Differentiability on perfect subsets P of $\mathbb{R}$; Smooth Peano functions from P onto P^2

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My work with Irek

RESEARCH

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CARDINAL INVARIANTS CONCERNING EXTENDABLE AND PERIPHERALLY CONTINUOUS FUNCTIONS

Abstract

Let $\mathcal{F}$ be a family of real functions, $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$. In the paper we will examine the following question. For which families $F \subseteq \mathbb{R}^\mathbb{R}$ does there exist $g \colon \mathbb{R} \to \mathbb{R}$ such that $f + g \in \mathcal{F}$ for all $f \in F$? More precisely, we will study a cardinal function $\mathrm{A}(\mathcal{F})$ defined as the smallest cardinality of a family $F \subseteq \mathbb{R}^\mathbb{R}$ for which there is no such g. We will prove that $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$ and $\mathrm{A}(\mathrm{PC}) = 2^\mathfrak{c}$, where Ext, PR and PC stand for the classes of extendable functions, functions with perfect road and peripherally continuous functions from $\mathbb{R}$ into $\mathbb{R}$, respectively. In particular, the equation $\mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+$ immediately implies that every real function is a sum of two extendable functions. This solves a problem of Gibson [6].

We miss you, Irek!

Outline

1. Differentiability on perfect subsets P of the real line
   Examples of results

2. Peano maps from perfect $P \subset \mathbb{R}$ onto P^2

3. Smooth Peano maps from compact $P \subset \mathbb{R}$ onto P^2

4. $\mathcal{C}^\infty$ Peano maps for unbounded perfect $P \subset \mathbb{R}$

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Scope: systematic study of partial differentiable maps

Derivative of f from perfect $P \subset \mathbb{R}$ into $\mathbb{R}$ is well defined

Still, theory behind is unpopular and/or underdeveloped

Example 1: Take

Theorem (Tietze Extension Thm)

For every closed subset X of $\mathbb{R}$ and $f : X \to \mathbb{R}$ with $f \in \mathcal{C}$ there is an $F : \mathbb{R} \to \mathbb{R}$ extending f such that $F \in \mathcal{C}$.

How well known is the answer for the following questions? (Related to Whitney extension theorem.)

Can, in the above, the class $\mathcal{C}$ of continuous functions be replaced with the class D^1 of differentiable functions?

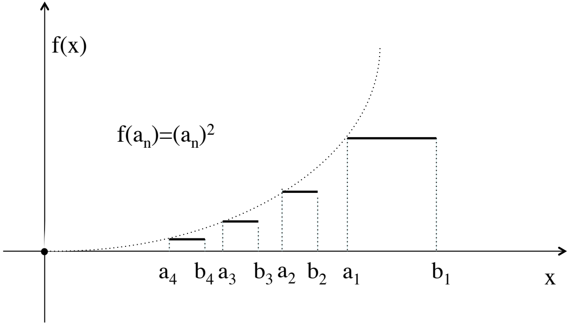
What about the class $\mathcal{C}^1$ of continuously differentiable functions?

Tietze Extension Theorem D^1 functions

Theorem ([Jarník 1923], also [Petruska, Laczkovich, 1974])

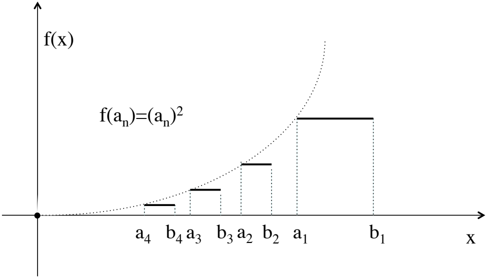
For every closed subset X of $\mathbb{R}$ and $f\colon X \to \mathbb{R}$ with $f \in D^1$ there is an $F\colon \mathbb{R} \to \mathbb{R}$ extending f such that $F \in D^1$.

No Tietze Extension Theorem for $\mathcal{C}^1$ functions:



$f'(x) = 0$ for all $x \in X = \{0\} \cup \bigcup_n [a_n, b_n]$

Calc 1 problem:



How to choose the intervals to insure there is no $\mathcal{C}^1$ extension?

1. Insure that $\lim_{n\to\infty} \frac{f(a_n)-f(b_{n+1})}{a_n-b_{n+1}} > 0$.
2. Apply Mean Value Theorem to notice that no D^1 extension of f can have continuous derivative at 0.

Example 2

$[\mathbb{R}]^{\mathfrak{c}}$ all $S \subset \mathbb{R}$ of cardinality continuum; $\mathcal{F} \subset \mathcal{C}$.

$Im^*(\mathcal{F})$: $\forall S \in [\mathbb{R}]^{\mathfrak{c}}\ \exists f \in \mathcal{F}$ such that $f[S] = [0,1]$.

$Im(\mathcal{F})$: $\forall S \in [\mathbb{R}]^{\mathfrak{c}}\ \exists f \in \mathcal{F}$ such that $f[S]$ contains a perfect set.

Clearly $Im(\mathcal{C}) \iff Im^*(\mathcal{C})$

Theorem ([A. Miller 1983])

It is consistent with ZFC that $Im^*(\mathcal{C})$ holds.
However, $Im^*(\mathcal{C})$ fails under the Continuum Hypothesis.
So, $Im(\mathcal{C})^*$ is independent from the ZFC axioms.

Theorem ([Ciesielski, Pawlikowski, 2003])

$Im(\mathcal{C})^*$ follows from the Covering Property Axiom CPA.

Example 2 continues

$Im(\mathcal{F})$: $\forall S \in [\mathbb{R}]^{\mathfrak{c}}$ $\exists f \in \mathcal{F}$ such that $f[S]$ contains a perfect set.

$Im^*(\mathcal{F})$: $\forall S \in [\mathbb{R}]^{\mathfrak{c}}$ $\exists f \in \mathcal{F}$ such that $f[S]$ contains $[0,1]$.

- $Im(\mathcal{C}) \iff Im^*(\mathcal{C})$
- $Im(\mathcal{C})$ and $Im^*(\mathcal{C})$ are independent from the ZFC axioms. Follows from CPA, contradicts CH.
- $Im^*(D^1)$ is false (Lusin's condition (N)).

What about $Im(D^1)$? $Im(\mathcal{C}^1)$? $Im(\mathcal{C}^\infty)$?

$Im(\mathcal{C}^\infty) \iff Im(\mathcal{C}) \iff Im^*(\mathcal{C})$

$Im(\mathcal{F})$: $\forall S \in [\mathbb{R}]^{\mathfrak{c}}\ \exists f \in \mathcal{F}$ such that $f[S]$ contains a perfect set.

Theorem ([Ciesielski, Nishura, 2012])

$Im(\mathcal{C}^\infty) \iff Im(\mathcal{C}) \iff Im^*(\mathcal{C})$; they are independent of ZFC.
$Im(\text{Analytic functions})$ is false.

Lemma (Key to the proof of the theorem)

For every continuous f from a closed $K \subset \mathbb{R}$ into a nowhere dense compact perfect $P \subset \mathbb{R}$ there exist:
a $\mathcal{C}^\infty$ function $g \colon \mathbb{R} \to \mathbb{R}$ and a homeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ such that $g \restriction K = h \circ f$.

$\forall$ cont $f \colon K \to P$ $\exists$ homeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ s.t. $h \circ f \in \mathcal{C}^\infty$

and $h \circ f$ can be extended to entire $\mathcal{C}^\infty$ function.

Outline

Which compact $P \subset \mathbb{R}$ can be mapped onto P^2

Theorem ([Ciesielski, Jasiński, to appear])

$P \subset \mathbb{R}$ – compact; κ=# of connected components in P.
$\exists$ a $\mathcal{C}^0$ Peano function $f\colon P \to P^2$ iff either $\kappa = 1$ or $\kappa = \mathfrak{c}$.

Proof: "$\Longleftarrow$" – easy; ($\kappa = 1$ – classic result of Peano)

"$\Longrightarrow$" – induction on *Cantor-Bendixon rank* $|X|_{CB}$; based on

Lemma

$|f[P]|_{CB} \leq |P|_{CB}$
for every countable compact $P \subset \mathbb{R}$ and continuous f.

Open problems

Question

- Characterize unbounded closed sets $P \subset \mathbb{R}$ admitting continuous f from P onto P^2
- Similarly, for arbitrary $P \subset \mathbb{R}$

Question (classic)

- Does there exist continuous f from $[0,1]$ onto $[0,1]^2$ with $f[a,b]$ convex for all $a \leq b$?

Known [J. Pach, C.A. Rogers 1983]:

$\exists f \in \mathcal{C}$ from $[0,1]$ onto $[0,1]^2$ s.t. $f[0,c]$ and $f[c,1]$ convex for all c

True Peano Curve?

Remarkable Portraits Made with a Single Sewing Thread Wrapped through Nails, by Kumi Yamashita

www.thisiscolossal.com/2012/06/

Outline

No $\mathcal{C}^1$ $f\colon \mathbb{R} \to \mathbb{R}$ with $f[P] = P^2$ for compact P

Remark (Morayne 1987, using Banach condition (T_2))

There is no D^1 function mapping P onto P^2 for $P = [0,1]$.

Similarly, for any P of positive Lebesgue measure.

Theorem ([Ciesielski, Jasiński, to appear])

*$P^2 \not\subset f[P]$ for **any** perfect compact $P \subset \mathbb{R}$ and $\mathcal{C}^1$ map $f\colon \mathbb{R} \to \mathbb{R}^2$*

Proof based on

Lemma

If $g\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$ and $P \subset \mathbb{R}$ is a compact perfect s.t. $P \subset g[P]$, then there exists an $x \in P$ with $|g'(x)| \geq 1$.

Not obvious: no Intermediate Value Theorem for $g \restriction P$.

The lemma and open problems

Lemma

If $g\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$ and $P \subset \mathbb{R}$ is a compact perfect s.t. $P \subset g[P]$, then there exists an $x \in P$ with $|g'(x)| \geq 1$.

Question

Does the lemma hold when

- only $g \restriction P$ is $\mathcal{C}^1$? (No extension thm for $\mathcal{C}^1$ functions!)
- g is D^1? (Same as $g \restriction P \in D^1$, by Jarník's thm.)

We do not even have a proof of the following

Conjecture

If $P \subset \mathbb{R}$ is a compact perfect and $g \restriction P \equiv 0$, then $P \not\subset g[P]$.

The theorem and open problems

Theorem ([Ciesielski, Jasiński, to appear])

$P^2 \neq f[P]$ for any compact $P \subset \mathbb{R}$ and $\mathcal{C}^1$ map $f\colon \mathbb{R} \to \mathbb{R}^2$.

Question

Does the theorem hold when

- only $f \restriction P$ is $\mathcal{C}^1$? (No extension thm for $\mathcal{C}^1$ functions!)
- f is D^1? (Same as $f \restriction P \in D^1$, by Jarník's thm.)

If the answer to any of these is negative,

How much smoothness a function from P onto P^2 may have?

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Can smooth Peano functions exist at all?

Theorem ([Ciesielski, Jasiński, to appear])

There exists a $\mathcal{C}^\infty$ function $f\colon \mathbb{R} \to \mathbb{R}^2$ and a perfect unbounded subset P of $\mathbb{R}$ such that $f[P] = P^2$.

An idea behind the proof

- P is a union of perfect sets $P_k \subseteq [3k, 3k+2]$, $k < \omega$, s.t.
- P_k can be mapped smoothly onto $P_\ell \times P_{\ell'}$ for any $\ell, \ell' < k$;
- the maps must be extendable to smooth entire functions.
- Then, a diagonal construction gives a desired f for such P.

Difficulty in constructing desired P_k's

- P_k can be mapped smoothly onto $P_\ell \times P_{\ell'}$ for any $\ell, \ell' < k$;
- the maps must be extendable to smooth entire functions.

Need a condition to insure extendability. It is given by

Lemma

Let $K \subset \mathbb{R}$ be compact nowhere dense and $g_0 \colon K \subset \mathbb{R}$ be s.t. for every $k < \omega$ there exists a $\delta_k \in (0,1)$ s.t. for all $x, y \in K$

- $|g_0(x) - g_0(y)| < |x-y|^{k+1}$ *provided* $0 < |x-y| < \delta_k$

Then g_0 can be extended to a $\mathcal{C}^\infty$ function $g \colon \mathbb{R} \to \mathbb{R}$.

More difficulties in constructing desired P_k's

- P_k can be mapped smoothly onto $P_\ell \times P_{\ell'}$ for any $\ell, \ell' < k$;

The standard h from 2^ω onto $(2^\omega)^2$ is $h = \langle h^{\mathrm{odd}}, h^{\mathrm{even}} \rangle$,

$h^{\mathrm{odd}}(s)(i) = s(2i+1)$ and $h^{\mathrm{even}}(s)(i) = s(2i)$.

For 2^ω identified with $C = \left\{ \sum_{i<\omega} \frac{2s(i)}{3^{i+1}} \colon s \in 2^\omega \right\}$,

$$\limsup_{s \to t} \left| \frac{h^{\mathrm{odd}}(s) - h^{\mathrm{odd}}(t)}{s - t} \right| = \infty!$$

caused by ‘compression’ of coordinates.

Constructing desired P_k's

- P_k can be mapped smoothly onto $P_\ell \times P_{\ell'}$ for any $\ell, \ell' < k$;

To compensate for the compression,

each P_k is created by appropriate "thickening" P_{k-1};

"Thickening" cannot be radical: P_k must be of measure zero.

This balancing act is the key of the proof.

Thank you for your attention!