On functions with continuous restrictions to various sets

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Outline

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Basic definitions; separate and linear continuities

We consider mainly functions $f\colon \mathbb{R}^n \to \mathbb{R}$, $n = 2, 3, 4, \ldots$ fixed.

For a fixed collection $\mathcal{F}$ of subsets of $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$

- f is $\mathcal{F}$-continuous iff $f \restriction F$ is continuous for every $F \in \mathcal{F}$

For $k \leq n$, $\mathcal{F}_{k,n}$: all k-dimensional flats (affine subspaces) of $\mathbb{R}^n$

$\mathcal{F}^+_{k,n}$: all $F \in \mathcal{F}_{k,n}$ parallel to spaces spanned by coordinate vectors

- f is separately continuous iff it is $\mathcal{F}^+_{1,n}$-continuous
- f is linearly continuous iff it is $\mathcal{F}_{1,n}$-continuous

Continuity vs $\mathcal{F}$-continuity: prehistory (for $n = 2$)

Cauchy, in 1821 book *Cours d'analyse*, incorrectly claimed:

separate continuity implies continuity!

Counterexamples:

- J. Thomae calculus text 1870 (and 1873), due to E. Heine:
  $F(x, y) = \sin\left(4 \arctan\left(\frac{y}{x}\right)\right)$ for $\langle x, y\rangle \neq \langle 0, 0\rangle$, $F(0, 0) = 0$.
- 1884 treatise on calculus by Genocchi and Peano:
  $P(x, y) = \frac{xy^2}{x^2+y^4}$ for $\langle x, y\rangle \neq \langle 0, 0\rangle$, $P(0, 0) = 0$.

Baire classification of separate continuous functions

Theorem ([Baire 1899] for $n = 1$, [Lebesgue 1905] for all n)

Every separately continuous function on $\mathbb{R}^n$ is Baire class $n-1$, but need not be of lower Baire class, as

- *for every Baire class $n-1$ function $g\colon [0,1] \to \mathbb{R}$ there is a separately continuous function F on $\mathbb{R}^n$ such that*

$$F(x, \ldots, x) = g(x) \text{ for all } x \in [0,1].$$

Corollary

Every linearly continuous function on $\mathbb{R}^n$ is Baire class $n-1$

Question (I believe open and very interesting)

Is the Baire class the best in the Corollary above?

Nothing is known for $n \geq 3$.

Baire classification of linearly continuous functions?

Corollary

Every linearly continuous function on $\mathbb{R}^n$ is Baire class $n-1$

Question (I believe open and very interesting)

Is the Baire class the best in the Corollary above?

Theorem (KC, very partial answer, preliminary work)

For every Baire class 1 function $g\colon [0,1] \to \mathbb{R}$ there is a linearly continuous function F on $\mathbb{R}^2$ such that

$$F(x, x^2) = g(x) \text{ for all } x \in [0,1].$$

Outline

1. Separate and linear continuity – prehistory
2. Discontinuity sets of separately/linearly continuous functions
3. Functions with continuous restrictions to k-flats
4. $\mathcal{F}$-continuity, allowing curvy surfaces in $\mathcal{F}$
5. When $\mathcal{F}$-continuity implies continuity?
6. Summary

Sets of discontinuity points for $\mathcal{F}$-continuous functions

$D(f)$ denotes the set of points of discontinuity of f

$\mathcal{D}(\mathcal{F}) = \{D(f)\colon f \text{ is } \mathcal{F}\text{-continuous}\}$

Theorem (Kershner 1943, characterization of $\mathcal{D}(\mathcal{F}^+_{1,n})$)

For any set $D \subset \mathbb{R}^n$

- *$D = D(f)$ for some separately continuous f on $\mathbb{R}^n$ iff*
- *D is an F_σ set and every orthogonal projection of D onto a coordinate hyperplane has first category image.*

Question (Kronrod 1944, still not fully answered)

Find a characterization $\mathcal{D}(\mathcal{F}_{1,n})$ (similar to that of Kershner) that is, of sets $D(f)$ for linearly continuous functions f

On sets $D(f)$ for linearly continuous functions f

Theorem (Slobodnik 1976: upper bound for $\mathcal{D}(\mathcal{F}_{1,n})$)

If $D \subset \mathbb{R}^n$ is the set of discontinuity points of some linearly continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, then

$$D = \bigcup_{i<\omega} D_i,$$

where each D_i is isometric to the graph of a Lipschitz function $\phi_i\colon K_i \to \mathbb{R}$ with K_i being compact nowhere dense in $\mathbb{R}^{n-1}$.

In particular, such D must have Hausdorff dimension $\leq n-1$,

while there is a separately continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ having positive Lebesgue (so, n-Hausdorff) measure.

New results on sets $D(f)$ for linearly continuous f

Theorem (KC and T. Glatzer: lower bound for $\mathcal{D}(\mathcal{F}_{1,n})$)

If D is a restriction of a convex $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ to a compact nowhere dense subset of $\mathbb{R}^{n-1}$, then $D = D(f)$ for a linearly continuous $f\colon \mathbb{R}^n \to \mathbb{R}$.

For $n = 2$ the results remains true when ϕ is $\mathcal{C}^2$ (continuously twice differentiable).

In particular, D may have positive $(n-1)$-Hausdorff measure.

Note a gap between classes of convex and Lipschitz functions

Outline

Can $\mathcal{F}_{k,n}$-continuity imply continuity?

Recall: $\mathcal{F}_{k,n}$ – all k-dimensional flats (affine subspaces) of $\mathbb{R}^n$

$P(x,y) = \frac{xy^2}{x^2+y^4}$ is discontinuous and $\mathcal{F}_{1,n}$-continuous.

Theorem (KC, submitted)

$$f_n(\vec{x}) = \frac{x_0(x_0)^{4^0}(x_1)^{4^1}\cdots(x_{n-1})^{4^{n-1}}}{(x_0)^{2^n}+(x_1)^{2^{n+1}}+\cdots+(x_{n-1})^{2^{n+(n-1)}}} = \frac{x_0\prod_{i=0}^{n-1}(x_i)^{2^{2i}}}{\sum_{i=0}^{n-1}(x_i)^{2^{n+i}}}$$

for $\vec{x} = \langle x_0, x_1, \ldots, x_{n-1}\rangle \neq \theta$, $f_n(\theta) = 0$, is $\mathcal{F}_{n-1,n}$-continuous but not continuous (on a path $\vec{p}(t) = \langle t^{2^n}, t^{2^{n-1}}, \ldots, t^{2^2}, t^{2^1}\rangle$).

- $f_2(x_0,x_1) = \frac{(x_0)(x_0)(x_1)^4}{(x_0)^4+(x_1)^8} = P((x_0)^2,(x_1)^2)$
- $f_3(x_0,x_1,x_2) = \frac{(x_0)(x_0)(x_1)^4(x_2)^{16}}{(x_0)^8+(x_1)^{16}+(x_2)^{32}}$, etc

Can $\mathcal{F}_{k,n}$-continuity imply continuity?

Here $\mathcal{F}_k$ denotes $\mathcal{F}_{k,n}$ and $\mathcal{F}_k^+$ denotes $\mathcal{F}_{k,n}^+$

$\mathcal{F}_n^+$- and $\mathcal{F}_n$-continuities are the standard continuity

Every function is $\mathcal{F}_0^+$- and $\mathcal{F}_0$-continuous

Theorem (KC and T. Glatzer, submitted)

For every $n \geq 2$,

$$\begin{array}{ccccccc} \mathcal{F}_n\text{-cont} & \Longrightarrow & \mathcal{F}_{n-1}\text{-cont} & \Longrightarrow & \cdots & \Longrightarrow & \mathcal{F}_1\text{-cont} \\ \Updownarrow & & \Downarrow & & & & \Downarrow \\ \mathcal{F}_n^+\text{-cont} & \Longrightarrow & \mathcal{F}_{n-1}^+\text{-cont} & \Longrightarrow & \cdots & \Longrightarrow & \mathcal{F}_1^+\text{-cont} \end{array}$$

None of the implications can be reversed

On the families $\mathcal{D}^+_{k,n} = \mathcal{D}(\mathcal{F}^+_{k,n})$, $\mathcal{F}^+_{k,n}$ – right k-flats

Theorem (KC and T. Glatzer, submitted)

For any $k < n$, $D \in \mathcal{D}^+_{k,n}$ iff D is an F_σ-set whose orthogonal projection $\pi_F[D]$ on any $(n-k)$-flat $F \in \mathcal{F}^+_{n-k}$ is of first category.

Corollary (KC and T. Glatzer)

$P^k \times \mathbb{R}^{n-k} \in \mathcal{D}^+_{k-1,n} \setminus \mathcal{D}^+_{k,n}$ for any nowhere dense perfect $P \subset \mathbb{R}$. In particular, these sets can have positive n-dimensional Lebesgue measure.

$$\{\emptyset\} = \mathcal{D}^+_{n,n} \subsetneq \mathcal{D}^+_{n-1,n} \subsetneq \cdots \subsetneq \mathcal{D}^+_{1,n} \subsetneq \mathcal{D}^+_{0,n}$$

Corollary (KC and T. Glatzer)

If $D \in \mathcal{D}_{k,n}$, then $\pi_F[D]$ is of first category for any $F \in \mathcal{F}_{n-k}$.

On the families $\mathcal{D}_{k,n} = \mathcal{D}(\mathcal{F}_{k,n})$, $\mathcal{F}_{k,n}$ – all k-flats

Theorem (KC and T. Glatzer, submitted)

For any $0 < k < n$ and $D \in \mathcal{D}_{k,n}$ there exists a sequence $\langle f_i \rangle_{i<\omega}$ of Lipschitz functions f_i from $V_i \in \mathcal{F}_{n-k}$ into a perpendicular k-flat whose graphs cover D.
So, every $D \in \mathcal{D}_{k,n}$ has Hausdorff dimension $\leq n - k$.

Proposition (KC and T. Glatzer)

$\{0\}^k \times P \times \mathbb{R}^{n-k-1} \in \mathcal{D}_{k,n}$ for any compact nowhere dense $P \subset \mathbb{R}$. In particular,
$\mathcal{D}_{k,n}$ contains the sets of positive $(n-k)$-Hausdorff measure.

$$\begin{array}{ccccccccccc} \{\emptyset\} & = & \mathcal{D}_{n,n} & \subsetneq & \mathcal{D}_{n-1,n} & \subsetneq & \cdots & \subsetneq & \mathcal{D}_{1,n} & \subsetneq & \mathcal{D}_{0,n} \\ & & \| & & \cap & & & & \cap & & \| \\ & & \mathcal{D}^+_{n,n} & \subsetneq & \mathcal{D}^+_{n-1,n} & \subsetneq & \cdots & \subsetneq & \mathcal{D}^+_{1,n} & \subsetneq & \mathcal{D}^+_{0,n} \end{array}$$

Characterization of $\mathcal{D}_{k,n} = \mathcal{D}(\mathcal{F}_{k,n})$ for $k \geq n/2$

Definition (Topology on $\mathcal{F}_{k,n}$)

Generated by a subbase formed by the sets
$\mathcal{F}(U) = \{F \in \mathcal{F}_k : F \cap U \neq \emptyset\}$, where U is an open set in $\mathbb{R}^n$.

Definition (Ideal $\mathcal{J}_{k,n}$)

$\mathcal{J}_{k,n}$ – all bounded sets $S \subset \mathbb{R}^n$ s.t. there is an increasing sequence $\langle \mathcal{L}_i : i < \omega \rangle$ of closed subsets of $\mathcal{F}_k$ such that $\bigcup_{i<\omega} \mathcal{L}_i = \mathcal{F}_k$ and, for every $i < \omega$,
S is disjoint with the interior $\mathrm{int}(\bigcup \mathcal{L}_i)$ of the set $\bigcup \mathcal{L}_i \subset \mathbb{R}^n$.

Theorem (KC and T. Glatzer, submitted)

Let $0 < k < n$ be such that $k \geq \frac{n}{2}$. A set $D \subset \mathbb{R}^n$ is in $\mathcal{D}_{k,n}$ iff D is a countable union of compact sets from $\mathcal{J}_{k,n}$.

Outline

More history; $\mathcal{F}$ consisting of graphs of functions

Scheefer 1890, Lebesgue 1905: for $\mathcal{A}$ =analytic functions

$\mathcal{A}$-continuity (for $n = 2$) does not imply continuity.

Theorem ([Rosenthal 1955])

- *D^2-continuity (for $n = 2$) does not imply continuity; however*
- *$\mathcal{C}^1$-continuity is equivalent to continuity (for every n),*

where $\mathcal{C}^1$ and D^2 are, respectively, continuously and twice differentiable functions.

Here, functions are with respect of any of coordinate hyperplanes, e.g., from x to y and from y to x.

On sets $D(f)$ for D^2-continuous functions f

Remember (Rosenthal) that $\mathcal{C}^1$-continuity implies continuity.

Theorem (KC and T. Glatzer)

There exists a D^2-continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$ for which $D(f)$ has positive one dimensional Hausdorff measure.

The example can be "lifted" to a D^2-continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ of positive $(n-1)$-Hausdorff measure.

Outline

For which families $\mathcal{F} \subset \mathcal{P}(\mathbb{R}^2)$, $\mathcal{D}(\mathcal{F}) = \emptyset$?

- $\mathcal{D}$(all converging sequences) $= \emptyset$.

Luzin's 1948 text: If $f_h(x) = f(x, h(x))$ is continuous for every continuous h, then $f(x, y)$ is continuous. In particular,

- $\mathcal{D}(\mathcal{C}(\mathbb{R})) = \emptyset$ (only graphs from x to y!)

Theorem (KC and Joseph Rosenblatt, submitted)

- $\mathcal{D}(\text{“}C^1\text{”}) = \emptyset$ *(we allow infinite derivatives)*
- $\mathcal{D}(D^1) \neq \emptyset$ *(basically, example of KC and TG)*

$T(h)$-continuity, $T(h)$ translations of single h

Theorem (KC and Joseph Rosenblatt, submitted)

- $\mathcal{D}(T(h)) \neq \emptyset$ *for every continuous* $h\colon \mathbb{R}^n \to \mathbb{R}$
- $\mathcal{D}(T(h)) = \emptyset$ *for a Baire class 1 function* $h\colon \mathbb{R}^n \to \mathbb{R}$*;*
  We can have $D(h) = P^n$ *with* P *compact measure 0.*

Theorem (KC and Joseph Rosenblatt)

- $\mathcal{D}(T(X)) = \emptyset$ *for any Borel set* $X \subset \mathbb{R}^n$ *which is either of positive measure or of the second category*
- $\mathcal{D}(T(P^n)) = \emptyset$ *for a compact* $P \subset \mathbb{R}$ *of measure zero.*

$I(h)$-continuity, $I(h)$ all isometric copies of h

Theorem (KC and Joseph Rosenblatt, submitted)

- *$T(h)$-continuity does not imply $I(h)$-continuity*

  For $h\colon \mathbb{R} \to \mathbb{Q}$, $h(x) = 0$ for all $x \notin \mathbb{Q} \cap [0, 1]$,

  $h \restriction \mathbb{Q} \cap [0, 1]$ having a dense graph in $[0, 1] \times \mathbb{R}$.

Question

- Does there exist a continuous $h\colon \mathbb{R} \to \mathbb{R}$ with $\mathcal{D}(I(h)) = \emptyset$?
- What can be said about the sets X with $\mathcal{D}(I(X)) = \emptyset$?

Outline

Summary of new results

- Big progress on characterization of sets of points of discontinuity of linearly continuous function
- Deep study of functions on $\mathbb{R}^n$ continuous when restricted to k-dimensional affine spaces
- Construction of D^2-continuous functions f with large set of points of discontinuity
- Discussion a theorem of Luzin
- Discussion of when $T(h)$-continuity implies continuity, for h being a graph of function

Thank you for your attention!