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NICE HAMEL BASES UNDER THE COVERING PROPERTY AXIOM

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Abstract. We prove that axiom $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, which follows from the Covering Property Axiom CPA and holds in the iterated perfect set model, implies that there exists a Hamel basis which is a union of less than continuum many pairwise disjoint perfect sets. We will also give two consequences of this last fact.

1. The result and its consequences

In this paper we will use standard set theoretic terminology as in [2]. We will consider the real line $\mathbf{R}$ as a linear space over the rationals $\mathbf{Q}$. Any linear base of this space will be referred to as a *Hamel base.* For $A \subset \mathbf{R}$ we will write $\mathrm{LIN}\,(A)$ to denote the linear subspace of $\mathbf{R}$ spanned by A.

Axiom $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ was introduced by the authors in [5], where it is shown that it holds in the iterated perfect set model. Also, $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ is a version of the axiom CPA which is described in the monograph [6].

It is known that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ captures, to a big extent, the essence of the iterated perfect set model. This follows from a recent result of J. Zapletal [13] who proved that for a "nice" cardinal invariant κ if $\kappa < \mathfrak{c}$ holds in any forcing extension then $\kappa < \mathfrak{c}$ follows already from $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$.

For the reader's convenience, we will restate $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, along with necessary definitions, in the next section.

The main result of this paper is the following theorem.

THEOREM 1.1. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists a family $\mathcal{H}$ of ω_1 pairwise disjoint perfect subsets of* $\mathbf{R}$ *such that $H = \bigcup \mathcal{H}$ is a Hamel basis.*

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This theorem will be proved in the following sections. For the rest of this section we will discuss its two consequences.

Let $\mathcal{I}$ be a translation invariant ideal on $\mathbf{R}$. We say that a subset X of $\mathbf{R}$ is *$\mathcal{I}$-rigid* provided $X, \mathbf{R}\setminus X \notin \mathcal{I}$ but $X \triangle (r+X) \in \mathcal{I}$ for every $r \in \mathbf{R}$. An easy inductive construction gives a non-measurable subset X of $\mathbf{R}$ without the Baire property which is $[\mathbf{R}]^{<\mathfrak{c}}$-rigid. (First such a construction, under CH, comes from Sierpiński [12]. Compare also [8].) Thus, under CH or MA there are $\mathcal{N}\cap\mathcal{M}$-rigid sets, where $\mathcal{N}$ and $\mathcal{M}$ stand for the ideals of measure zero and of the ideal meager subsets of $\mathbf{R}$, respectively. Recently these sets have been studied by Laczkovich [11] and Cichoń, Jasiński, Kamburelis, and Szczepaniak [1]. In particular, Laczkovich [11, Theorem 2] implies that there is no $\mathcal{N}\cap\mathcal{M}$-rigid set in the random and Cohen models. The next corollary shows that the existence of such sets follows from $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$.

COROLLARY 1.2. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies there exists an $\mathcal{N}\cap\mathcal{M}$-rigid set X which is neither measurable nor does it have the Baire property.*

PROOF. Let $\mathcal{H} = \{Q_\xi : \xi < \omega_1\}$ be from Theorem 1.1 and for every $\xi < \omega_1$ let $L_\xi = \mathrm{LIN}\left(\bigcup_{\eta<\xi} Q_\eta\right)$. Then $\mathbf{R}$ is an increasing union of L_ξ's and each L_ξ belongs to $\mathcal{N}\cap\mathcal{M}$, since it is a proper Borel subgroup of $\mathbf{R}$.

Since, under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$, the cofinalities of the ideals $\mathcal{N}$ and $\mathcal{M}$ are equal to ω_1 (see [4] or [6]), there is a family $\{C_\xi : \xi < \omega_1\}$ such that every $S \in \mathcal{M}\cup\mathcal{N}$ is a subset of some C_ξ. By induction choose $X_0 = \{x_\xi : \xi < \omega_1\} \subset \mathbf{R}$ such that

$$x_\xi \notin C_\xi \cup \mathrm{LIN}\left(L_\xi \cup \{x_\zeta : \zeta < \xi\}\right).$$

Then X_0 intersects the complement of every set from $\mathcal{M}\cup\mathcal{N}$. Define

$$X = \bigcup_{\xi<\omega_1} (x_\xi + L_\xi)$$

and notice that $X_0 \subset X$ and $2X_0 \subset \mathbf{R}\setminus X$. Thus, both X and $\mathbf{R}\setminus X$ intersect the complement of every set from $\mathcal{M}\cup\mathcal{N}$. In particular, $X, \mathbf{R}\setminus X \notin \mathcal{M}\cup\mathcal{N}$.

Next notice that for every $r \in L_\zeta$

$$X \triangle (r+X) \subset \bigcup_{\xi<\zeta} \left[(x_\xi + L_\xi) \cup (r + x_\xi + L_\xi)\right] \in \mathcal{N}\cap\mathcal{M}.$$

Thus, X is $\mathcal{N}\cap\mathcal{M}$-rigid, but also $\mathcal{N}$-rigid and $\mathcal{M}$-rigid. These last two facts imply that X is neither measurable nor does it have the Baire property. $\square$

Our second application of Theorem 1.1 is the following result.

COROLLARY 1.3. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies there exists a function* $f: \mathbf{R} \to \mathbf{R}$ *such that for every* $h \in \mathbf{R}$ *the difference function* $\Delta_h(x) = f(x+h) - f(x)$ *is Borel; however, for every* $\alpha < \omega_1$ *there is an* $h \in \mathbf{R}$ *such that* Δ_h *is not of Borel class* α.

Note that answering a question of Laczkovich [10], Filipów and Recław [7] gave an example of such an f under CH. Recław also asked (private communication) whether such a function can be constructed in absence of CH. Corollary 1.3 gives an affirmative answer to this question. It is an open question whether such a function exists in ZFC.

PROOF. The proof is quite similar to that of Corollary 1.2.

Let $\mathcal{H} = \{Q_\xi : \xi < \omega_1\}$ be from Theorem 1.1. For every $\xi < \omega_1$ define $L_\xi = \mathrm{LIN}\left(\bigcup_{\eta<\xi} Q_\eta\right)$ and choose a Borel subset B_ξ of Q_ξ of Borel class greater than ξ. Define

$$X = \bigcup_{\xi<\omega_1} (B_\xi + L_\xi)$$

and let f be the characteristic function χ_X of X.

To see that f is as required note that

$$\Delta_{-h}(x) = \left[\chi_{(h+X)\setminus X} - \chi_{X\setminus(h+X)}\right](x).$$

So, it is enough to show that each of the sets $(h+X)\setminus X$ and $X\setminus(h+X)$ is Borel, though they can be of arbitrary high class. For this, notice that for every $h \in L_{\alpha+1} \setminus L_\alpha$ we have

$$h + X = h + \bigcup_{\xi<\omega_1} (B_\xi + L_\xi) = \bigcup_{\xi\leqq\alpha} (h + B_\xi + L_\xi) \cup \bigcup_{\alpha<\xi<\omega_1} (B_\xi + L_\xi)$$

and that the sets $\bigcup_{\xi\leqq\alpha}(h + B_\xi + L_\xi) \subset L_{\alpha+1}$ and $\bigcup_{\alpha<\xi<\omega_1}(B_\xi + L_\xi)$ are disjoint. So

$$(h+X)\setminus X = \bigcup_{\xi\leqq\alpha} (h + B_\xi + L_\xi) \setminus X = \bigcup_{\xi\leqq\alpha} (h + B_\xi + L_\xi) \Big\backslash \bigcup_{\xi\leqq\alpha} (B_\xi + L_\xi)$$

is Borel, since each set $B_\xi + L_\xi$ is Borel. (It is a subset of $Q_\xi + L_\xi$, which is homeomorphic to $Q_\xi \times L_\xi$ via addition function.) Similarly, set $X \setminus (h+X)$ is Borel.

Finally notice that for $h \in Q_\alpha \setminus B_\alpha$ the set

$$(h+X)\setminus X = \bigcup_{\xi\leqq\alpha} (h + B_\xi + L_\xi)$$

is of Borel class greater than α, since so is $(h+Q_\alpha)\cap\big[(h+X)\setminus X\big]=h+B_\alpha$. Thus, $\Delta_h(x)$ can be of an arbitrarily high Borel class. $\square$

2. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ and how it implies the theorem

In what follows the Cantor set 2^ω will be denoted by the symbol $\mathfrak{C}$. For a Polish space X (i.e., a complete separable metric space) $\mathrm{Perf}\,(X)$ will stand for a collection of all subsets of X homeomorphic to the Cantor set $\mathfrak{C}$.

The main notion behind the formulation of a $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ is that of a prism in a Polish space X and of its subprism. A *prism* in X is a perfect set $P\in\mathrm{Perf}\,(X)$ which comes with (implicitly given) coordinate system, that is, a homeomorphism from $\mathfrak{C}^\alpha$, $0<\alpha<\omega_1$, onto P. If P is a prism with a coordinate function $f:\mathfrak{C}^\alpha\to P$ then its *subprism* is any set of the form $f[E]$, where E is an *iterated perfect set*, that is, it belongs to the family $\mathbf{P}_\alpha$ to be defined later.

In addition, we consider every singleton as a (trivial) prism, whose only subprism is itself. We also define $\mathrm{Perf}^*(X)$ as the family of all sets P such that either $P\in\mathrm{Perf}\,(X)$ or P is a singleton.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ is expressed in terms of the following game $\mathrm{GAME}_{\mathrm{prism}}(X)$ of length ω_1. The game has two players, Player I and Player II. At each stage $\xi<\omega_1$ of the game Player I plays a prism $P_\xi\in\mathrm{Perf}^*(X)$ and Player II must respond with a subprism Q_ξ of P_ξ. The game $\big\langle\langle P_\xi,Q_\xi\rangle:\xi<\omega_1\big\rangle$ is won by Player I provided

$$\bigcup_{\xi<\omega_1}Q_\xi=X;$$

otherwise the game is won by Player II.

By a strategy for Player II we will understand any function S such that $S\big(\big\langle\langle P_\eta,Q_\eta\rangle:\eta<\xi\big\rangle,P_\xi\big)$ is a subprism of P_ξ, where $\big\langle\langle P_\eta,Q_\eta\rangle:\eta<\xi\big\rangle$ is any partial game. (We abuse here slightly the notation, since the function S depends also on the implicitly given coordinate functions making each P_η a prism.) A game $\big\langle\langle P_\xi,Q_\xi\rangle:\xi<\omega_1\big\rangle$ is played according to a strategy S for Player II provided $Q_\xi=S\big(\big\langle\langle P_\eta,Q_\eta\rangle:\eta<\xi\big\rangle,P_\xi\big)$ for every $\xi<\omega_1$. A strategy S for Player II is a *winning strategy* for Player II provided Player II wins any game played according to the strategy S.

Now, we can formulate the axiom.

$\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$: $\mathfrak{c}=\omega_2$ and for any Polish space X Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{prism}}(X)$.

Now, Theorem 1.1 follows quite easily from the axiom and the following lemma, whose proof will occupy the remainder of this paper.

LEMMA 2.1. *Let $M \subset \mathbf{R}$ be sigma-compact and linearly independent. Then for every prism P in $\mathbf{R}$ there exist a subprism Q of P and a compact subset R of $P \setminus M$ such that $M \cup R$ is a maximal linearly independent subset of $M \cup Q$.*

PROOF OF THEOREM 1.1. For a linearly independent sigma-compact set $M \subset \mathbf{R}$ and a prism P in $\mathbf{R}$ let $Q(M,P) = Q$ and $R(M,P) = R \subset P \setminus M$ be as in Lemma 2.1. Consider Player II strategy S given by

$$S\big(\big\langle \langle P_\eta, Q_\eta\rangle : \eta < \xi \big\rangle, P_\xi\big) = Q\Big(\bigcup\{R_\eta : \eta < \xi\}, P_\xi\Big),$$

where the R_η's are defined inductively by $R_\eta = R\big(\bigcup\{R_\zeta : \zeta < \eta\}, P_\eta\big)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ strategy S is not a winning strategy for Player II. So there exists a game $\big\langle \langle P_\xi, Q_\xi\rangle : \xi < \omega_1 \big\rangle$ played according to S in which Player II loses, that is, $\mathbf{R} = \bigcup_{\xi<\omega_1} Q_\xi$.

Let $\mathcal{H} = \{R_\xi : \xi < \omega_1\}$ and notice that $\bigcup \mathcal{H}$ is a Hamel basis. Indeed, clearly $\bigcup \mathcal{H}$ is linearly independent. To see that it spans $\mathbf{R}$ it is enough to notice that $\mathrm{LIN}\left(\bigcup_{\eta<\xi} R_\eta\right) = \mathrm{LIN}\left(\bigcup_{\eta<\xi} Q_\eta\right)$ for every $\xi < \omega_1$.

Although sets in $\mathcal{H}$ need not be perfect, they are clearly pairwise disjoint and compact. Thus, the theorem follows immediately from the following remark. □

REMARK 2.2. If there exists a family $\mathcal{H}$ of ω_1 pairwise disjoint compact subsets of $\mathbf{R}$ such that $\bigcup \mathcal{H}$ is a Hamel basis then there exists such an $\mathcal{H}$ with $\mathcal{H} \subset \mathrm{Perf}(\mathbf{R})$.

PROOF. Let $\mathcal{H}_0$ be a family of ω_1 pairwise disjoint compact subsets of $\mathbf{R}$ such that $\bigcup \mathcal{H}_0$ is a Hamel basis. Partitioning each $H \in \mathcal{H}_0$ into its perfect part and singletons from scattered part we can assume that $\mathcal{H}_0$ contains only perfect sets and singletons. To get $\mathcal{H}$ as required fix a perfect set $P_0 \in \mathcal{H}_0$ and an $x \in P_0$ and notice that if we replace each $P \in \mathcal{H}_0 \setminus \{P_0\}$ with $px + qP$ for some $p, q \in \mathbf{Q} \setminus \{0\}$ then the resulting family will still be pairwise disjoint with union being a Hamel basis. Thus, without loss of generality, we can assume that every open interval in $\mathbf{R}$ contains ω_1 perfect sets from $\mathcal{H}_0$. Now, for every singleton $\{x\}$ in $\mathcal{H}_0$ we can choose a sequence $P^x_1 > P^x_2 > P^x_3 > \cdots$ from $\mathcal{H}_0$ converging to x, and replace a family $\{x\} \cup \{P^x_n : n < \omega\}$ with its union. (We assume that we choose different sets P^x_n for different singletons.) If $\mathcal{H}$ is such a modification of $\mathcal{H}_0$ then $\mathcal{H}$ is as desired. □

3. Iterated perfect sets and fusion lemmas for prisms

Let $0<\alpha<\omega_1$. To define $\mathbf{P}_\alpha$ we need to consider the family $\Phi_{\rm prism}(\alpha)$ of all continuous injections $f\colon \mathfrak{C}^\alpha\to\mathfrak{C}^\alpha$ with the property that

$$f(x)\restriction\beta=f(y)\restriction\beta \;\Leftrightarrow\; x\restriction\beta=y\restriction\beta \quad \text{for all } \beta<\alpha \text{ and } x,y\in\mathfrak{C}^\alpha \tag{1}$$

or, equivalently, such that for every $\beta<\alpha$

$$f\restriction\restriction\beta \stackrel{\rm def}{=} \left\{\langle x\restriction\beta, y\restriction\beta\rangle : \langle x,y\rangle\in f\right\}$$

is a one-to-one function from $\mathfrak{C}^\beta$ into $\mathfrak{C}^\beta$. For example, if $\alpha=3$ then $f\in\Phi_{\rm prism}(\alpha)$ provided there exist continuous functions $f_0\colon\mathfrak{C}\to\mathfrak{C}$, $f_1\colon\mathfrak{C}^2\to\mathfrak{C}$, and $f_2\colon\mathfrak{C}^3\to\mathfrak{C}$ such that $f(x_0,x_1,x_2)=\left\langle f_0(x_0),f_1(x_0,x_1),f_2(x_0,x_1,x_2)\right\rangle$ for all $x_0,x_1,x_2\in\mathfrak{C}$ and the maps f_0, $\langle f_0,f_1\rangle$, and f are one-to-one. The functions f from $\Phi_{\rm prism}(\alpha)$ were first introduced, in a more general setting, in [9] where they were called *projection-keeping homeomorphisms.* Note that

$$\Phi_{\rm prism}(\alpha) \text{ is closed under the compositions} \tag{2}$$

and that for every $0<\beta<\alpha$

$$\text{if } f\in\Phi_{\rm prism}(\alpha) \text{ then } f\restriction\restriction\beta\in\Phi_{\rm prism}(\beta). \tag{3}$$

We define $\mathbf{P}_\alpha$ as

$$\mathbf{P}_\alpha=\left\{\text{range}\,(f) : f\in\Phi_{\rm prism}(\alpha)\right\}.$$

The simplest possible elements of $\mathbf{P}_\alpha$ are the *perfect cubes,* that is, sets of the form $\prod_{\beta<\alpha}C_\beta$, where $C_\beta\in\mathfrak{C}$ for every $\beta<\alpha$. (If f_β is a continuous injection from $\mathfrak{C}$ onto P_β and $f\colon\mathfrak{C}^\alpha\to\mathfrak{C}^\alpha$ is given by $f(x)(\beta)=f_\beta(x_\beta)$ then $f\in\Phi_{\rm prism}(\alpha)$ and $\text{range}\,(f)=\prod_{\beta<\alpha}C_\beta$.)

Note also that

$$\text{if } f\in\Phi_{\rm prism}(\alpha) \text{ and } P\in\mathbf{P}_\alpha \text{ then } f[P]\in\mathbf{P}_\alpha. \tag{4}$$

Indeed, if $P=g[\mathfrak{C}^\alpha]$ for some $g\in\Phi_{\rm prism}(\alpha)$ then, by condition (2), we have $f[P]=f\left[g[\mathfrak{C}^\alpha]\right]=(f\circ g)[\mathfrak{C}^\alpha]\in\mathbf{P}_\alpha$.

In what follows for a fixed $0<\alpha<\omega_1$ and $0<\beta\leqq\alpha$ the symbol π_β will stand for the projection from $\mathfrak{C}^\alpha$ onto $\mathfrak{C}^\beta$. We will always consider $\mathfrak{C}^\alpha$ with the following standard metric ρ: fix an enumeration $\{\langle\beta_k, n_k\rangle : k < \omega\}$ of $\alpha \times \omega$ and for distinct $x, y \in \mathfrak{C}^\alpha$ define

$$\rho(x,y) = 2^{-\min\{k<\omega : x(\beta_k)(n_k) \neq y(\beta_k)(n_k)\}}. \tag{5}$$

The open ball in $\mathfrak{C}^\alpha$ with a center at $z \in \mathfrak{C}^\alpha$ and radius $\varepsilon > 0$ will be denoted by $B_\alpha(z,\varepsilon)$. Notice that in this metric any two open balls are either disjoint or one is a subset of another. Also for every $\gamma < \alpha$ and $\varepsilon > 0$

$$\pi_\gamma\big[B_\alpha(x,\varepsilon)\big] = \pi_\gamma\big[B_\alpha(y,\varepsilon)\big] \quad \text{for every } x, y \in \mathfrak{C}^\alpha \text{ with } x \restriction \gamma = y \restriction \gamma. \tag{6}$$

It is also easy to see that any $B_\alpha(z,\varepsilon)$ is a clopen set and, in fact, it is a perfect cube in $\mathfrak{C}^\alpha$, so it belongs to $\mathbf{P}_\alpha$.

For a fixed $0 < \alpha < \omega_1$ let $\{\langle\beta_k, n_k\rangle : k < \omega\}$ be an enumeration of $\alpha \times \omega$ used in the definition (5) of the metric ρ and let

$$A_k = \{\langle\beta_i, n_i\rangle : i < k\} \quad \text{for every } k < \omega. \tag{7}$$

In what follows we will need the following simple fusion lemma, which can be found in [5]. For the reader's convenience we include here its short proof.

LEMMA 3.1. *Let $0 < \alpha < \omega_1$ and for $k < \omega$ let $\mathcal{E}_k = \{E_s \in \mathbf{P}_\alpha : s \in 2^{A_k}\}$. Assume that for every $k < \omega$, $s, t \in 2^{A_k}$, and $\beta < \alpha$ we have:*

(i) *the diameter of E_s is less than or equal to 2^{-k},*

(ii) *if $i < k$ then $E_s \subset E_{s\restriction i}$,*

(ag) *(agreement) if $s \restriction (\beta \times \omega) = t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] = \pi_\beta[E_t]$,*

(sp) *(split) if $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] \cap \pi_\beta[E_t] = \emptyset$.*

Then $Q = \bigcap_{k<\omega} \bigcup \mathcal{E}_k$ belongs to $\mathbf{P}_\alpha$.

PROOF. For $x \in \mathfrak{C}^\alpha$ let $\bar{x} \in 2^{\alpha\times\omega}$ be defined by $\bar{x}(\beta, n) = x(\beta)(n)$.

First note that, by conditions (i) and (sp), for every $k < \omega$ the sets in $\mathcal{E}_k$ are pairwise disjoint and each of diameter at most 2^{-k}. Thus, taking into account (ii), the function $h : \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ defined by

$$h(x) = r \iff \{r\} = \bigcap_{k<\omega} E_{\bar{x}\restriction A_k}$$

is well defined and is one-to-one. It is also easy to see that h is continuous and that $Q = h[\mathfrak{C}^\alpha]$. Thus, we need to prove only that $h \in \Phi_{\text{prism}}(\alpha)$, that is, that h is projection-keeping.

To show this fix $\beta < \alpha$, put $S = \bigcup_{i<\omega} 2^{A_i}$, and notice that, by (i) and (ag), for every $x \in \mathfrak{C}^\alpha$ we have

$$\left\{ h(x) \restriction \beta \right\} = \pi_\beta \left[\bigcap \{ E_{\bar{x} \restriction A_k} : k < \omega \} \right] = \bigcap \left\{ \pi_\beta [E_{\bar{x} \restriction A_k}] : k < \omega \right\}$$

$$= \bigcap \left\{ \pi_\beta [E_s] : s \in S \ \& \ s \subset \bar{x} \right\} = \bigcap \left\{ \pi_\beta [E_s] : s \in S \ \& \ s \restriction (\beta \times \omega) \subset \bar{x} \right\}.$$

Now, if $x \restriction \beta = y \restriction \beta$ then for every $s \in S$

$$s \restriction (\beta \times \omega) \subset \bar{x} \quad \Leftrightarrow \quad s \restriction (\beta \times \omega) \subset \bar{y}$$

so $h(x) \restriction \beta = h(y) \restriction \beta$.

On the other hand, if $x \restriction \beta \neq y \restriction \beta$ then there exists a $k < \omega$ big enough such that for $s = \bar{x} \restriction A_k$ and $t = \bar{y} \restriction A_k$ we have $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$. But then $\left\{ h(x) \restriction \beta \right\}$ and $\left\{ h(y) \restriction \beta \right\}$ are subsets of $\pi_\beta[E_s]$ and $\pi_\beta[E_t]$, respectively, which, by (sp), are disjoint. So, $h(x) \restriction \beta \neq h(y) \restriction \beta$. $\square$

In what follows we will also need the following simple fact, which follows from the fact that every dense G_δ subset of a Polish space $X \times X$ contains a product $G \times P$, where G is dense G_δ in X and $P \in \operatorname{Perf}(X)$. For the proof see e.g. [4] or [6].

Claim 3.2. *Let $0 < \alpha < \omega_1$. If G is a second category Borel subset of $\mathfrak{C}^\alpha$ then G contains a perfect cube $\prod_{\beta<\alpha} P_\beta$.*

We will also use the following variant of the Kuratowski–Ulam theorem, which can be deduced from the classical Kuratowski–Ulam theorem via a simple closure argument. Its proof can be found in [3] or [6].

Lemma 3.3. *Let $0 < \alpha < \omega_1$. For every comeager set $H \subset \mathfrak{C}^\alpha$ there exists a comeager set $G \subset H$ such that for every $x \in G$ and $\beta < \alpha$ the set*

$$G_{x \restriction \beta} = \left\{ y \in \mathfrak{C}^{\alpha \setminus \beta} : (x \restriction \beta) \cup y \in G \right\}$$

is comeager in $\mathfrak{C}^{\alpha \setminus \beta}$.

4. Proof of Lemma 2.1

Let X be a Polish space, $0 < n < \omega$, and $F \subset X^n$ be an n-ary relation. We say that a set $S \subset X$ is *F-independent* provided $F\big(x(0), \ldots, x(n-1)\big)$ does not hold for any one-to-one $x : n \to S$. For a family $\mathcal{F}$ of finitary relations on X (i.e., relations $F \subset X^n$ where $0 < n < \omega$) we say that $S \subset X$ is $\mathcal{F}$-independent provided S is F-independent for every $F \in \mathcal{F}$. We will use the term *unary relation* for any 1-ary relation.

PROPOSITION 4.1. *Let $0 < \alpha < \omega_1$ and $\mathcal{F}$ be a countable family of closed finitary relations on $\mathfrak{C}^\alpha$. Assume that every unary relation in $\mathcal{F}$ is nowhere dense in $\mathfrak{C}^\alpha$ and that for every $F \in \mathcal{F}$ there exists a comeager subset G_F of $\mathfrak{C}^\alpha$ such that*

(ex) *for every F-independent finite set $S \subset G_F$, $x \in S$, and $\beta < \alpha$ the set*

$$\left\{ z \in \mathfrak{C}^{\alpha\setminus\beta} : \ S \cup \{z \cup x \restriction \beta\} \subset G_F \text{ is } F\text{-independent} \right\}$$

is dense in $\mathfrak{C}^{\alpha\setminus\beta}$.

Then there is an $E \in \mathbf{P}_\alpha$ which is $\mathcal{F}$-independent.

Note that without the assumption that the unary relations in $\mathcal{F}$ are nowhere dense the proposition is false: the unary relation $F = \mathfrak{C}^\alpha$ satisfies the condition (ex) (with $G_F = \mathfrak{C}^\alpha$) and no non-empty set is F-independent. On the other hand, for any n-ary relation $F \in \mathcal{F}$ with $n > 1$ condition (ex) implies that F is nowhere dense in $(\mathfrak{C}^\alpha)^n$. However, not every nowhere dense binary relation satisfies (ex). For example $F = \left\{ \langle x, y \rangle : x(0) = y(0) \right\}$ is nowhere dense and it does not satisfy (ex) if $\alpha > 1$.

PROOF. First notice that applying Lemma 3.3, if necessary, we can assume that for every $F \in \mathcal{F}$, $x \in G_F$, and $\beta < \alpha$ the set $(G_F)_{x\restriction\beta}$ is comeager in $\mathfrak{C}^{\alpha\setminus\beta}$. But this implies that each set from the condition (ex) is comeager in $\mathfrak{C}^{\alpha\setminus\beta}$ since it is an intersection of $(G_F)_{x\restriction\beta}$ and an open set $\left\{ z \in \mathfrak{C}^{\alpha\setminus\beta} : S \cup \{z \cup x \restriction \beta\} \text{ is } F\text{-independent} \right\}$. In particular, if we put $G = \bigcap_{F\in\mathcal{F}} G_F$ then G is comeager in $\mathfrak{C}^\alpha$ and it is easy to see that it satisfies the following condition.

(EX) For every $\mathcal{F}$-independent finite set $S \subset G$, $x \in S$, and $\beta < \alpha$ the set

$$\left\{ z \in \mathfrak{C}^{\alpha\setminus\beta} : \ S \cup \{z \cup x \restriction \beta\} \subset G \text{ is } \mathcal{F}\text{-independent} \right\}$$

is dense in $\mathfrak{C}^{\alpha\setminus\beta}$.

Let $\{F_k : k < \omega\}$ be an enumeration of $\mathcal{F}$ with infinite repetitions. Also, for $k < \omega$ let $A_k = \left\{ \langle \beta_i, n_i \rangle : i < k \right\}$ be as in the condition (7). By induction on $k < \omega$ we will construct two sequences: $\langle \varepsilon_k > 0 : k < \omega \rangle$ converging to 0 and $\left\langle \{x_s \in G : s \in 2^{A_k}\} : k < \omega \right\rangle$ of $\mathcal{F}$-independent sets such that for every $\beta < \alpha$, $k < \omega$, and $s, t \in 2^{A_k}$

(a) $x_s \restriction \beta = x_t \restriction \beta$ if and only if $s \restriction \beta \times \omega = t \restriction \beta \times \omega$;

(b) if $E_s = B_\alpha(x_s, \varepsilon_k)$ and $\mathcal{E}_k = \{E_s : s \in 2^{A_k}\}$ then $\mathcal{E}_k$'s satisfy (ii), (ag), and (sp) from Lemma 3.1;

(c) if F_k is an n-ary relation then $F_k(z_0, \ldots, z_{n-1})$ does not hold provided each z_i is chosen from a different ball from $\mathcal{E}_k$.

Before we construct such sequences, let us first note that $E=\bigcap_{k<\omega}\bigcup\mathcal{E}_k$ is as desired. Indeed, $E\in\mathbf{P}_\alpha$ by Lemma 3.1. To see that E is $\mathcal{F}$-independent pick an n-ary relation $F\in\mathcal{F}$, $\{z_0,\dots,z_{n-1}\}\in[E]^n$, and find a $k<\omega$ with $F_k=F$ which is big enough so that ε_k is smaller than the distance between z_i and z_j for all $i<j<n$. Then the z_i's must belong to distinct elements of $\mathcal{E}_k$ so, by (c), $F(z_0,\dots,z_{n-1})$ does not hold.

For $k=0$ we pick an arbitrary $\mathcal{F}$-independent $x_\emptyset\in G$ by choosing an arbitrary element of G which does not belong to any nowhere dense unary relation from $\mathcal{F}$. Also, we choose an $\varepsilon_0\in(0,1]$ ensuring (c), which can be done since F_0 is closed. (This is a non-trivial requirement only when F_0 is a unary relation.) Clearly (a)–(c) are satisfied.

Assume that for some $k<\omega$ the construction is done up to the level k. For $s\in 2^{A_k}$ and $j<2$ let $s\hat{}j=s\cup\left\{\left\langle\langle\beta_k,n_k\rangle,j\right\rangle\right\}\in 2^{A_{k+1}}$ and define $x_{s\hat{}0}=x_s$. Let $\{s_i:i<2^k\}$ be an enumeration of 2^{A_k} and put $S=\left\{x_{s\hat{}0}:s\in 2^{A_k}\right\}$. The points $x_{s_i\hat{}1}\in G\cap E_{s_i}$ will be chosen by induction on $i\leqq 2^k$ such that the set $S_i=S\cup\left\{x_{s_j\hat{}1}:j<i\right\}$ is $\mathcal{F}$-independent and the condition (a) is satisfied for the elements of S_i. Clearly, by the inductive assumption (a) is satisfied for the elements of $S_0=S$. So, assume that for some $i\leqq 2^k$ the set S_i is already constructed. We need to find an appropriate $x_{s_i\hat{}1}\in G\cap E_{s_i}$. Let $\beta<\alpha$ be maximal such that there is an $s\in\left\{s\hat{}0:s\in 2^{A_k}\right\}\cup\{s_j\hat{}1:j<i\}$ with $s\restriction\beta\times\omega=(s_i\hat{}1)\restriction\beta\times\omega$ and let $x=x_s\restriction\beta$. We will choose $x_{s_i\hat{}1}$ extending x and such that $x_{s_i\hat{}1}(\beta)\neq x_t(\beta)$ for all $x_t\in S_i$. Notice that this will ensure that the condition (a) is satisfied for the elements of S_{i+1}. Surprisingly, a more difficult condition to ensure will be that $x_{s_i\hat{}1}\in E_{s_i}=B_\alpha(x_{s_i\hat{}0},\varepsilon_k)$, since at the first glance it is not even obvious that

$$B_\alpha(x_{s_i\hat{}0},\varepsilon_k)\text{ contains an extension of }x. \tag{8}$$

To argue for this first notice that maximality of β ensures that $\beta\geqq\beta_k$, since $s_i\hat{}0\in S_i$ and $(s_i\hat{}0)\restriction\beta_k\times\omega=(s_i\hat{}1)\restriction\beta_k\times\omega$. If $\beta=\beta_k$ we have $x=x_{s_i\hat{}0}\restriction\beta$ and (8) is obvious. So, assume that $\beta>\beta_k$. Then there is a $j<i$ such that $s=s_j\hat{}1$. We also have $s_j\restriction\beta\times\omega=s_i\restriction\beta\times\omega$ so, by the inductive assumption, $x_{s_j}\restriction\beta=x_{s_i}\restriction\beta$.

Now, let $n<\omega$ be the smallest such that $2^{-n}<\varepsilon_k$. Then, by the definition of the metric on $\mathfrak{C}^\alpha$, the fact that $x_s=x_{s_j\hat{}1}\in E_{s_j}=B_\alpha(x_{s_j},\varepsilon_k)$ means that $x_s(\gamma)(m)=x_{s_j}(\gamma)(m)$ for every $\langle\gamma,m\rangle\in A_n$. Therefore, we have $x(\gamma)(m)=x_s(\gamma)(m)=x_{s_j}(\gamma)(m)=x_{s_i}(\gamma)(m)$ for every $\langle\gamma,m\rangle\in A_n$ with $\gamma<\beta$. Thus, we can extend x to an element $y\in\mathfrak{C}^\alpha$ for which $y(\gamma)(m)=x_{s_i}(\gamma)(m)$ for every $\langle\gamma,m\rangle\in A_n$. But this y witnesses (8).

To finish the construction of $x_{s_i\hat{}1}$ notice that by (8) we can find an open ball B in $\mathfrak{C}^{\alpha\setminus\beta}$ such that $\{x\}\times B\subset B_\alpha(x_{s_i\hat{}0},\varepsilon_k)$. Decreasing B, if necessary, we can also insure that $y(\beta)\neq x_t(\beta)$ for every $t\in S_i$ and $y\in\{x\}\times B$. By condition (EX) we can find a $z\in B$ such that $S_i\cup\{x\cup z\}\subset G$ is $\mathcal{F}$-independent. We put $x_{s_i\hat{}1}=x\cup z$.

Thus, we constructed an $\mathcal{F}$-independent set $\left\{x_{s\hat{}j}: s\in 2^{A_k}\ \&\ j<2\right\}\subset G$ satisfying (a) and such that $x_{s\hat{}0},x_{s\hat{}1}\in E_s$ for every $s\in 2^{A_k}$. To finish the construction ensuring (a)–(c) we need to choose an $\varepsilon_{k+1}\leqq 2^{-(k+1)}$ small enough to guarantee the following properties.

- $E_{s\hat{}j}=B_\alpha(x_{s\hat{}0},\varepsilon_{k_1})\subset E_s$ for every $s\in 2^{A_k}$ and $j<2$. This will ensure condition (ii).
- Condition (sp) holds. This can be done, since (a) is satisfied.
- Condition (c) is satisfied. This can be done since $\left\{x_s: s\in 2^{A_{k+1}}\right\}$ is $\mathcal{F}$-independent and F_{k+1} is a closed relation.

Note that (ag) is guaranteed by (a) and our definition of E_s's. This finishes the proof of Proposition 4.1. □

We say that an n-ary relation F on a Polish space X is *symmetric* provided for any sequence $\langle x_i\in X: i<n\rangle$ and any permutation π of n

$$F(x_0,\dots,x_{n-1}) \text{ holds if and only if } F\left(x_{\pi(0)},\dots,x_{\pi(n-1)}\right) \text{ holds.}$$

For such an F and $A\subset X$ we put

$$F*A=A\cup\left\{x\in X: (\exists a_1,\dots,a_{n-1}\in A)\ F(x,a_1,\dots,a_{n-1})\right\}.$$

If F is a unary relation we interpret the above as $F*A=A\cup F$. If $\mathcal{F}$ is a family of symmetric finitary relations on X then we put $\mathcal{F}*A=\bigcup_{F\in\mathcal{F}}F*A$. Also, an *$\mathcal{F}$-closure of* A, denoted by $\mathrm{cl}_{\mathcal{F}}(A)$, is the least $B\subset X$ containing A such that $\mathcal{F}*B=B$. Note that $\mathrm{cl}_{\mathcal{F}}(A)=\bigcup_{n<\omega}\mathcal{F}^n*A$, where $\mathcal{F}^0*A=A$ and $\mathcal{F}^{n+1}*A=\mathcal{F}*(\mathcal{F}^n*A)$. Thus, if $\mathcal{F}$ is a countable family of closed symmetric finitary relations then $\mathrm{cl}_{\mathcal{F}}(A)$ is F_σ in X for a sigma-compact $A\subset X$ since $F*K$ is closed for every $F\in\mathcal{F}$ and compact $K\subset X$.

We are most interested in these notions when we are concerned with either linear independence (over $\mathbf{Q}$) or algebraic independence in $\mathbf{R}$. In the first case $\mathcal{F}=\mathcal{F}_{\mathrm{lin}}$ is defined as the family of all relations F_ℓ of all $\langle x_0,\dots,x_{n-1}\rangle$ for which

$$\ell\left(x_{\pi(0)},\dots,x_{\pi(n-1)}\right)=0 \text{ for some permutation } \pi \text{ of } n, \tag{9}$$

where ℓ is a non-zero linear function with rational coefficients. In this case $\mathcal{F}$-independence stands for linear independence (over $\mathbf{Q}$) and $\mathrm{cl}_{\mathcal{F}}(A)$ is the linear span of A. When $\mathcal{F}$ is the family of all relations F_ℓ, where ℓ spans over all non-zero polynomials with rational coefficients, then $\mathcal{F}$-independence stands for algebraic independence, while $\mathrm{cl}_{\mathcal{F}}(A)$ is the algebraic closure of $\mathbf{Q}(A)$.

We will need also one more notion. For a family $\mathcal{F}$ of closed symmetric finitary relations on X and an $M \subset X$ we define $\mathcal{F}_M$ as the collection of all possible projections of the relations from $\mathcal{F}$ along M. In other words, $\mathcal{F}_M$ is the collection of all (symmetric) relations

$$(10)\quad \bigl\{\langle x_0,\dots,x_{k-1}\rangle : (\exists a_k,\dots,a_{n-1}\in M)\ F(x_0,\dots,x_{k-1},a_k,\dots,a_{n-1})\bigr\},$$

where $F \in \mathcal{F}$ is an n-ary relation and $0 < k \leqq n$. Note that if M is compact then each relation in $\mathcal{F}_M$ is still closed and for every $A \subset X$ we have

$$(11)\qquad \mathrm{cl}_{\mathcal{F}}(M\cup A)=\mathrm{cl}_{\mathcal{F}_M}(A).$$

Also, if M is $\mathcal{F}$-independent then

(12) $A \cup M$ is $\mathcal{F}$-independent provided A is $\mathcal{F}_M$-independent.

LEMMA 4.2. *Let $\mathcal{F}$ be an arbitrary family of closed symmetric finitary relations in a Polish space X. Then for every prism P in X there exists a subprism Q of P and a compact $\mathcal{F}$-independent set $R \subset P$ such that $Q \subset \mathrm{cl}_{\mathcal{F}}(R)$.*

PROOF. For $0 < \alpha < \omega_1$ let I_α be the statement:

I_α: the lemma holds for any prism P with witness function $f : \mathfrak{C}^\alpha \to P$.

We will prove I_α by induction on α. First notice that I_α implies the following:

I^*_α: for every $k < \omega$ and continuous functions $g_0,\dots,g_k : \mathfrak{C}^\alpha \to X$ there exist an $E \in \mathbf{P}_\alpha$ and a compact $\mathcal{F}$-independent set $R \subset \bigcup_{i\leqq k} g_i[\mathfrak{C}^\alpha]$ such that $\bigcup_{i\leqq k} g_i[E] \subset \mathrm{cl}_{\mathcal{F}}(R)$.

To see that I^*_α holds true for $k = 0$, for every n-ary relation $F \in \mathcal{F}$ define $F^0 = \bigl\{\langle x_0,\dots,x_{n-1}\rangle \in (\mathfrak{C}^\alpha)^n : F\bigl(g_0(x_0),\dots,g_0(x_{n-1})\bigr)\bigr\}$. By I_α applied to $\mathcal{F}_0 = \{F^0 : F \in \mathcal{F}\}$ we can find an $\mathcal{F}_0$-independent set $R_0 \subset \mathfrak{C}^\alpha$ and an $E \in \mathbf{P}_\alpha$ such that $E \subset \mathrm{cl}_{\mathcal{F}_0}(R)$. But then $R = g_0[R_0]$ is compact, $\mathcal{F}$-independent, and $g_0[E] \subset \mathrm{cl}_{\mathcal{F}}\bigl(g_0[R_0]\bigr) = \mathrm{cl}_{\mathcal{F}}(R)$.

To make an inductive step assume that I^*_α holds for some $k < \omega$ and take continuous functions $g_0,\dots,g_{k+1} : \mathfrak{C}^\alpha \to X$. By the inductive assumption we can find an $E_0 \in \mathbf{P}_\alpha$ and a compact $\mathcal{F}$-independent set $R_0 \subset \bigcup_{i\leqq k} g_i[\mathfrak{C}^\alpha]$ such that $\bigcup_{i\leqq k} g_i[E_0] \subset \mathrm{cl}_{\mathcal{F}}(R_0)$. Let $h \in \Phi_{\mathrm{prism}}(\alpha)$ be a mapping from $\mathfrak{C}^\alpha$ onto E_0. Using the case $k = 0$ to the function $g_{k+1} \circ h$ and the family $\mathcal{F}_{R_0}$ we can find an $E_1 \in \mathbf{P}_\alpha$ and a compact $\mathcal{F}_{R_0}$-independent set $R_1 \subset (g_{k+1} \circ h)[\mathfrak{C}^\alpha]$ such that $(g_{k+1} \circ h)[E_1] \subset \mathrm{cl}_{\mathcal{F}_{R_0}}(R_1)$. Then, by (12), we conclude that $R = R_0 \cup R_1$ is $\mathcal{F}$-independent. Put $E = h[E_1] \in \mathbf{P}_\alpha$. Then, by (11), we have $g_{k+1}[E] \subset \mathrm{cl}_{\mathcal{F}_{R_0}}(R_1) = \mathrm{cl}_{\mathcal{F}}(R_0 \cup R_1) = \mathrm{cl}_{\mathcal{F}}(R)$, while clearly $\bigcup_{i \leqq k} g_i[E] \subset \bigcup_{i \leqq k} g_i[E_0] \subset \mathrm{cl}_{\mathcal{F}}(R_0) \subset \mathrm{cl}_{\mathcal{F}}(R)$. Thus, E and R satisfy I^*_α.

Now, we are ready to prove I_α. So, fix $0 < \alpha < \omega_1$ and assume that I_γ is true for all $0 < \gamma < \alpha$. Let P be a prism in X with witness function $f : \mathfrak{C}^\alpha \to P$. We need to find appropriate Q and R.

Let W be the set of all $\beta \leqq \alpha$ for which there exists an $E \in \mathbf{P}_\alpha$ and an $F \in \mathcal{F}$ such that for every $z \in \pi_\beta[E]$ there is a finite set $R_z \subset P$ for which

$$f\big[\{x \in E : z \subset x\}\big] \subset F * R_z. \tag{13}$$

Notice that W is non-empty since $\alpha \in W$. So $\beta = \min W$ is well defined. Let $E \in \mathbf{P}_\alpha$ be such that (13) holds for β. Replacing f with its composition with an appropriate function from $\Phi_{\mathrm{prism}}(\alpha)$ (compare (4)), if necessary, we can assume that $E = \mathfrak{C}^\alpha$.

If $\beta = 0$ then $f[\mathfrak{C}^\alpha] \subset \mathrm{cl}_{\mathcal{F}}(R_0)$ for some finite set $R_0 \subset P$, and we can find an $\mathcal{F}$-independent finite $R \subset R_0$ with $f[\mathfrak{C}^\alpha] \subset \mathrm{cl}_{\mathcal{F}}(R)$. (Note that if T is $\mathcal{F}$-independent and $x \in X \setminus \mathrm{cl}_{\mathcal{F}}(T)$ then $T \cup \{x\}$ is also $\mathcal{F}$-independent.) Thus, $Q = f[\mathfrak{C}^\alpha]$ and R satisfy I_α. So, for the rest of the proof we will assume that $\beta > 0$.

Next, assume that $0 < \beta < \alpha$. Let $\mathcal{B}_\beta$ be a countable basis of $\mathfrak{C}^{\alpha \setminus \beta}$ consisting of non-empty clopen sets and assume that F satisfying (13) is $(n+1)$-ary. For every $B \in \mathcal{B}_\beta$ consider the set

$$K_B = \Big\{z \in \mathfrak{C}^\beta : \big(\exists \langle x_1, \dots, x_n\rangle \in P^n\big)\,(\forall y \in B)\ F\big(f(z \cup y), x_1, \dots, x_n\big)\Big\}.$$

It is easy to see that each set K_B is closed. Notice also that

$$\mathfrak{C}^\beta = \bigcup_{B \in \mathcal{B}_\beta} K_B. \tag{14}$$

To see this, fix a $z \in \mathfrak{C}^\beta$. By (13), there exists a finite set $S_z \subset \mathfrak{C}^\alpha$ such that $\mathfrak{C}^{\alpha\setminus\beta} = \bigcup_{x_1,\dots,x_n \in f[S_z]} \big\{y \in \mathfrak{C}^{\alpha\setminus\beta} : F\big(f(z \cup y), x_1, \dots, x_n\big)\big\}$. Since each set $\big\{y \in \mathfrak{C}^{\alpha\setminus\beta} : F\big(f(z \cup y), x_1, \dots, x_n\big)\big\}$ is closed, one of them must contain a $B \in \mathcal{B}_\beta$, and so $z \in K_B$.

Thus, by (14), there exists a $B \in \mathcal{B}_\beta$ such that K_B has a non-empty interior. In particular, there is a non-empty clopen set $U \subset K_B$. But then for every $z \in U$ there exists a $g(z) = \langle g_1(z), \ldots, g_n(z) \rangle \in P^n$ such that $F\big(f(z \cup y), g_1(z), \ldots, g_n(z)\big)$ holds for every $y \in B$. Now

$$T = \Big\{ \langle z, \bar{p} \rangle \in U \times P^n : (\forall y \in B)\ F\big(f(z \cup y), \bar{p}\big) \Big\}$$

is a compact subset of $U \times P^n$ and g constitutes a selector of T. Thus, we can choose g to be Borel. In particular, there is a dense G_δ subset W of U such that $g \restriction W$ is continuous. So, by Claim 3.2, we can find a perfect cube $C \subset W \subset \mathfrak{C}^\beta$. Now, identifying C with $\mathfrak{C}^\beta$, we conclude that the functions $g_1, \ldots, g_n : \mathfrak{C}^\beta \to P$ are continuous and that $F\big(f(z \cup y), g_1(z), \ldots, g_n(z)\big)$ holds for every $z \in \mathfrak{C}^\beta$ and $y \in B$.

Since, by the inductive hypothesis, I_β is true, condition I^*_β holds as well. Thus, there exist an $E \in \mathbf{P}_\beta$ and a compact $\mathcal{F}$-independent set $R \subset P$ such that $\bigcup_{i=1}^n g_i[E] \subset \mathrm{cl}_{\mathcal{F}}(R)$. Since $Q = f[E \times B]$ is a subprism of P, we just need to show that $Q \subset \mathrm{cl}_{\mathcal{F}}(R)$. To see this just note that for every $z \in E$ we have $f\big[\{z\} \times B\big] \subset F * \big\{g_1(z), \ldots, g_n(z)\big\} \subset \mathrm{cl}_{\mathcal{F}}\big(\bigcup_{i=1}^n g_i[E]\big) \subset \mathrm{cl}_{\mathcal{F}}(R)$. This finishes the proof of the case $0 < \beta < \alpha$.

For the remainder of the proof we will assume that $\beta = \alpha$. This means that there is no $E \in \mathbf{P}_\alpha$ such that for some $F \in \mathcal{F}$ and $\beta < \alpha$

$$(15) \qquad \big(\forall z \in \pi_\beta[E]\big)\big(\exists R_z \in [P]^{<\omega}\big)\ f\big[\{x \in E : z \subset x\}\big] \subset F * R_z.$$

For every n-ary $F \in \mathcal{F}$ let $F^* = \Big\{\langle x_0, \ldots, x_{n-1}\rangle : F\big(f(x_0), \ldots, f(x_{n-1})\big)\Big\}$ and let $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$. We will apply Proposition 4.1 to find an $\mathcal{F}^*$-independent $E \in \mathbf{P}_\alpha$. Then $Q = f[E]$ is an $\mathcal{F}$-independent subprism of P and together with $R = Q$ they satisfy the lemma.

To see that the assumptions of Proposition 4.1 are satisfied, first notice that unary relations in $\mathcal{F}^*$ are nowhere dense. Indeed, otherwise there is a unary relation $F^* \in \mathcal{F}^*$ and a non-empty clopen set $E \subset F^*$. But then E contradicts (15), as $f[E] \subset F * \emptyset$. Thus, we just need to show that the condition (ex) is satisfied.

So, fix an $F \in \mathcal{F}$. For $0 < \beta < \alpha$ and $B \in \mathcal{B}_\beta$ let

$$K(B) = \Big\{ z \in \mathfrak{C}^\beta : \big(\exists R_z \in [P]^{<\omega}\big)\ f\big[\{z\} \times B\big] \subset F * R_z \Big\}.$$

Clearly $K(B)$ is F_σ. Notice also that it is meager, since otherwise there would exist a non-empty clopen $U \subset K(B)$ and $E = U \times B$ would contradict (15). Thus, each set $K_\beta = \bigcup_{B \in \mathcal{B}_\alpha} K(B)$ is meager. Also, for every $z \in \mathfrak{C}^\beta \setminus K_\beta$ and for every finite $R \subset P$ the set $\big\{ y \in \mathfrak{C}^{\alpha \setminus \beta} : f(z \cup y) \notin F * R \big\}$

is dense and open. In particular, if R is a finite F-independent subset of P then

$$W_R = \left\{ y \in \mathfrak{C}^{\alpha \setminus \beta} : R \cup \left\{ f(z \cup y) \right\} \text{ is } F\text{-independent} \right\} \tag{16}$$

is dense and open. Let

$$H = \bigcap_{0<\beta<\alpha} \left((\mathfrak{C}^{\beta} \setminus K_{\beta}) \times \mathfrak{C}^{\alpha \setminus \beta} \right)$$

and notice that H is comeager since each K_β is meager in $\mathfrak{C}^\beta$. By Lemma 3.3 we can find a comeager set $G \subset H$ such that

$$G_{x \restriction \beta} = \left\{ y \in \mathfrak{C}^{\alpha \setminus \beta} : (x \restriction \beta) \cup y \in G \right\}$$

is comeager for every $x \in G$ and $\beta < \alpha$. To finish the proof it is enough to show that G satisfies (ex) for F^*. So, take an F^*-independent finite set $S \subset G$, an $x \in S$, and a $\beta < \alpha$.

First let us assume that $\beta > 0$. Then $x \in S \subset G \subset H$ implies that $z = x \restriction \beta \in \mathfrak{C}^\beta \setminus K_\beta$. In particular, the set $W_{f[S]}$ from (16) is comeager, and so is $W_{f[S]} \cap G_{x\restriction\beta}$. To get (ex) it is enough to notice that $W_{f[S]} \cap G_{x\restriction\beta}$ is a subset of $\left\{ y \in \mathfrak{C}^{\alpha\setminus\beta} : S \cup \{y \cup z\} \subset G \text{ is } F^*\text{-independent} \right\}$.

Finally assume that $\beta = 0$. We need to show that the set

$$\left\{ y \in G : S \cup \{y\} \text{ is } F^*\text{-independent} \right\}$$

is dense. But this set must be comeager, since otherwise its complement would contain a non-empty clopen set E which would contradict (15) with $\beta = 0$. □

Proof of Lemma 2.1. Let $\mathcal{F} = \mathcal{F}_{\text{lin}}$ be the linear independence family defined in (9) and let $\bar{M} = \langle M_n : n < \omega \rangle$ be an increasing family of compact sets such that $M = \bigcup_{n<\omega} M_n$. Let $\mathcal{F}_{\bar{M}} = \bigcup_{n<\omega} \mathcal{F}_{M_n}$, where each $\mathcal{F}_{M_n}$ is defined in (10), that is, $\mathcal{F}_{M_n}$ is the the collection of all possible projections of the relations from $\mathcal{F}$ along M_n.

If $M \cap P$ is of second category in P then we can choose a subprism Q of P with $Q \subset M$. Then Q and $R = \emptyset$ have the desired properties. On the other hand, if $M \cap P$ is of first category in P then, by Claim 3.2, we can find a subprism P_1 of P disjoint with M.

Now, applying Lemma 4.2 we can find a subprism Q of P_1 and a compact $\mathcal{F}_{\bar{M}}$-independent set $R \subset P_1 \subset P \setminus M$ such that $Q \subset \text{cl}_{\mathcal{F}_{\bar{M}}}(R)$. But then $M \cup R$ is $\mathcal{F}$-independent, see (12). Moreover,

$$Q \subset \text{cl}_{\mathcal{F}_{\bar{M}}}(R) = \text{cl}_{\mathcal{F}}(M \cup R) = \text{LIN}\,(M \cup R).$$

So, $M \cup Q \subset \mathrm{LIN}\,(M \cup R)$ proving that Q and R are as desired. $\square$

5. Remarks

It is worth to notice that in case when $M = \emptyset$ the proof of Lemma 2.1 is easier, and gives a stronger result.

PROPOSITION 5.1. *Every prism P in $\mathbf{R}$ there is a subprism Q which is linearly independent.*

PROOF. This follows from Proposition 4.1 used with $\mathcal{F} = \mathcal{F}_{\mathrm{lin}}$. $\square$

REMARK 5.2. Note that Proposition 5.1 is false if we require Q to be a subcube of prism P, that is, $Q = f[C]$, where C is a perfect cube in $\mathfrak{C}^\alpha$ and $f : \mathfrak{C}^\alpha \to P$ is a coordinate function making P a prism. Indeed, let P_1 and P_2 be disjoint perfect subsets of $\mathbf{R}$ such that $P_1 \cup P_1$ is linearly independent over $\mathbf{Q}$. Let $f : P_1 \times P_2 \to \mathbf{R}$ be defined by the formula $f(x_1, x_2) = x_1 + x_2$. Identifying P_1 and P_2 with $\mathfrak{C}$ we think about f as defined on $\mathfrak{C}^2$ and treat P as a prism. To see that P has no linearly independent subcube let $Q = Q_1 \times Q_2$ be a subcube of P and choose different $a_1, b_1 \in Q_1$ and $a_2, b_2 \in Q_2$. Then $\{a_1 + a_2, a_1 + b_2, b_1 + a_2, b_1 + a_2\} \subset Q$ and they are clearly linearly dependent.

REMARK 5.3. In Lemma 2.1 we cannot require $R = Q$. Namely, let P_1, P_2, and f be as in Remark 5.2. If $M = P_2$ then P has no subprism Q such that $M \cup Q$ is linearly independent, since any vertical section of Q is a translation of a portion of M.

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