & \text{if } x \in \mathbb{R}^n \setminus \Delta_n \\ g(d_n^{-1}(x)) & \text{if } x \in \Delta_n. \end{cases}$$

f works, as: $g = f \circ d_n$, f is separately continuous on $\mathbb{R}^n \setminus \Delta_n$, and f is separately continuous on Δ_n with respect to the ith variable, since $f \restriction D_i = h_i \restriction D_i$.

Proof of Claim

$d_n(x) = \langle x, \ldots, x \rangle$, $\{\varphi_k\}_k$ is a partition of unity of $\mathbb{R}^n \setminus \Delta_n$
$g_k \to_k g$; $f_k : \mathbb{R}^{n-1} \to \mathbb{R}$ separately cont. & $g_k = f_k \circ d_{n-1}$.

$$h_i(x) = \begin{cases} \sum_k \varphi_k(x) f_k(q_i(x)) & \text{if } x \in \mathbb{R}^n \setminus \Delta_n \\ g(d_n^{-1}(x)) & \text{if } x \in \Delta_n. \end{cases}$$

Claim

h_i is separately continuous on $\mathbb{R}^n \setminus \Delta_n$ – obvious
h_i is continuous on Δ_n with respect to the ith coordinate

Fix $t \in \mathbb{R}$. For $x \in \mathbb{R}^n \setminus \Delta_n$ with $q_i(x) = q_i(d_n(t))$:
$|h_i(x) - h_i(d_n(t))| = |\sum_k \varphi_k(x) f_k(q_i(x)) - g(t)| =$
$|\sum_k \varphi_k(x) g_k(t) - g(t)| = |\sum_k \varphi_k(x) g_k(t) - \sum_k \varphi_k(x) g(t)| \leq$
$\sum_k \varphi_k(x) |g_k(t) - g(t)| \leq \sum_k \varphi_k(x) \varepsilon = \varepsilon$ whenever x is so close to $d_n(t)$ that $\varphi_k(x) = 0$ for every $k < m$, where m is such that $|g_k(t) - g(t)| \leq \varepsilon$ for every $k \geq m$.

Outline

Summary of new results

- Big progress on characterization of sets of points of discontinuity of linearly continuous function
- Deep study of functions on $\mathbb{R}^n$ continuous when restricted to k-dimensional affine spaces
- Construction of D^2-continuous functions f with large set of points of discontinuity
- Discussion a theorem of Luzin
- Discussion of when $T(h)$-continuity implies continuity, for h being a graph of function

Thank you for your attention!