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A big symmetric planar set with small category projections

by

Krzysztof Ciesielski (Morgantown, WV) and
Tomasz Natkaniec (Gdańsk)

Abstract. We show that under appropriate set-theoretic assumptions (which follow from Martin's axiom and the continuum hypothesis) there exists a nowhere meager set $A \subset \mathbb{R}$ such that

(i) the set $\{c \in \mathbb{R}: \pi[(f + c) \cap (A \times A)]$ is not meager$\}$ is meager for each continuous nowhere constant function $f: \mathbb{R} \to \mathbb{R}$,
(ii) the set $\{c \in \mathbb{R}: (f + c) \cap (A \times A) = \emptyset\}$ is nowhere meager for each continuous function $f: \mathbb{R} \to \mathbb{R}$.

The existence of such a set also follows from the principle CPA, which holds in the iterated perfect set model. We also prove that the existence of a set A as in (i) cannot be proved in ZFC alone even when we restrict our attention to homeomorphisms of $\mathbb{R}$. On the other hand, for the class of real-analytic functions a Bernstein set A satisfying (ii) exists in ZFC.

1. The results. Let $\mathcal{M}$ denote the class of all meager subsets of $\mathbb{R}$ and let $\pi: \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. For a function $f: \mathbb{R} \to \mathbb{R}$ and a set $E \subset \mathbb{R}^2$ the *f-category projection* of E is the set $\{c \in \mathbb{R}: \pi[(f + c) \cap E] \notin \mathcal{M}\}$. (See [CG].) In papers [Da], [N1], and [N2] their authors considered the second category sets $A \subset \mathbb{R}$ such that an f-category projection of $A \times A$ has an empty interior for every linear function $f(x) = ax + b$. Inspired by these results Bartoszyński and Halbeisen [BH] recently constructed a second category set $A \subset \mathbb{R}$ with an even stronger property: for each polynomial p which is neither constant nor the identity

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function the set $p \cap (A \times A)$ is finite. In particular, if p is a non-constant polynomial then the p-category projection of $A \times A$ has an empty interior.

These results lead to the following, more general question: for which classes $\mathcal{F}$ of continuous functions does there exist a "big" set $A \subset \mathbb{R}$ such that the f-category projection is "small" for every $f \in \mathcal{F}$? In particular, what happens for the classes $\mathcal{A}$ of real-analytic functions, $\mathcal{C}_0$ of all nowhere constant functions from $\mathbb{R}$ to $\mathbb{R}$, and for the entire class $\mathcal{C}$ of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

In this note we answer these questions. We use standard terminology as in [Ci]. We consider only real-valued functions of one variable, unless otherwise specified. No distinction is made between a function and its graph. For functions f, g put $[f = g] = \{x \in \mathbb{R}\colon f(x) = g(x)\}$. A set $A \subset \mathbb{R}$ is *nowhere meager* if $A \cap I \notin \mathcal{M}$ for each non-degenerate interval I. The symbol $B(x, r)$ denotes the open ball centered at x and with radius r, and id stands for the identity function.

The main general theorem in the positive direction is the following result.

THEOREM 1. *Let $\mathcal{J} \supset [\mathbb{R}]^{\le\omega}$ be a translation invariant ideal on $\mathbb{R}$ and $\mathcal{B}_0$ be a family of Borel sets containing all Borel non-meager sets such that*

(1) $$B \setminus \bigcup \mathcal{G} \neq \emptyset \quad \textit{for every } B \in \mathcal{B}_0 \textit{ and } \mathcal{G} \in [\mathcal{J}]^{<\mathfrak{c}}.$$

If $\mathcal{F} \subset \mathcal{C}$ is such that for every $f \in \mathcal{F}$,

(2) $$\textit{the set } L_f = \{y \in \mathbb{R}\colon f^{-1}(y) \notin \mathcal{J}\} \textit{ belongs to } \mathcal{J}$$

then there exists an $A \subset \mathbb{R}$ intersecting every $B \in \mathcal{B}_0$ such that for every $f \in \mathcal{F}$,

(a) *there exists a $Z \in [\mathbb{R}]^{<\mathfrak{c}}$ such that $f \cap (A \times A) \subset (A \times Z) \cup \mathrm{id}$.*

Moreover, *if* (2) *holds for every $f \in (\pm\mathcal{F}) \cup (\mathrm{id} - \mathcal{F})$ then we can also have the following property for every $f \in \mathcal{F}$*:

(b) *the set $\{c \in \mathbb{R}\colon (f + c) \cap (A \times A) = \emptyset\}$ intersects every $B \in \mathcal{B}_0$.*

We leave the proof of this theorem to the next section. In the remainder of this section we discuss several of its corollaries. In particular, applying Theorem 1 to the ideal $[\mathbb{R}]^{\le\omega}$ of countable sets and the family $\mathcal{B}_0$ of all uncountable Borel sets we get the following result.

COROLLARY 2. *Let $\mathcal{F} \subset \mathcal{C}$ be such that the set $L_f = \{y \in \mathbb{R}\colon |f^{-1}(y)| > \omega\}$ is empty for every $f \in \mathcal{F}$. Then there exists a Bernstein set $A \subset \mathbb{R}$ such that for every $f \in \mathcal{F}$,*

(a) *there exists a $Z \in [\mathbb{R}]^{<\mathfrak{c}}$ such that $f \cap (A \times A) \subset (A \times Z) \cup \mathrm{id}$.*

In addition, *if $L_f = \emptyset$ for every $f \in (\pm\mathcal{F}) \cup (\mathrm{id} - \mathcal{F})$ then we can also have*

(b) *$\{c \in \mathbb{R}\colon (f + c) \cap (A \times A) = \emptyset\}$ contains a Bernstein set for every $f \in \mathcal{F}$.*

In particular,

- *there exists a set A satisfying* (a) *for every countable-to-one function $f\in\mathcal{C}$*;
- *there exists a set A satisfying* (a) *and* (b) *for every analytic function $f\in\mathcal{A}$*; *moreover*, *if $f\neq \mathrm{id}$ is not constant then $|f\cap(A\times A)|<\mathfrak{c}$.*

Proof. Clearly the ideal $\mathcal{J}=[\mathbb{R}]^{\le\omega}$ and the family $\mathcal{B}_0$ of all uncountable Borel sets satisfy (1) from Theorem 1. Since we assume that (2) is also satisfied, the first assertion of the corollary follows from Theorem 1. To see the "additional" part for analytic functions first notice that $\mathcal{F}=\mathcal{A}$ satisfies the assumption for (b), since $L_f=\emptyset$ for every non-constant $f\in\mathcal{A}$. In particular, if $f\neq\mathrm{id}$, then by (a) the set $\pi[f\cap(A\times A)]$ is contained in $(f-\mathrm{id})^{-1}(0)\cup\bigcup_{z\in Z}f^{-1}(z)$, which is a union of less than $\mathfrak{c}$ countable sets. ∎

In the proof of the next corollary we need the following simple fact:

(3) If $f\in\mathcal{C}$ then the set $\{c\colon [f+c=\mathrm{id}]\notin\mathcal{M}\}$ is countable.

To see it notice that for each $c\in\mathbb{R}$ the set $[f+c=\mathrm{id}]$ is closed. Therefore, if $[f+c=\mathrm{id}]\notin\mathcal{M}$, then there exists a non-empty open interval I_c with $f(x)=x-c$ for each $x\in I_c$. It is easy to observe that $I_c\cap I_d=\emptyset$ if $c\neq d$.

Corollary 3. *There exists a Bernstein set $A\subset\mathbb{R}$ such that for every homeomorphism $f\colon\mathbb{R}\to\mathbb{R}$ for all but countably many $c\in\mathbb{R}$ the set $\pi[(f+c)\cap(A\times A)]$ is a union of a meager set and a set of cardinality less than $\mathfrak{c}$. In particular*, *if $[\mathbb{R}]^{<\mathfrak{c}}\subset\mathcal{M}$ then the set*

$$\{c\in\mathbb{R}\colon \pi[(f+c)\cap(A\times A)]\notin\mathcal{M}\}$$

is countable (*so meager*) *for every homeomorphism $f\colon\mathbb{R}\to\mathbb{R}$.*

Proof. Let A be as in Corollary 2. So for every homeomorphism $f\colon\mathbb{R}\to\mathbb{R}$ and $c\in\mathbb{R}$ there exists a $Z\in[\mathbb{R}]^{<\mathfrak{c}}$ such that $(f+c)\cap(A\times A)\subset(A\times Z)\cup\mathrm{id}$. Notice that $\pi[(f+c)\cap(A\times A)]\subset[f+c=\mathrm{id}]\cup\pi[(f+c)\cap(A\times Z)]$ and $\pi[(f+c)\cap(A\times Z)]\subset f^{-1}(Z-c)\in[\mathbb{R}]^{<\mathfrak{c}}$. So there exists a $D\in[\mathbb{R}]^{<\mathfrak{c}}$ such that $\pi[(f+c)\cap(A\times A)]\subset[f+c=\mathrm{id}]\cup D$. The conclusion follows from (3). ∎

Notice that the set-theoretic assumption in Corollary 3 is essential.

Theorem 4. *It is relatively consistent with ZFC that for every nowhere meager set $A\subset\mathbb{R}$ there exists a homeomorphism $f\colon\mathbb{R}\to\mathbb{R}$ such that the set $\{c\in\mathbb{R}\colon\pi[(f+c)\cap(A\times A)]\notin\mathcal{M}\}$ is nowhere meager.*

Proof. This follows immediately from Theorem 12 and Proposition 13, which will be proven in Section 4. ∎

Applying Theorem 1 to the ideal $\mathcal{M}$ and the family $\mathcal{B}_0$ of non-meager Borel sets we also get the following result. (Recall that the set-theoretic assumption about covering by meager sets follows form Martin's axiom MA and from the continuum hypothesis CH.)

COROLLARY 5. *Assume that less than continuum many meager sets do not cover* $\mathbb{R}$. *Then there exists a nowhere meager set* $A \subset \mathbb{R}$ *such that for every* $f \in \mathcal{C}$ *the set* $\{c \in \mathbb{R}\colon (f+c) \cap (A \times A) = \emptyset\}$ *is nowhere meager in* $\mathbb{R}$. *In particular*, *the* f*-category projection of* $A \times A$ *has an empty interior.*

Proof. The set-theoretic assumption ensures that the ideal $\mathcal{J} = \mathcal{M}$ and the family $\mathcal{B}_0$ of non-meager Borel sets satisfy assumption (1) of Theorem 1. Since for $\mathcal{J} = \mathcal{M}$, (2) holds for every $f \in \mathcal{C}$, there is a set A satisfying (b) from Theorem 1. Clearly, it has the desired properties. ■

For the class $\mathcal{C}_0$ of nowhere constant functions we have yet another corollary. In its proof we will use the following simple fact:

(4) $$f^{-1}(M) \in \mathcal{M} \quad \text{for every } f \in \mathcal{C}_0 \text{ and } M \in \mathcal{M}.$$

Indeed, if sets F_n are closed and nowhere dense in $\mathbb{R}$ such that $M \subset \bigcup_{n<\omega} F_n$ then $f^{-1}(M) \subset \bigcup_{n<\omega} f^{-1}(F_n)$. It is enough to notice that $f^{-1}(F_n)$ is closed and nowhere dense for every $f \in \mathcal{C}_0$.

COROLLARY 6. *Assume that less than continuum many meager sets do not cover* $\mathbb{R}$ *and that* $[\mathbb{R}]^{<\mathfrak{c}} \subset \mathcal{M}$. *Then there exists a nowhere meager set* $A \subset \mathbb{R}$ *such that for every* $f \in \mathcal{C}_0$ *the set* $\{c \in \mathbb{R}\colon \pi[(f+c) \cap (A \times A)] \notin \mathcal{M}\}$ *is countable.*

Proof. Let $f \in \mathcal{C}_0$ and $c \in \mathbb{R}$. By Theorem 1(a) there exists a $Z \in [\mathbb{R}]^{<\mathfrak{c}}$ such that $\pi[(f+c) \cap (A \times A)] \subset [f + c = \mathrm{id}] \cup \pi[(f+c) \cap (A \times Z)]$. But, by (4), $\pi[(f+c) \cap (A \times Z)] \subset f^{-1}(Z - c) \in \mathcal{M}$ since $Z - c \in [\mathbb{R}]^{<\mathfrak{c}} \subset \mathcal{M}$. Thus, $\pi[(f+c) \cap (A \times A)] \in \mathcal{M}$ as long as $[f + c = \mathrm{id}] \in \mathcal{M}$. So the result follows immediately from (3). ■

We believe that the conclusion of Corollary 6 cannot be proved in ZFC; we state this below as a conjecture. (See also the last section of the paper for some comments on it.)

CONJECTURE 1. *It is relatively consistent with ZFC that for every nowhere meager set* $A \subset \mathbb{R}$ *there is an* $f \in \mathcal{C}_0$ *such that* $\pi[(f+c) \cap (A \times A)] \notin \mathcal{M}$ *for every* $c \in \mathbb{R}$.

It is worth noting that a set A as in Corollaries 5 and 6 can also be constructed under the Covering Property Axiom CPA, which extracts the essence of the iterated perfect set model. (See [CP2, CMP, CP3].) This is of interest, since under CPA the set-theoretic assumptions of each of these corollaries are false: CPA implies that $\mathfrak{c} = \omega_2$ and that $\mathbb{R}$ can be covered by ω_1 meager sets. In fact, in the theorem we will use only a simpler version of CPA known as $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$.

THEOREM 7. *Assume that* $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ *holds. Then there exists a nowhere meager set* $A \subset \mathbb{R}$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that for every* $f \in \mathcal{C}$ *there exists a countable set* $Z \subset A$ *such that*

$$f \cap (A \times A) \subset (A \times Z) \cup \mathrm{id}.$$

In particular, for every $f \in \mathcal{C}_0$ *the set* $\{c \in \mathbb{R}\colon \pi[(f+c) \cap (A \times A)] \notin \mathcal{M}\}$ *is countable and the set* $\{c \in \mathbb{R}\colon (f+c) \cap (A \times A) = \emptyset\}$ *is the complement of a set of cardinality* $\omega_1 < \mathfrak{c}$, *so it contains a Bernstein set.*

Proof. The first assertion will be proved in Section 3. To prove the second assertion, fix an $f \in \mathcal{C}_0$. The proof that $\{c \in \mathbb{R}\colon \pi[(f+c)\cap(A\times A)] \notin \mathcal{M}\}$ is countable is exactly the same as for Corollary 6. To see that the complement D of $\{c \in \mathbb{R}\colon (f+c) \cap (A \times A) = \emptyset\}$ has cardinality ω_1 notice that if $(f+c) \cap (A \times A) \neq \emptyset$ then there are $x, y \in A$ such that $f(x) + c = y$, that is, $c = y - f(x) \in A - f[A]$. So D equals $A - f[A]$ and has cardinality ω_1. ∎

It is also important to notice that the set A in Corollaries 5 and 6 cannot have the Baire property.

PROPOSITION 8. *Suppose that* $f \in \mathcal{C}_0$ *and that* $A \subset \mathbb{R}$ *is a non-meager set having the Baire property. Then the* f*-category projection of* $A \times A$ *has a non-void interior.*

Proof. Let G be a non-empty open set with $M = G \setminus A \in \mathcal{M}$. Fix $x_0 \in G \cap A$ and $c_0 \in \mathbb{R}$ with $f(x_0) + c_0 \in G \cap A$, an $\varepsilon > 0$ such that $B(f(x_0) + c_0, 2\varepsilon) \subset G$, and a $\delta > 0$ such that $f[B(x_0, \delta)] \subset B(f(x_0), \varepsilon)$. Let $c \in B(c_0, \varepsilon)$. Since $A_0 = A \cap B(x_0, \delta) \notin \mathcal{M}$ and $f + c \in \mathcal{C}_0$, condition (4) implies $(f+c)(A_0) \notin \mathcal{M}$. Since $(f+c)(A_0) \subset G$ and $(f+c)(A_0) \cap A \notin \mathcal{M}$, it follows that c belongs to the f-category projection of $A \times A$. ∎

We finish this section with the following result of Bartoszyński and Halbeissen, which was one of the starting points for this note.

THEOREM 9 ([BH]). *There exists a set* $A \subset \mathbb{R}$ *intersecting every perfect set such that for each non-constant polynomial* $p \neq \mathrm{id}$ *the set* $p \cap (A \times A)$ *is finite.*

Note that Corollary 2 implies immediately a weaker version of Theorem 9: there exists a Bernstein set A for which each set $p \cap (A \times A)$ has cardinality less than $\mathfrak{c}$. However, we see no easy way to deduce the full version of the theorem from the results presented above. Nevertheless we wish to include here a very short proof of Theorem 9, since it is considerably simpler and completely different from the argument presented in [BH].

Proof. First notice that if A is a transcendental base of $\mathbb{R}$ (over $\mathbb{Q}$) then the set $p \cap (A \times A)$ is finite for every polynomial p which is neither constant nor the identity function. Indeed, if $K \in [A]^{<\omega}$ is such that $p \in \overline{\mathbb{Q}(K)}[x]$, where $\overline{\mathbb{Q}(K)}$ stands for the algebraic closure of $\mathbb{Q}(K)$ in $\mathbb{R}$, then for every

$a \in A \setminus K$ we have $p(a) \in \overline{\mathbb{Q}(K \cup \{a\})} \setminus \overline{\mathbb{Q}(K)}$, since A is algebraically independent. (See e.g. [Ku, Lemma 2, p. 99].) So if $p(a) \in A$ then $p(a) = a$. But this is impossible, since $p(a) = a$ implies that a is a root of a non-zero polynomial $p - \mathrm{id} \in \overline{\mathbb{Q}(K)}[x]$. So $\pi[p \cap (A \times A)] \subset K$. It is well known that there are transcendental bases A that are also Bernstein sets ([Ci, Corollary 7.3.6 and Exercise 2 on page 126]) and any such base satisfies the conclusion. ∎

2. Proof of Theorem 1. Let $\{\langle f_\alpha, B_\alpha\rangle : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{F} \times \mathcal{B}_0$. For each $\alpha < \mathfrak{c}$ we will choose, by induction on $\alpha < \mathfrak{c}$, points $x_\alpha \in B_\alpha$ and $c_\alpha \in B_\alpha$ aiming for $A = \{x_\alpha : \alpha < \mathfrak{c}\}$. We will set up the induction in such a way that for every $\alpha < \mathfrak{c}$ the set Z satisfying (a) for f_α will be $A_\alpha = \{x_\beta : \beta < \alpha\}$ and, if (2) holds for every $f \in (\pm\mathcal{F}) \cup (\mathrm{id} - \mathcal{F})$, that $(f_\alpha + c_\alpha) \cap (A \times A) = \emptyset$. So assume that for some $\alpha < \mathfrak{c}$ the sets $\{x_\beta : \beta < \alpha\}$ and $\{c_\beta : \beta < \alpha\}$ are already constructed. If we need only ensure (a), we put $c_\alpha = x_\alpha$ and choose an $x_\alpha \in B_\alpha \setminus \bigcup_{\beta \le \alpha} L_{f_\beta}$ such that

$$f_\beta \cap (\{x_\alpha\} \times (A_\alpha \setminus A_\beta)) = \emptyset = f_\beta \cap (A_\alpha \times \{x_\alpha\})$$

for all $\beta \le \alpha$. This is possible by (1), since $\{L_{f_\beta} : \beta \le \alpha\} \subset \mathcal{J}$, the singletons $f_\beta[\{a\}]$ are in $\mathcal{J}$ for $\beta \le \alpha$ and $a \in A_\alpha$, and $\{x \in \mathbb{R} : f_\beta(x) = x_\gamma\} = f_\beta^{-1}(x_\gamma) \in \mathcal{J}$ for every $\beta \le \gamma < \alpha$ since $x_\gamma \notin L_{f_\beta}$. It is easy to see that such a choice implies that $f_\alpha \cap (A \times A) \subset (A \times A_\alpha) \cup \mathrm{id}$ for every $\alpha < \mathfrak{c}$. So in what follows we assume that (2) holds for every $f \in (\pm\mathcal{F}) \cup (\mathrm{id} - \mathcal{F})$. We also assume that the following inductive conditions hold for every $\alpha < \mathfrak{c}$:

(I) $(\mathrm{id} - f_\alpha)^{-1}(c_\alpha) \cup f_\alpha^{-1}(x_\beta - c_\alpha) \in \mathcal{J}$ for every $\beta < \alpha$,
(II) $f_\beta^{-1}(x_\alpha) \cup f_\beta^{-1}(x_\alpha - c_\beta) \in \mathcal{J}$ for every $\beta \le \alpha$.

First choose a $c_\alpha \in B_\alpha \setminus (L_{\mathrm{id}-f_\alpha} \cup \bigcup_{\beta<\alpha}(x_\beta + L_{-f_\alpha}))$ outside the set

$$\{c \in \mathbb{R} : (f_\alpha + c) \cap (A_\alpha \times A_\alpha) \ne \emptyset\} = A_\alpha - f_\alpha[A_\alpha] \in [\mathbb{R}]^{<\mathfrak{c}}.$$

Such a choice is possible by (1), since the singletons and the sets $L_{\mathrm{id}-f_\alpha}$ and $x_\beta + L_{-f_\alpha}$ belong to $\mathcal{J}$. Clearly $c_\alpha \notin L_{\mathrm{id}-f_\alpha}$ ensures that $(\mathrm{id}-f_\alpha)^{-1}(c_\alpha) \in \mathcal{J}$. Similarly, $c_\alpha \notin x_\beta + L_{-f_\alpha}$ implies that $c_\alpha - x_\beta \notin L_{-f_\alpha}$ and hence the set $f_\alpha^{-1}(x_\beta - c_\alpha) = (-f_\alpha)^{-1}(c_\alpha - x_\beta)$ belongs to $\mathcal{J}$. Thus, condition (I) is satisfied. Note also that $c_\alpha \notin A_\alpha - f_\alpha[A_\alpha]$ implies that

(i) $(f_\alpha + c_\alpha) \cap (A_\alpha \times A_\alpha) = \emptyset$.

Next choose an $x_\alpha \in B_\alpha \setminus \bigcup_{\beta \le \alpha}(L_{f_\beta} \cup (c_\beta + L_{f_\beta}))$ such that

$$(f_\beta + c_\beta) \cap (\{x_\alpha\} \times A_\alpha) = \emptyset = (f_\beta + c_\beta) \cap (A_\alpha \times \{x_\alpha\}), \tag{5}$$

$x_\alpha \notin \bigcup_{\gamma \le \alpha}(\mathrm{id} - f_\gamma)^{-1}(c_\gamma)$, and

$$f_\beta \cap (\{x_\alpha\} \times (A_\alpha \setminus A_\beta)) = \emptyset = f_\beta \cap (A_\alpha \times \{x_\alpha\}) \tag{6}$$

for every $\beta \leq \alpha$. This is possible by (1), since $\{L_{f_\beta} \cup (c_\beta + L_{f_\beta})\colon \beta \leq \alpha\} \subset \mathcal{J}$ and by inductive assumptions (I)&(II) for every $a \in A_\alpha$ and $\beta \leq \gamma < \alpha$ the following sets belong to $\mathcal{J}$:

- $\{x \in \mathbb{R}\colon (f_\beta + c_\beta)(x) = a\} = f_\beta^{-1}(a - c_\beta)$;
- $(f_\beta + c_\beta)[\{a\}]$;
- $(\mathrm{id} - f_\gamma)^{-1}(c_\gamma)$;
- $\{x \in \mathbb{R}\colon f_\beta(x) = x_\gamma\} = f_\beta^{-1}(x_\gamma)$;
- $f_\beta[\{a\}]$.

Note that $x_\alpha \notin \bigcup_{\beta \leq \alpha} L_{f_\beta}$ ensures $f_\beta^{-1}(x_\alpha) \in \mathcal{J}$, and $x_\alpha \notin \bigcup_{\beta \leq \alpha}(c_\beta + L_{f_\beta})$ implies that $x_\alpha - c_\beta \notin L_{f_\beta}$, so the set $f_\beta^{-1}(x_\alpha - c_\beta)$ is in $\mathcal{J}$. In particular, (II) is satisfied. This finishes the inductive construction.

Clearly A is nowhere meager, since it meets every non-meager Borel set. To see that (b) holds notice that by (i) we have $(f_\beta + c_\beta) \cap (A_\beta \times A_\beta) = \emptyset$ while $(f_\beta + c_\beta) \cap (A_\alpha \times A_\alpha) = \emptyset$ for $\alpha > \beta$ is ensured by the choice of x_α as in (5) and the fact that $(f_\beta + c_\beta)(x_\alpha) \neq x_\alpha$ since $x_\alpha \notin (\mathrm{id} - f_\beta)^{-1}(c_\beta)$. To see that (a) holds pick an $f \in \mathcal{F}$ and let $\beta < \mathfrak{c}$ be such that $f = f_\beta$. The choice of x_α for $\alpha > \beta$ as in (6) implies that $f \cap (A \times A) \subset (A \times A_\beta) \cup \mathrm{id}$. ■

3. Set A from CPA. To formulate axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ we need a few definitions. Let $\mathfrak{C}$ denote the Cantor set 2^ω. For a Polish space X we use $\mathrm{Perf}(X)$ to denote the family of all subsets of X homeomorphic to $\mathfrak{C}$. A subset C of a product $\mathfrak{C}^\omega$ of the Cantor set is said to be a *perfect cube* if $C = \prod_{n\in\omega} C_n$, where $C_n \in \mathrm{Perf}(\mathfrak{C})$ for each n. For a fixed Polish space X let $\mathcal{F}_{\mathrm{cube}}$ stand for the family of all continuous injections from a perfect cube $C \subset \mathfrak{C}^\omega$ onto a set P from $\mathrm{Perf}(X)$. We consider each function $f \in \mathcal{F}_{\mathrm{cube}}$ from C onto P as a coordinate system imposed on P.

We say that $P \in \mathrm{Perf}(X)$ is a *cube* if we consider it with (implicitly given) witness function $f \in \mathcal{F}_{\mathrm{cube}}$ onto P, and Q is a *subcube* of a cube $P \in \mathrm{Perf}(X)$ provided $Q = f[C]$, where $f \in \mathcal{F}_{\mathrm{cube}}$ is a witness function for P and $C \subset \mathrm{dom}(f) \subset \mathfrak{C}^\omega$ is a perfect cube. (Here and in what follows $\mathrm{dom}(f)$ stands for the domain of f.)

We say that a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *cube dense* in $\mathrm{Perf}(X)$ if every cube $P \in \mathrm{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \mathrm{Perf}(X)$ is cube dense provided

$$\forall f \in \mathcal{F}_{\mathrm{cube}}\ \exists g \in \mathcal{F}_{\mathrm{cube}}\ (g \subset f\ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{7}$$

We also need a notion of a *constant cube*: the family $\mathcal{C}_{\mathrm{cube}}(X)$ of constant "cubes" is defined as the family of all constant functions from a perfect cube $C \subset \mathfrak{C}^\omega$ to X. We define $\mathcal{F}^*_{\mathrm{cube}}(X)$ as

$$\mathcal{F}^*_{\mathrm{cube}} = \mathcal{F}_{\mathrm{cube}} \cup \mathcal{C}_{\mathrm{cube}}. \tag{8}$$

Thus, $\mathcal{F}^*_{\rm cube}$ is the family of all continuous functions from a perfect cube $C \subset \mathfrak{C}^\omega$ into X which are either one-to-one or constant. Now the range of every $f \in \mathcal{F}^*_{\rm cube}$ belongs to the family $\mathrm{Perf}^*(X)$ of all sets P such that either $P \in \mathrm{Perf}(X)$ or P is a singleton. The meaning of "$P \in \mathrm{Perf}^*(X)$ is a cube" and "Q is a subcube of a cube $P \in \mathrm{Perf}^*(X)$" is defined in a natural way.

For a Polish space X consider the following game $\mathrm{GAME}_{\rm cube}(X)$ of length ω_1. The game has two players, I and II. At each stage $\xi < \omega_1$ of the game Player I can play an arbitrary cube $P_\xi \in \mathrm{Perf}^*(X)$ and Player II must respond with a subcube Q_ξ of P_ξ. The game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ is won by Player I provided

$$\bigcup_{\xi<\omega_1} Q_\xi = X;$$

otherwise the game is won by Player II. A *strategy* for Player II is any function S such that $S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi)$ is a subcube of P_ξ, where $\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle$ is any partial game. (We abuse here slightly the notation, since the function S also depends on the implicitly given coordinate functions $f_\eta : \mathfrak{C}^\omega \to P_\eta$ making each P_η a cube.) A game $\langle\langle P_\xi, Q_\xi\rangle : \xi < \omega_1\rangle$ is played according to a strategy S provided $Q_\xi = S(\langle\langle P_\eta, Q_\eta\rangle : \eta < \xi\rangle, P_\xi)$ for every $\xi < \omega_1$. A strategy S for Player II is a *winning strategy* for Player II provided Player II wins any game played according to S. Now we can formulate the following axiom (see [CP3]):

$\mathrm{CPA}^{\rm game}_{\rm cube}$: $\mathfrak{c} = \omega_2$ and for any Polish space X Player II has no winning strategy in the game $\mathrm{GAME}_{\rm cube}(X)$.

All we need to know about cube-dense families is the following fact.

FACT 10. *Let X be a Polish space and let $\mathcal{E} \subset \mathrm{Perf}^*(X)$ contain all singletons. If for every $P \in \mathrm{Perf}(X)$ and every Borel probability measure μ on P there exists a $Q \in \mathrm{Perf}(P) \cap \mathcal{E}$ such that $\mu(Q) > 0$, then $\mathcal{E}$ is cube-dense.*

Proof. This follows immediately from [CP1, Claim 3.2]. (See also [CP3, Claim 1.1.5] or [CP2, Claim 2.3].) ∎

We will apply this fact to $X = \mathcal{C}$, where $\mathcal{C}$ is considered with the sup norm. Notice that for every $Q \subset \mathcal{C}$ the set $\bigcup Q \subset \mathbb{R}^2$ is the union of the graphs of all functions belonging to Q, since functions are identified with their graphs. In what follows, for a set $K \subset \mathbb{R}^2$ and $x \in \mathbb{R}$ we denote by K_x the vertical section of K above x, that is, $K_x = \{y : \langle x, y\rangle \in K\}$. Similarly, $K^x = \{x : \langle x, y\rangle \in K\}$. For $A \in [\mathbb{R}]^{\le\omega}$ let $\mathcal{E}(A)$ be the family of all $Q \in \mathrm{Perf}^*(\mathcal{C})$ such that

- $\bigcup Q$ is nowhere dense in $\mathbb{R}^2$,
- $[\bigcup Q]_x$ is nowhere dense in $\mathbb{R}$ for every $x \in A$.

LEMMA 1. *The family $\mathcal{E}(A)$ is cube-dense for every $A \in [\mathbb{R}]^{\leq\omega}$.*

Proof. Without loss of generality we can assume that A is dense in $\mathbb{R}$. Clearly every singleton belongs to $\mathcal{E}(A)$. So let $P \in \mathrm{Perf}(\mathcal{C})$ and let μ be a Borel probability measure μ on P. By Fact 10 it is enough to show that there exists a $Q \in \mathrm{Perf}(P) \cap \mathcal{E}(A)$ such that $\mu(Q) > 0$. To see this, fix a countable base $\mathcal{B}$ for $\mathbb{R}$ and let $\langle\langle a_n, J_n\rangle: n < \omega\rangle$ be an enumeration of $A \times \mathcal{B}$. Notice that for every $n < \omega$ there exists a non-empty open set $U_n \subset J_n$ such that

$$\mu(\{f \in P\colon f(a_n) \in U_n\}) < 2^{-(n+2)}. \tag{9}$$

Indeed, if $\mathcal{U}_n$ is an infinite family of non-empty pairwise disjoint open subsets of J_n then for each $U \in \mathcal{U}_n$ the set $\{f \in P\colon f(a_n) \in U\}$ is open in P (so μ-measurable) and so condition (9) must hold for some $U \in \mathcal{U}_n$. Let $W = \bigcup_{n<\omega}\{f \in \mathcal{C}\colon f(a_n) \in U_n\}$. It is clear that W is open and dense in $\mathcal{C}$. So $Q = P \setminus W = P \setminus \bigcup_{n<\omega}\{f \in P\colon f(a_n) \in U_n\}$ is nowhere dense (and therefore $\bigcup Q$ is nowhere dense in $\mathbb{R}^2$), and by (9), it has μ-measure at least $1 - \sum_{n<\omega} 2^{-(n+2)} = 2^{-1} > 0$. It is also clear that for every $x \in A$ the set $\bigcup\{U_n\colon a_n = x\}$ is dense open in $\mathbb{R}$ and it is disjoint from $[\bigcup Q]_x$. Thus $Q \in \mathcal{E}(A)$. ■

PROPOSITION 11. *Assume that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ holds, let X be a Polish space, and let S be a mapping associating to every $\overline{P} \in \bigcup_{\alpha<\omega_1}(\mathrm{Perf}^*(X))^\alpha$ a cube-dense family $\mathcal{E}(\overline{P}) \subset \mathrm{Perf}^*(X)$. Then there exists a sequence $\langle\langle P_\xi, Q_\xi\rangle: \xi < \omega_1\rangle$ such that $Q_\xi \in \mathrm{Perf}^*(P_\xi) \cap \mathcal{E}(\langle P_\zeta\colon \zeta < \xi\rangle)$ for every $\xi < \omega_1$ and $X = \bigcup_{\xi<\omega_1} Q_\xi$.*

Proof. This follows easily from $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$. More precisely, it is enough to apply $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ to the strategy S^* such that $S^*(\langle\langle P_\eta, Q_\eta\rangle: \eta < \xi\rangle, P_\xi)$ is a subcube of P_ξ from $S(\langle P_\eta\colon \eta < \xi\rangle)$. ■

Proof of Theorem 7. First recall that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ implies that the cofinality of the ideal of meager sets is equal to $\omega_1 < \mathfrak{c}$, that is, there exists an $\mathcal{M}_0 \in [\mathcal{M}]^{\omega_1}$ such that every meager set is contained in some $M \in \mathcal{M}_0$. (See e.g. [CP1, Sec. 4] or [CP3].) Let $\mathcal{B}_0$ be a countable base for $\mathbb{R}$ and let $\{\langle M_\xi, J_\xi\rangle: \xi < \omega_1\}$ be an enumeration of $\mathcal{M}_0 \times \mathcal{B}_0$. By simultaneous induction on $\xi < \omega_1$, using Lemma 1, we will define functions S, Q, and k on $(\mathrm{Perf}^*(\mathcal{C}))^\xi$ such that

(i) $S(\langle P_\zeta\colon \zeta < \xi\rangle) = \mathcal{E}(\{a_\zeta\colon \zeta < \xi\})$, where $a_\zeta = k(\langle P_\eta\colon \eta \leq \zeta\rangle) \in \mathbb{R}$, and $Q_\xi = Q(\langle P_\zeta\colon \zeta < \xi\rangle) \in \mathcal{E}(\{a_\zeta\colon \zeta < \xi\})$,

(ii) $k(\langle P_\zeta\colon \zeta \leq \xi\rangle)$ belongs to J_ξ and to the residual set

$$\bigcap_{\zeta\leq\xi}\{z \in \mathbb{R}\colon (\textstyle\bigcup Q_\zeta)^z \text{ and } (\bigcup Q_\zeta)_z \text{ are nowhere dense in } \mathbb{R}\},$$

(iii) $k(\langle P_\zeta\colon \zeta\le\xi\rangle)$ does not belong to the meager set

$$M_\xi \cup \bigcup_{\eta\le\xi} \left(\{(\bigcup Q_\eta)_{a_\zeta}\colon \zeta<\xi\} \cup \{(\bigcup Q_\eta)^{a_\zeta}\colon \eta\le\zeta<\xi\}\right).$$

The set as in (ii) is residual by the Kuratowski–Ulam theorem, since each set $\bigcup Q_\zeta$ is nowhere dense, as Q_ζ belongs to some $\mathcal{E}(A)$. In (iii) for every $\eta\le\zeta<\xi$ the set $(\bigcup Q_\eta)_{a_\zeta}\cup(\bigcup Q_\eta)^{a_\zeta}$ is nowhere dense by the choice of $a_\zeta = k(\langle P_\eta\colon \eta\le\zeta\rangle)$ as in (ii). Finally, for $\zeta<\eta$ the set $(\bigcup Q_\eta)_{a_\zeta}$ is nowhere dense since, by (i), Q_η belongs to $S(\langle P_\zeta\colon \zeta<\eta\rangle)=\mathcal{E}(\{a_\zeta\colon \zeta<\eta\})$.

Now, by axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ and Proposition 11, there exists a sequence $\langle\langle P_\xi, Q_\xi, a_\xi\rangle\colon \xi<\omega_1\rangle$ such that $\mathcal{C}=\bigcup_{\xi<\omega_1} Q_\xi$ and conditions (i)–(iii) are satisfied. We claim that $A=\{a_\xi\colon \xi<\omega_1\}$ satisfies the conclusion of Theorem 7.

Clearly, A is nowhere meager since for every non-empty open set $U\subset\mathbb{R}$ and every meager set M there exists a $\xi<\omega_1$ such that $J_\xi\subset U$ and $M\subset M_\xi$. But then $a_\xi\in(A\cap J_\xi)\setminus M_\xi\subset(A\cap U)\setminus M$, so $A\cap U\ne M$. To see the first assertion of Theorem 7 take an $f\in\mathcal{C}$. Then there exists an $\eta<\omega_1$ such that $f\in Q_\eta$. We claim that for $Z=\{a_\beta\colon \beta<\eta\}$ we have $f\cap(A\times A)\subset(A\times Z)\cup\mathrm{id}$. Indeed, let $\eta\le\xi<\omega_1$ and $\zeta<\omega_1$ be such that $\zeta\ne\xi$. We need to show that $\langle a_\zeta, a_\xi\rangle\notin f$. But if $\zeta<\xi$ then, by (iii), a_ξ does not belong to $[\bigcup Q_\eta]_{a_\zeta}\ni f(a_\zeta)$, so $\langle a_\zeta, a_\xi\rangle\notin f$. Similarly, if $\xi<\zeta$ then, again by (iii), a_ζ does not belong to $[\bigcup Q_\eta]^{a_\xi}\supset f^{-1}(a_\xi)$ and once more $\langle a_\zeta, a_\xi\rangle\notin f$. ∎

4. Main consistency result. The main goal of this section is to prove the following theorem.

THEOREM 12. *It is relatively consistent with ZFC that $\mathfrak{c}=\omega_2$ and the following two conditions hold simultaneously*:

(A) *For every family $\{B_\xi\colon \xi<\omega_1\}$ of pairwise disjoint nowhere meager subsets of $\mathbb{R}^2$ there exists an increasing homeomorphism $f\colon\mathbb{R}\to\mathbb{R}$ such that $\pi[f\cap B_\xi]$ is nowhere meager for every $\xi<\omega_1$.*

(B) *Every nowhere meager set $B\subset\mathbb{R}$ contains a nowhere meager subset of cardinality ω_1.*

We use Theorem 12, in conjunction with the following proposition, to deduce Theorem 4.

PROPOSITION 13. *Assume that $\mathfrak{c}>\omega_1$ and that* (A) *and* (B) *hold. Then for every nowhere meager set $A\subset\mathbb{R}$ there is a homeomorphism $f\colon\mathbb{R}\to\mathbb{R}$ such that $\{c\in\mathbb{R}\colon \pi[(f+c)\cap(A\times A)]\notin\mathcal{M}\}$ is nowhere meager.*

Proof. By (B) we can assume that $|A|=\omega_1<\mathfrak{c}$. Let B be a Bernstein set such that $(b+A)\cap(b'+A)=\emptyset$ for any distinct $b,b'\in B$. Let $\{c_\xi\colon \xi<\omega_1\}$

be a nowhere meager subset of B, let $B_\xi = A \times (c_\xi + A)$ for $\xi < \omega_1$, and let $f\colon \mathbb{R} \to \mathbb{R}$ be as in (A). Then $\{c \in \mathbb{R}\colon \pi[(f+c)\cap(A\times A)] \notin \mathcal{M}\}$ contains a nowhere meager set $\{-c_\xi\colon \xi < \omega_1\}$ since for every $\xi < \omega_1$ we have

$$\pi[(f-c_\xi)\cap(A\times A)] = \pi[f\cap B_\xi] \notin \mathcal{M},$$

finishing the proof. ∎

The proof of Theorem 12 is a slight modification of the proof of the main result (Theorem 2) from [CS]. Also, Theorem 12 easily implies [CS, Theorem 2]. We will use the terminology and notation of [CS]. In particular, according to the machinery used in that paper, Theorem 12 follows in a standard way from the following lemma. (More precisely, condition (A) is ensured by the lemma, while (B) and $\mathfrak{c} = \omega_2$ are guaranteed by the iteration procedure.)

Lemma 2. *For every family $\mathcal{B} = \{B_\xi\colon \xi < \omega_1\}$ of pairwise disjoint nowhere meager subsets of $\mathbb{R}^2$ and for every ω_1-oracle $\mathcal{M}$ there exists an $\mathcal{M}$-cc forcing notion $Q_{\mathcal{B}}$ of cardinality ω_1 such that $Q_{\mathcal{B}}$ forces*

> *there exists an increasing homeomorphism $f\colon \mathbb{R}\to\mathbb{R}$ such that $\pi[f\cap B_\xi]$ is nowhere meager for every $\xi<\omega_1$.*

In what follows we present the proof of Lemma 2. Let

$$\Gamma = \{\lambda < \omega_1\colon \lambda \text{ is a limit ordinal}\}.$$

Recall that an ω_1-*oracle* is any sequence $\mathcal{M} = \langle M_\delta\colon \delta\in\Gamma\rangle$, where M_δ is a countable transitive model of ZFC$^-$ (that is, ZFC without the power set axiom) with the property that $\delta+1 \subset M_\delta$, δ is countable in M_δ, and the set $\{\delta\in\Gamma\colon A\cap\delta\in M_\delta\}$ is stationary in ω_1 for every $A\subset\omega_1$. With each ω_1-oracle $\mathcal{M} = \langle M_\delta\colon \delta\in\Gamma\rangle$ there is associated a filter $D_{\mathcal{M}}$ generated by the sets $I_{\mathcal{M}}(A) = \{\delta\in\Gamma\colon A\cap\delta\in M_\delta\}$ for $A\subset\omega_1$. It is proved in [Sh, Claim 1.4] that $D_{\mathcal{M}}$ is a proper normal filter containing every closed unbounded subset of Γ. We will also need the following fact, which, for our purposes, can be viewed as the definition of the $\mathcal{M}$-cc property.

Fact 14 ([CS, Fact 4]). *Let P be a forcing notion of cardinality $\le\omega_1$, $e\colon P\to\omega_1$ be one-to-one, and $\mathcal{M} = \langle M_\delta\colon \delta\in\Gamma\rangle$ be an ω_1-oracle. If there exists a $C\in D_{\mathcal{M}}$ such that for every $\delta\in\Gamma\cap C$,*

> *$e^{-1}(E)$ is predense in P for every set $E\in M_\delta\cap\mathcal{P}(\delta)$ for which $e^{-1}(E)$ is predense in $e^{-1}(\{\gamma\colon \gamma<\delta\})$,*

then P has the $\mathcal{M}$-cc property. ∎

Let $\mathcal{K}$ be the family of all sequences $\bar{h} = \langle h_\xi\colon \xi\in\Gamma\rangle$ such that each h_ξ is a function from a countable set $D_\xi\subset\mathbb{R}$ onto $R_\xi\subset\mathbb{R}$ such that h_ξ is dense in $\mathbb{R}^2$ and

$$D_\xi\cap D_\eta = R_\xi\cap R_\eta = \emptyset \quad \text{for any distinct } \xi,\eta\in\Gamma.$$

For each $\bar{h} \in \mathcal{K}$ we will define a forcing notion $Q_{\bar{h}}$. The forcing $Q_{\mathcal{B}}$ required in Lemma 2 will be chosen as $Q_{\bar{h}}$ for some $\bar{h} \in \mathcal{K}$. So let $\mathcal{H}$ be the family of all strictly increasing functions from finite subsets of $\mathbb{R}$ into $\mathbb{R}$ and fix an $\bar{h} \in \mathcal{K}$. Then $Q_{\bar{h}}$ is defined as

$$Q_{\bar{h}} = \Big\{ h \in \mathcal{H}\colon h \subset \bigcup_{\xi \in \Gamma} h_\xi \ \&\ |h \cap h_\xi| \le 1 \text{ for every } \xi \in \Gamma \Big\}$$

and is ordered by reverse inclusion. In what follows we will use the following basic property of $Q_{\bar{h}}$.

Fact 15. *Let $\bar{h} = \langle h_\xi\colon \xi \in \Gamma\rangle \in \mathcal{K}$ and $f = \bigcup H$, where H is a V-generic filter over $Q_{\bar{h}}$. Then f is a strictly increasing function from a dense subset D of $\mathbb{R}$ onto a dense subset of $\mathbb{R}$. In particular, f can be uniquely extended to an increasing homeomorphism $\widetilde{f}$ of $\mathbb{R}$.*

Proof. Clearly f is a strictly increasing function from a subset D of $\mathbb{R}$ onto a subset R of $\mathbb{R}$. Thus, it is enough to show that D and R are dense in $\mathbb{R}$. So let $U \neq \emptyset$ be open in $\mathbb{R}$ and notice that the set

$$D = \{h \in Q_{\bar{h}}\colon \operatorname{dom}(h) \cap U \neq \emptyset\}$$

is dense in $Q_{\bar{h}}$. Indeed, if $h_0 \in Q_{\bar{h}}$ is such that $\operatorname{dom}(h_0) \cap U = \emptyset$ then we can find $\xi \in \Gamma$ such that $h \cap h_\xi = \emptyset$. Since the graph of h_ξ is dense in $\mathbb{R}^2$ we can find $\langle x, y\rangle \in h_\xi$ such that $x \in U$ and $h = h_0 \cup \{\langle x, y\rangle\}$ is strictly increasing. Then $h \in D$ extends h_0. Similarly we can prove that the set

$$\{h \in Q_{\bar{h}}\colon \operatorname{range}(h) \cap U \neq \emptyset\}$$

is dense in $Q_{\bar{h}}$. The rest follows from the genericity of H. ■

Now let $\mathcal{B} = \{B_\xi\colon \xi < \omega_1\}$ be as in the lemma and fix an ω_1-oracle $\mathcal{M} = \langle M_\delta\colon \delta \in \Gamma\rangle$. By Fact 15 in order to prove Lemma 2 it is enough to find an $\bar{h} = \langle h_\xi\colon \xi \in \Gamma\rangle \in \mathcal{K}$ such that

$$Q_{\mathcal{B}} = Q_{\bar{h}} \text{ is } \mathcal{M}\text{-cc} \tag{10}$$

and $Q_{\bar{h}}$ forces that, in $V[H]$,

$$\text{the set } \pi(f \cap B_\xi) \text{ is nowhere meager for every } \xi < \omega_1, \tag{11}$$

where the function f is as in Fact 15. To define $\bar{h}$ we will construct a sequence $\langle\langle x_\alpha, y_\alpha\rangle \in \mathbb{R}^2\colon \alpha < \omega_1\rangle$ aiming at $h_\xi = \{\langle x_{\xi+n}, y_{\xi+n}\rangle\colon n < \omega\}$, where $\xi \in \Gamma$. So let $\mathcal{U} \not\ni \emptyset$ be a standard countable basis for $\mathbb{R}$ and for every $\xi \in \Gamma$ let $\langle\langle U_n^\xi, V_n^\xi, \zeta_n^\xi\rangle\colon n < \omega\rangle$ be a fixed enumeration of $\mathcal{U} \times \mathcal{U} \times \xi$. Points $\langle x_{\xi+n}, y_{\xi+n}\rangle$ are chosen inductively in such a way that

(i) $\langle x_{\xi+n}, y_{\xi+n}\rangle$ is a Cohen real over $M_\delta[\langle\langle x_\alpha, y_\alpha\rangle\colon \alpha < \xi+n\rangle]$ for every $\delta \le \xi$, $\delta \in \Gamma$, that is, $\langle x_{\xi+n}, y_{\xi+n}\rangle$ is outside all meager subsets of $\mathbb{R}^2$ which are coded in $M_\delta[\langle\langle x_\alpha, y_\alpha\rangle\colon \alpha < \xi+n\rangle]$;

(ii) $\langle x_{\xi+n}, y_{\xi+n}\rangle \in (U_n^\xi \times V_n^\xi) \cap B_{\zeta_n^\xi}$.

The choice of $\langle x_{\xi+n}, y_{\xi+n}\rangle$ is possible since the sets $(U_n^\xi \times V_n^\xi) \cap B_{\zeta_n^\xi}$ are non-meager and each time we need to avoid only countably many meager sets. Condition (ii) guarantees that the graph of each h_ξ will be dense in $\mathbb{R}^2$. Note also that if $\Gamma \ni \delta \le \alpha_0 < \ldots < \alpha_{k-1}$, where $k < \omega$, then (by the product lemma in M_δ)

$$\langle\langle x_{\alpha_i}, y_{\alpha_i}\rangle : i < k\rangle \text{ is an } M_\delta\text{-generic Cohen real in } (\mathbb{R}^2)^k. \tag{12}$$

For $h \in \mathcal{H}$ and $0 < k < \omega$ let $U(h,k)$ stand for the set of all sequences $\langle\langle a_i, b_i\rangle \notin h : i < k\rangle \in (\mathbb{R}^2)^k$ such that $h \cup \{\langle a_i, b_i\rangle : i < k\} \in \mathcal{H}$. Clearly $U(h,k)$ is an open subset of $(\mathbb{R}^2)^k$. In fact, it can be easily proved that if $h = \{\langle x_j, y_j\rangle : 0 < j < m\}$, where $x_1 < \ldots < x_{m-1}$, then $\langle\langle a_i, b_i\rangle : i < k\rangle$ belongs to $U(h,k)$ if and only if $\{\langle a_i, b_i\rangle : i < k\} \in \mathcal{H}$ is disjoint from h and

$$\{\langle a_i, b_i\rangle : i < k\} \subset \bigcup_{j<m} (x_j, x_{j+1}) \times (y_j, y_{j+1}),$$

where $x_0 = y_0 = -\infty$ and $x_m = y_m = \infty$. In particular, if $0 < j_0 < \ldots < j_{k-1} < m$ and sets $W_i \ni \langle x_{j_i}, y_{j_i}\rangle$, $i < k$, are open in $\mathbb{R}^2$ then there are non-empty open sets $V_i \subset W_i$ such that

$$\prod_{i<k} V_i \subset U(h,k). \tag{13}$$

For $\delta \in \Gamma$ let $(Q_{\bar{h}})^\delta = \{h \in Q_{\bar{h}} : h \subset \bigcup_{\zeta<\delta} h_\zeta\}$.

FACT 16. *Let $\delta \in \Gamma$ and let $E \in M_\delta$ be a predense subset of $(Q_{\bar{h}})^\delta$. Then for every $k < \omega$ and $h \in (Q_{\bar{h}})^\delta$ the open set*

$$B_h^k = \bigcup\{U(g,k) : g \in (Q_{\bar{h}})^\delta \text{ extends } h \text{ and some } h_0 \in E\} \tag{14}$$

is dense in $U(h,k)$.

Proof. Let $\langle\langle a_i, b_i\rangle : i < k\rangle \in U(h,k) \subset (\mathbb{R}^2)^k$ and let W be an open subset of $U(h,k)$ containing $\langle\langle a_i, b_i\rangle : i < k\rangle$. We need to show that W intersects $U(g,k)$ for some $g \in Q_{\bar{h}}$ extending h and an $h_0 \in E$. Decreasing W if necessary, we can assume that it is of the form $\prod_{i<k} W_i$. Since $h_1 = h \cup \{\langle a_i, b_i\rangle : i < k\} \in \mathcal{H}$, by (13) there are open sets $V_i \subset W_i$ for which $\prod_{i<k} V_i \subset U(h,k) \cap W$. In particular, for any choice of points $\langle c_i, d_i\rangle \in V_i$ we have $h_1 \cup \{\langle c_i, d_i\rangle : i < k\} \in \mathcal{H}$. Since all functions h_ξ are dense in $\mathbb{R}^2$, we can choose points $\langle c_i, d_i\rangle$ from distinct functions h_ξ in such a way that $h_2 = h \cup \{\langle c_i, d_i\rangle : i < k\} \in (Q_{\bar{h}})^\delta$. Since E is predense in $(Q_{\bar{h}})^\delta$, there exists a $g \in (Q_{\bar{h}})^\delta$ extending $h_2 \le h$ and some $h_0 \in E$. But $\{\langle c_i, d_i\rangle : i < k\} \subset g$, so by (13), there are non-empty open sets $U_i \subset V_i$ for which $\prod_{i<k} U_i \subset U(g,k) \cap W$. ∎

Now we are ready to prove (10), that is, that $Q_{\bar{h}}$ is $\mathcal{M}$-cc. So fix a bijection e: $Q_{\bar{h}} \to \omega_1$ and let

$$C = \{\delta \in \Gamma\colon (Q_{\bar{h}})^\delta = e^{-1}(\delta) \in M_\delta\}.$$

Then $C \in D_{\mathcal{M}}$. (See e.g. [Sh, Claim 1.4(4)].) Take a $\delta \in C$ and fix an $E \subset \delta$, $E \in M_\delta$, for which $e^{-1}(E)$ is predense in $(Q_{\bar{h}})^\delta$. By Fact 14 it is enough to show that

$$e^{-1}(E) \text{ is predense in } Q_{\bar{h}}.$$

So take h_0 from $Q_{\bar{h}}$, put $h = h_0 \restriction \bigcup_{\eta<\delta} D_\eta$ and $h_1 = h_0 \setminus h$, and notice that the condition h belongs to $(Q_{\bar{h}})^\delta$. Assume that $h_1 = \{\langle x_i, y_i\rangle\colon i < k\}$, where $x_0 < \ldots < x_{k-1}$. So $\langle\langle x_i, y_i\rangle\colon i < k\rangle \in U(h,k)$. By Fact 16 the set $U(h,k) \setminus B^k_h$ is nowhere dense and belongs to M_δ (as it is defined from $(Q_{\bar{h}})^\delta \in M_\delta$). Hence, by (12), $\langle\langle x_i, y_i\rangle\colon i < k\rangle$ cannot belong to this set, so $\langle\langle x_i, y_i\rangle\colon i < k\rangle \in B^k_h$. In particular, there is a $g \in (Q_{\bar{h}})^\delta$ extending h and some $h_0 \in e^{-1}(E)$ such that $\langle\langle x_i, y_i\rangle\colon i < k\rangle \in U(g,k)$. But then $g \cup h_1$ belongs to $Q_{\bar{h}}$ and extends h and h_0. This finishes the proof of (10). The proof of (11) is similar.

So fix a $\zeta < \omega_1$. We will prove that $\pi(f \cap B_\zeta)$ is nowhere meager in $\mathbb{R}$. Suppose not. Then there exists a $U^* \in \mathcal{U}$ such that $\pi(f \cap B_\zeta)$ is meager in U^*. Let a condition $h^* \in Q_{\bar{h}}$ and $Q_{\bar{h}}$-names $\dot{U}_m$, for $m < \omega$, be such that

$$h^* \Vdash_{Q_{\bar{h}}} \text{“each } \dot{U}_m \text{ is an open dense subset of } U^* \text{ and } \pi(f \cap B_\zeta) \cap \textstyle\bigcap_{m<\omega} \dot{U}_m = \emptyset.\text{”}$$

For each $m < \omega$, since h^* forces that $\dot{U}_m$ is an open dense subset of U^*, for every subset $U \in \mathcal{U}$ of U^* there is a subset $V \in \mathcal{U}$ of U and a maximal antichain $\langle h^m_{V,k}\colon k < \kappa^m_V\rangle$ in $Q_{\bar{h}}$ such that each $h^m_{V,k}$ forces that $V \subset \dot{U}_m$. Note that each of these antichains must be countable, since the forcing notion $Q_{\bar{h}}$ is $\mathcal{M}$-cc and therefore ccc. Combining all these antichains we find a $\mathcal{V} \subset \mathcal{U}$ and a sequence $\langle h^m_{V,k} \in Q_{\bar{h}}\colon m < \omega,\ V \in \mathcal{V},\ k < \kappa^m_V\rangle$ such that

- $\kappa^m_V \le \omega$,
- $h^m_{V,k} \Vdash_{Q_{\bar{h}}}$ “$V \subseteq \dot{U}_m$”,
- for every $m < \omega$, $h \in Q_{\bar{h}}$ extending h^*, and every subset $U \in \mathcal{U}$ of U^* there is a subset $V \in \mathcal{V}$ of U and a $k < \kappa^m_V$ such that the conditions h and $h^m_{V,k}$ are compatible.

Note that for sufficiently large $\delta \in \Gamma$ we have $h^m_{V,k} \in (Q_{\bar{h}})^\delta$ for all $m < \omega$, $V \in \mathcal{V}$, and $k < \kappa^m_V$. Now, by the definition of ω_1-oracle, the set B_0 of all $\delta \in \Gamma$ for which

$$\langle h^m_{V,k} \in Q_{\bar{h}}\colon m < \omega,\ V \in \mathcal{V},\ k < \kappa^m_V\rangle \in M_\delta \quad \text{and} \quad (Q_{\bar{h}})^\delta \in M_\delta$$

is stationary in ω_1. Thus, choose a $\delta > \zeta$ from B_0 and let $U, W \in \mathcal{U}$ be such that $U \subset U^*$ and $h^* \cup \{\langle x, y\rangle\} \in \mathcal{H}$ for every $\langle x, y\rangle \in U \times W$. Using clause

(ii) of the choice of x_α's we may find an $n<\omega$ such that $\langle x_{\delta+n}, y_{\delta+n}\rangle \in (U\times W)\cap B_\zeta$. Then $h_0 = h^*\cup\{\langle x_{\delta+n}, y_{\delta+n}\rangle\}\in Q_{\bar h}$ extends h^* and $h_0 \Vdash$ "$x_{\delta+n}\in U^*\cap\pi(f\cap B_\zeta)$" since $\langle x_{\delta+n}, y_{\delta+n}\rangle\in f$. We will show that

$$h_0 \Vdash \text{“}x_{\delta+n}\in \bigcap_{m<\omega}\dot U_m\text{”},$$

which contradicts the choice of h^*. Assume that this is not the case. Then there exist $i<\omega$ and $h_1\in Q_{\bar h}$ stronger than h_0 such that $h_1\Vdash$ "$x_{\delta+n}\notin \dot U_i$". Define $h = h_1\restriction\{x_\alpha\colon \alpha<\delta\}\in (Q_{\bar h})^\delta$ and $h_1\setminus h = \{\langle a_l, b_l\rangle\colon l<m\}$, where $a_0<\ldots<a_{m-1}$. Let $j<m$ be such that $\langle x_{\delta+n}, y_{\delta+n}\rangle = \langle a_j, b_j\rangle$. Consider the set Z of all $\langle\langle z_l, z_l'\rangle\colon l<m\rangle\in(\mathbb{R}^2)^m$ for which

- there exist $V\in\mathcal{V}$, $k<\kappa_V^i$, and $g\in(Q_{\bar h})^\delta$ such that $z_j\in V$, g extends h and $h^i_{V,k}$, and $\langle\langle z_l, z_l'\rangle\colon l<m\rangle\in U(g,m)$.

Claim. *The set Z belongs to the model M_δ and it is an open dense subset of $K=\{\langle\langle z_l, z_l'\rangle\colon l<m\rangle\in U(h,m)\colon z_j\in U^*\}$.*

Proof. It should be clear that Z is (coded) in M_δ. (Remember the choice of δ.) It is also obvious that Z is open. To show that it is dense in $U(h,m)$ we proceed as in the proof of Fact 16. Let $\langle\langle c_l, d_l\rangle\colon l<m\rangle\in K$ and W be an open subset of K containing $\langle\langle c_l, d_l\rangle\colon l<m\rangle$. We need to show that $W\cap Z\neq\emptyset$. As in the proof of Fact 16 we can find an open set $\prod_{l<m}(V_l\times U_l)\subset K\cap W$ and points $\langle c_l, d_l\rangle\in V_l\times U_l$ such that $h_2 = h\cup\{\langle c_l, d_l\rangle\colon l<m\}\in(Q_{\bar h})^\delta$. Since $h_2\in(Q_{\bar h})^\delta$ extends h^* and V_j is an open subset of U^*, there is a subset $V\in\mathcal{V}$ of V_j and a $k<\kappa_V^i$ such that the conditions h_2 and $h^i_{V,k}$ are compatible. Let $g\in Q_{\bar h}$ extend h_2 and $h^i_{V,k}$. Since $h_2, h^i_{V,k}\in(Q_{\bar h})^\delta$ we can assume that $g\in(Q_{\bar h})^\delta$. But $\{\langle c_l, d_l\rangle\colon l<m\}\subset g$, so by (13), there are non-empty open sets $U_l'\subset U_l$ and $V_l'\subset V_l$ for which $\prod_{l<m}(U_i'\times V_i')\subset U(g,m)\cap\prod_{l<m}(V_l\times U_l)$. Thus, $\emptyset\neq\prod_{l<m}(U_i'\times V_i')\subset Z\cap W$. This completes the proof of the Claim. ∎

Now, $\langle\langle a_l, b_l\rangle\colon l<m\rangle$ belongs to K. Since by the Claim, $K\setminus Z\in M_\delta$ is nowhere dense, by (12) we conclude that this point does not belong to $K\setminus Z$. So $\langle\langle a_l, b_l\rangle\colon l<m\rangle\in Z$. But this means that there exist $g\in(Q_{\bar h})^\delta$ and $V\in\mathcal{V}$ such that:

- $g\le h$, $g\Vdash$ "$V\subseteq\dot U_i$",
- $\langle\langle a_l, b_l\rangle\colon l<m\rangle\in U(h,m)$, and $x_{\delta+n}=a_j\in V$.

But then $h_3 = g\cup\{\langle a_l, b_l\rangle\colon l<m\}$ belongs to $Q_{\bar h}$ and extends both g and h_1. So h_3 forces that $x_{\delta+n}=a_j\in V\subseteq\dot U_i$, contradicting our assumption that $h_1\Vdash$ "$x_{\delta+n}\notin\dot U_i$". This finishes the proof of (11) and of Lemma 2. ∎

5. A conjecture. We believe that Corollary 5 cannot be proved in ZFC. We believe the following is consistent with ZFC.

(C) For every nowhere meager set $A \subset [0,1]$ there exists a continuous Peano-like function p from $[0,1]$ onto $[0,1]^2$ such that for every $x \in [0,1]$ the set $p[A] \cap (\{x\} \times [0,1])$ is non-meager in $\{x\} \times [0,1]$.

PROPOSITION 17. *If* (C) *holds then for every nowhere meager set* $A \subset \mathbb{R}$ *there is an* $f \in \mathcal{C}_0$ *such that* $\pi[(f+c) \cap (A \times A)] \notin \mathcal{M}$ *for every* $c \in \mathbb{R}$.

Proof. Let $A \subset \mathbb{R}$ be nowhere meager, f be as in condition (C), and let $f_0 = \pi \circ p$: $[0,1] \to [0,1]$. Since there exists a meager set $M \subset [0,1]$ such that $p{\restriction}[0,1] \setminus M$ is a homeomorphism between $[0,1] \setminus M$ and $[0,1]^2 \setminus p[M]$, for every $c \in [0,1]$ we have

$$\pi[(f_0 + c) \cap (A \times A)] = \pi[f_0 \cap (A \times (A - c))] = A \cap (f_0)^{-1}(A - c) \notin \mathcal{M}$$

since $A \cap (f_0)^{-1}(A-c) = A \cap p^{-1}(\pi^{-1}(A-c)) = A \cap p^{-1}((A-c) \times [0,1])$ and this last set is equal, modulo $\mathcal{M}$, to $p^{-1}(p[A] \cap ((A-c) \times [0,1]))$ while, by (C), the set $p[A] \cap ((A-c) \times [0,1])$ is not meager (by the Kuratowski–Ulam theorem). To get a function from $\mathbb{R}$ to $\mathbb{R}$ glue countably many shifts of f_0. ∎

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Department of Mathematics
West Virginia University
Morgantown, WV 26506-6310, U.S.A.
E-mail: K_Cies@math.wvu.edu
Web: http://www.math.wvu.edu/~kcies

Department of Mathematics
Gdańsk University
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: mattn@math.univ.gda.pl

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