On functions on $\mathbb{R}^n$ continuous when restricted to nice curves or surfaces

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Objects considered in this talk

We consider only functions $f\colon \mathbb{R}^n \to \mathbb{R}$ with $n = 2, 3, 4, \ldots$ fixed.

For a collection $\mathfrak{C}$ of subsets of (curves in) $\mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$:

- f is $\mathfrak{C}$-continuous iff $f \restriction C$ is continuous for every $C \in \mathfrak{C}$;
- f is linearly continuous iff it is $\mathcal{L}$-continuous,
  where $\mathcal{L}$ is the family of all lines in $\mathbb{R}^n$;
- f is separately continuous iff it is $\mathcal{L}^{\#}$-continuous,
  where $\mathcal{L}^{\#} = \{L \in \mathcal{L}\colon L \text{ is parallel to one of the axes}\}$;
- for a class $\mathcal{F}$ of functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$
  f is $\mathcal{F}$-continuous iff it is $\mathcal{F}^*$-continuous,
  where $\mathcal{F}^*$ = all isometric copies of the graphs of $h \in \mathcal{F}$.

Continuity vs $\mathfrak{C}$-continuity: prehistory (for $n = 2$)

Cauchy, in 1821 book *Cours d'analyse*, incorrectly claimed:

separate continuity implies continuity!

Counterexamples:

- J. Thomae calculus text 1870 (and 1873), due to E. Heine:
  $F(x,y) = \sin\left(4\arctan\left(\frac{y}{x}\right)\right)$ for $\langle x,y\rangle \neq \langle 0,0\rangle$, $F(0,0) = 0$.
- 1884 treatise on calculus by Genocchi and Peano:
  $P(x,y) = \frac{xy^2}{x^2+y^4}$ for $\langle x,y\rangle \neq \langle 0,0\rangle$, $P(0,0) = 0$.

Continuity vs $\mathfrak{C}$-continuity: more history

Scheefer 1890, Lebesgue 1905: for $\mathcal{A}$ analytic functions

$\mathcal{A}$-continuity (for $n = 2$) does not imply continuity.

Theorem ([Rosenthal 1955])

- *D^2-continuity (for $n = 2$) does not imply continuity; however*
- *$\mathcal{C}^1$-continuity is equivalent to continuity (for every n).*

Here: $\mathcal{C}^1$ and D^2 are, respectively, continuously and twice differentiable functions.

Digression on $P(x, y) = \frac{xy^2}{x^2+y^4}$ example

Clearly a function $P_n(x_0, \ldots, x_{n-1}) = P(x_0, x_1)$ is:

- discontinuous and linearly continuous for any $n \geq 2$;
- it is continuous on every proper hyperplane in $\mathbb{R}^n$ iff $n = 2$.

If $\mathcal{H} =$ all proper hyperplanes (i.e., affine subspaces) of $\mathbb{R}^n$,

- P_n is $\mathcal{H}$-continuous iff $n = 2$.

Q: Does $\mathcal{H}$-continuity imply continuity for $n > 2$?

A: It is easy to find counterexamples, e.g., by induction.

Q: Are there nice counterexamples, similar to P_2, for $n > 2$?

Nice discotinuous $\mathcal{H}$-continuous functions for all n

Theorem (KC 2012)

$$f_n(\vec{x}) = \frac{x_0(x_0)^{4^0}(x_1)^{4^1}\cdots(x_{n-1})^{4^{n-1}}}{(x_0)^{2^n}+(x_1)^{2^{n+1}}+\cdots+(x_{n-1})^{2^{n+(n-1)}}} = \frac{x_0\prod_{i=0}^{n-1}(x_i)^{2^{2i}}}{\sum_{i=0}^{n-1}(x_i)^{2^{n+i}}}$$

for $\vec{x} = \langle x_0, x_1, \ldots, x_{n-1}\rangle \neq$ *origin* θ, $f_n(\theta) = 0$, *is* $\mathcal{H}$*-continuous but not continuous (on a path* $\vec{p}(t) = \langle t^{2^n}, t^{2^{n-1}}, \ldots, t^{2^2}, t^{2^1}\rangle$*).*

- $f_2(x_0, x_1) = \frac{(x_0)(x_0)(x_1)^4}{(x_0)^4+(x_1)^8} = P((x_0)^2, (x_1)^2)$
- $f_3(x_0, x_1, x_2) = \frac{(x_0)(x_0)(x_1)^4(x_2)^{16}}{(x_0)^8+(x_1)^{16}+(x_2)^{32}}$
- $f_4(x_0, x_1, x_2, x_3) = \frac{(x_0)(x_0)(x_1)^4(x_2)^{16}(x_3)^{64}}{(x_0)^{16}+(x_1)^{32}+(x_2)^{64}+(x_3)^{128}}$, etc

Back to the history of separate and linear continuity

Theorem ([Baire 1899] for $n = 1$, [Lebesgue 1905] for all n)

Every separately continuous function on $\mathbb{R}^n$ is Baire calss $n-1$, but need not be of lower Baire class, as

- *for every Baire class $n-1$ function $g\colon [0,1] \to \mathbb{R}$ there is a separately continuous function F on $\mathbb{R}^n$ such that*

$$F(x, \ldots, x) = g(x) \text{ for all } x \in [0,1].$$

Corollary

Every linearly continuous function on $\mathbb{R}^n$ is Baire class $n-1$

Question (I believe open)

Is the Baire class the best in the Corollary above?

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Theorem (KC, very partial answer, preliminary work)

For every Baire class 1 function $g\colon [0,1] \to \mathbb{R}$ there is a separately continuous function F on $\mathbb{R}^2$ such that

$$F(x, x^2) = g(x) \text{ for all } x \in [0,1].$$

Sets of discontinuity points for $\mathfrak{C}$-continuous functions

$D(f)$ denotes the set of points of discontinuity of f

Theorem (Kershner 1943)

For any set $D \subset \mathbb{R}^n$

- *$D = D(f)$ for some separately continuous f on $\mathbb{R}^n$ iff*
- *D is an F_σ set and every orthogonal projection of D onto a coordinate hyperplane has first category image.*

Question (Kronrod 1976, still not fully answered)

Find a characterization of sets $D(f)$ (similar to that of Kershner) for linearly continuous functions f

On sets $D(f)$ for linearly continuous functions f

Theorem (Slobodnik 1976: upper bound for $D(f)$'s)

If $D \subset \mathbb{R}^n$ is the set of discontinuity points of some linearly continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, then

$$D = \bigcup_{i=1}^{\infty} D_i,$$

where each D_i is isometric to the graph of a Lipschitz function $\phi_i\colon K_i \to \mathbb{R}$ with K_i being compact nowhere dense in $\mathbb{R}^{n-1}$.

In particular, such D must have Hausdorff dimension $\leq n-1$,

while there is a separately continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ having positive Lebesgue (so, n-Hausdorff) measure.

New results on sets $D(f)$ for linearly continuous f

Theorem (KC and T. Glatzer: lower bound for $D(f)$'s)

If D is a restriction of a convex $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ to a compact nowhere dense subset of $\mathbb{R}^{n-1}$, then $D = D(f)$ for a linearly continuous $f\colon \mathbb{R}^n \to \mathbb{R}$.

For $n = 2$ the results remains true when ϕ is $\mathcal{C}^2$ (continuously twice differentiable).

In particular, D may have positive $(n-1)$-Hausdorff measure.

Note a gap between classes of convex and Lipschitz functions

On sets $D(f)$ for D^2-continuous functions f

Remember (Rosenthal) that $\mathcal{C}^1$-continuity implies continuity.

Theorem (KC and T. Glatzer, to appear)

There exists a D^2-continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$ for which $D(f)$ has positive one dimensional Hausdorff measure.

The example can be "lifted" to a D^2-continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ of positive $(n-1)$-Hausdorff measure.

Summary of new results

- For every $n \geq 2$ there is a simple discontinuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ with continuous restriction to every proper hyperplane.
- For every convex $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ and compact nowhere dense $K \subset \mathbb{R}^{n-1}$, there is a linearly continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f) = \phi \restriction K$.
- For $n = 2$, the same is true for $\mathcal{C}^2$ functions ϕ.
- There exists a D^2-continuous $f\colon \mathbb{R}^n \to \mathbb{R}$ with $D(f)$ of positive $(n-1)$-Hausdorff measure.

Thank you for your attention!