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Small Coverings with Smooth Functions under the Covering Property Axiom

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Abstract. In the paper we formulate a Covering Property Axiom, $\text{CPA}_{\text{prism}}$, which holds in the iterated perfect set model, and show that it implies the following facts, of which (a) and (b) are the generalizations of results of J. Steprāns.

(a) There exists a family $\mathcal{F}$ of less than continuum many $\mathcal{C}^1$ functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\mathbb{R}^2$ is covered by functions from $\mathcal{F}$, in the sense that for every $\langle x, y\rangle \in \mathbb{R}^2$ there exists an $f \in \mathcal{F}$ such that either $f(x) = y$ or $f(y) = x$.

(b) For every Borel function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a family $\mathcal{F}$ of less than continuum many "$\mathcal{C}^1$" functions (*i.e.,* differentiable functions with continuous derivatives, where derivative can be infinite) whose graphs cover the graph of f.

(c) For every $n > 0$ and a D^n function $f\colon \mathbb{R} \to \mathbb{R}$ there exists a family $\mathcal{F}$ of less than continuum many $\mathcal{C}^n$ functions whose graphs cover the graph of f.

We also provide the examples showing that in the above properties the smoothness conditions are the best possible. Parts (b), (c), and the examples are closely related to work of A. Olevskiĭ.

1 Basic Notation

Our set theoretic terminology is standard and follows that of [9]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. The Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. We use the term *Polish space* for a complete separable metric space *without isolated points*. A subset of a Polish space is *perfect* if it is closed and contains no isolated points. For a Polish space X symbol $\text{perf}(X)$ will stand for a collection of all subsets of X homeomorphic to the Cantor set $\mathfrak{C}$. Thus, in general, $\text{perf}(X)$ is just a (co-initial) subfamily of the family of perfect subsets of X, though these two collections coincide if X is zero dimensional. For a fixed $0 < \alpha < \omega_1$ and $0 < \beta \le \alpha$, a symbol π_β will stand for the projection from $\mathfrak{C}^\alpha$ onto $\mathfrak{C}^\beta$. We will always consider $\mathfrak{C}^\alpha$ with the following standard metric ρ: fix an enumeration $\{\langle \beta_k, n_k\rangle \colon k < \omega\}$ of $\alpha \times \omega$ and for distinct $x, y \in \mathfrak{C}^\alpha$ define

$$\rho(x, y) = 2^{-\min\{k<\omega\colon\, x(\beta_k)(n_k)\neq y(\beta_k)(n_k)\}}. \tag{1}$$

An open ball in $\mathfrak{C}^\alpha$ with a center at $z \in \mathfrak{C}^\alpha$ and radius $\varepsilon > 0$ will be denoted by $B_\alpha(z, \varepsilon)$. Notice that in this metric any two open balls are either disjoint or one is a

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subset of the other. Also for every $\gamma < \alpha$

$$\pi_\gamma[B_\alpha(s,\varepsilon)] = \pi_\gamma[B_\alpha(t,\varepsilon)] \quad \text{for every } s,t \in \mathfrak{C}^\alpha \text{ with } s \restriction \gamma = t \restriction \gamma. \tag{2}$$

It is also easy to see that any $B_\alpha(z,\varepsilon)$ is a clopen set.

We will use standard notation for the classes of differentiable partial functions from $\mathbb{R}$ into $\mathbb{R}$. Thus, if X is an arbitrary subset of $\mathbb{R}$ without isolated points we will write $\mathcal{C}^0(X)$ or $\mathcal{C}(X)$ for the class of all continuous functions $f\colon X \to \mathbb{R}$ and $D^1(X)$ for the class of all differentiable functions $f\colon X \to \mathbb{R}$, that is, those for which the limit

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x \in X}} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite for all $x_0 \in X$. Also, for $0 < n < \omega$, we will write $D^n(X)$ to denote the class of all functions $f\colon X \to \mathbb{R}$ which are n-times differentiable with all derivatives being finite and $\mathcal{C}^n(X)$ for the class of all $f \in D^n(X)$ whose nth derivative $f^{(n)}$ is continuous. The symbol $\mathcal{C}^\infty(X)$ will be used for all infinitely many times differentiable functions from X into $\mathbb{R}$. In addition, we say that a function $f\colon X \to \mathbb{R}$ is in the class "$D^n(X)$" if $f \in \mathcal{C}^{n-1}(X)$ and it has an nth derivative, which can be infinite; f is in the class "$\mathcal{C}^n(X)$" when f is in "$D^n(X)$" and its nth derivative is continuous when its range $[-\infty,\infty]$ is considered with the standard topology. "$\mathcal{C}^\infty(X)$" will stand for all functions $f\colon X \to \mathbb{R}$ which are either in $\mathcal{C}^\infty(X)$ or, for some $0 < n < \omega$, they are in "$\mathcal{C}^n(X)$" and $f^{(n)}$ is constant equal to ∞ or $-\infty$. (Thus, in general, "$\mathcal{C}^\infty(X)$" is not a subclass of "$\mathcal{C}^n(X)$".) In addition, we assume that functions defined on a singleton are in the $\mathcal{C}^\infty$ class, that is, $\mathcal{C}^\infty(\{x\}) = \mathbb{R}^{\{x\}}$. We will use these symbols mainly for X's which are either in the class perf($\mathbb{R}$) or are the singletons. In particular, $\mathcal{C}^n_{\text{perf}}$ will stand for the union of all $\mathcal{C}^n(P)$ for which $P \subset \mathbb{R}$ is either in perf($\mathbb{R}$) or a *singleton*. The classes D^n_{perf}, $\mathcal{C}^\infty_{\text{perf}}$, and "$\mathcal{C}^\infty_{\text{perf}}$" are defined the similar way. We will drop parameter X if $X = \mathbb{R}$. In particular, $D^n = D^n(\mathbb{R})$ and $\mathcal{C}^n = \mathcal{C}^n(\mathbb{R})$. The relations between these classes for $n < \omega$ are given in a chart below, where arrows $\to$ indicate the strict inclusions $\subsetneq$.

$$\begin{array}{ccccc}
 & & D^{n+1} & \longleftarrow & \mathcal{C}^{n+1} \\
 & & \downarrow & & \downarrow \\
\mathcal{C}^n & \longleftarrow & \text{“}D^{n+1}\text{”} & \longleftarrow & \text{“}\mathcal{C}^{n+1}\text{”}
\end{array}$$

Chart 1.

In addition, for $F \subset \mathbb{R}^2$ we define $F^{-1} = \{\langle y,x\rangle \colon \langle x,y\rangle \in F\}$, and for $\mathcal{F} \subset \mathcal{P}(\mathbb{R}^2)$ we put $\mathcal{F}^{-1} = \{F^{-1} \colon F \in \mathcal{F}\}$.

2 Axiom $\mathrm{CPA}_{\mathrm{prism}}$

Axiom $\mathrm{CPA}_{\mathrm{prism}}$ is a simpler version of the axiom CPA which is described in [13]. The main notion needed for the axiom is that of a prism and prism-density.

Let A be a non-empty countable set of ordinal numbers and let $\Phi_{\text{prism}}(A)$ be the family of all continuous injections $f\colon \mathfrak{C}^A \to \mathfrak{C}^A$ with the property that

$$(3) \qquad f(x) \restriction \alpha = f(y) \restriction \alpha \iff x \restriction \alpha = y \restriction \alpha \quad \text{ for all } \alpha \in A \text{ and } x, y \in \mathfrak{C}^A$$

or, equivalently, such that for every $\alpha \in A$

$$f \restriction\restriction \alpha \stackrel{\text{def}}{=} \{\langle x \restriction \alpha, y \restriction \alpha\rangle \colon \langle x, y\rangle \in f\}$$

is a one-to-one function from $\mathfrak{C}^{A\cap\alpha}$ into $\mathfrak{C}^{A\cap\alpha}$. Functions f from $\Phi_{\text{prism}}(A)$ were first introduced in a more general setting in [19], where they are called *projection-keeping homeomorphisms*. Note that

$$(4) \qquad \Phi_{\text{prism}}(A) \text{ is closed under compositions}$$

and that for every ordinal number $\alpha > 0$

$$(5) \qquad \text{if } f \in \Phi_{\text{prism}}(A) \text{ then } f \restriction\restriction \alpha \in \Phi_{\text{prism}}(A \cap \alpha).$$

For $0 < \alpha < \omega_1$ let

$$\mathbb{P}_\alpha = \{\text{range}(f) \colon f \in \Phi_{\text{prism}}(\alpha)\}.$$

Note that

$$(6) \qquad \text{if } f \in \Phi_{\text{prism}}(\alpha) \text{ and } P \in \mathbb{P}_\alpha \text{ then } f[P] \in \mathbb{P}_\alpha.$$

Indeed, if $P = g[\mathfrak{C}^\alpha]$ for some $g \in \Phi_{\text{prism}}(\alpha)$ then, by condition (4), we have $f[P] = f[g[\mathfrak{C}^\alpha]] = (f \circ g)[\mathfrak{C}^\alpha] \in \mathbb{P}_\alpha$.

We will write Φ_{prism} for $\bigcup_{0<\alpha<\omega_1} \Phi_{\text{prism}}(\alpha)$ and define

$$\mathbb{P}_{\omega_1} \stackrel{\text{def}}{=} \bigcup_{0<\alpha<\omega_1} \mathbb{P}_\alpha = \{\text{range}(f) \colon f \in \Phi_{\text{prism}}\}.$$

Following [19] we will refer to elements of $\mathbb{P}_{\omega_1}$ as *iterated perfect sets*.

The simplest elements of $\mathbb{P}_{\omega_1}$ are *cubes* (in $\mathfrak{C}^A$), that is, the sets of the form $C = \prod_{a\in A} C_a$, where $C_a \in \text{perf}(\mathfrak{C})$ for each $a \in A$. (This is justified by a function $f = \langle f_a\rangle_{a\in A} \in \Phi_{\text{prism}}(A)$, where each f_a is a homeomorphism from $\mathfrak{C}$ onto C_a.) In particular, since any open ball $B_\alpha(z, \varepsilon)$ (in the metric given by (1)) is a cube in $\mathfrak{C}^\alpha$, it belongs to $\mathbb{P}_\alpha$. In fact, more can be said:

$$(7) \qquad \text{if } \mathcal{B}_\alpha \stackrel{\text{def}}{=} \{B \subset \mathfrak{C}^\alpha \colon B \text{ is clopen in } \mathfrak{C}^\alpha\}, \text{ then } \mathcal{B}_\alpha \subset \mathbb{P}_\alpha.$$

This is the case, since any clopen E in $\mathfrak{C}^\alpha$ is a finite union of disjoint open balls, each of which belongs to $\mathbb{P}_\alpha$, and it is easy to see that $\mathbb{P}_\alpha$ is closed under finite unions of open balls.

In general, the structure of elements of $\mathbb{P}_{\omega_1}$ can be considerably more complex. However, there is only one non-trivial fact about $\mathbb{P}_{\omega_1}$ that we will use in this paper: the family $\mathbb{P}_{\omega_1}$ satisfies the following fusion lemma.

Lemma 2.1 (Fusion Lemma) *Let $0 < \alpha < \omega_1$, $\mathcal{A} \in \{\mathcal{B}_\alpha, \mathbb{P}_\alpha\}$, and let $\langle \mathcal{D}_k \subset [\mathcal{A}]^{<\omega} \colon k < \omega \rangle$ be such that for every $k < \omega$ the following holds.*

(P1) *($\mathcal{D}_k$ is $\mathcal{A}$-open): If $\{E_0, \dots, E_n\} \in \mathcal{D}_k$ and $E'_0, \dots, E'_n \in \mathcal{A}$ are such that $E'_i \subset E_i$ for every $i \le n$ then $\{E'_0, \dots, E'_n\} \in \mathcal{D}_k$.*

(P2) *(sequence splits): If $\{E_0, \dots, E_n\} \in \mathcal{D}_k$ and $\{E^i_0, E^i_1\} \in \mathcal{D}_{k+1}$ for every $i \le n$ is such that $E^i_0 \cup E^i_1 \subset E_i$ then $\{E^i_j \colon i \le n \,\&\, j < 2\} \in \mathcal{D}_{k+1}$.*

(P3) *($\mathcal{D}_k$ is nicely $\mathcal{A}$-dense): For every $E \in \mathcal{A}$ and $\gamma < \alpha$ there are disjoint $E_0, E_1 \in \mathcal{A}$ such that $E_0 \cup E_1 \subset E$, $\{E_0, E_1\} \in \mathcal{D}_k$, and $\pi_\gamma[E_0] = \pi_\gamma[E_1]$.*

Then there exists a sequence $\langle \mathcal{E}_k \in \mathcal{D}_k \colon k < \omega \rangle$ with the property that its fusion $Q = \bigcap_{k<\omega} \bigcup \mathcal{E}_k$ belongs to $\mathbb{P}_\alpha$.

Although the lemma looks quite complicated, it should be stressed that in all its applications we will be checking only condition (P3), since the other two conditions will be trivially satisfied. The proof of Lemma 2.1 will be postponed to the end of this paper.

The only other fact we will use on $\mathbb{P}_{\omega_1}$ (or, more precisely, on cubes) is the following:

Claim 2.2 *If $G \subset \mathfrak{C}^\omega$ is comeager in $\mathfrak{C}^\omega$ then it contains a perfect cube $\prod_{i<\omega} P_i$.*

Proof It follows easily, by induction on coordinates, from the following well-known fact.

> For every comeager subset H of $\mathfrak{C} \times \mathfrak{C}$ there are perfect set $P \subset \mathfrak{C}$ and a comeager subset $\hat{H}$ of $\mathfrak{C}$ such that $P \times \hat{H} \subset H$.

(See [20, Exercise 19.3]. Its version for $\mathbb{R}^2$ is also proved in [14, condition ($\star$), p. 416].) ■

To state $\mathrm{CPA}_{\mathrm{prism}}$ we need a few more definitions. For a fixed Polish space X let $\mathcal{F}_{\mathrm{prism}}(X)$ (or just $\mathcal{F}_{\mathrm{prism}}$, if X is clear from the context) be the family of all continuous injections $f \colon E \to X$, where $E \in \mathbb{P}_{\omega_1}$. Each such injection f is called a *prism* in X and is considered as a coordinate system imposed on $P = \mathrm{range}(f)$.[1] We will usually abuse this terminology and refer to P itself as a *prism* (in X) and to f as a *witness function* for P. A function $g \in \mathcal{F}_{\mathrm{prism}}$ is *subprism* of f provided $g \subset f$. In the above spirit, we call $Q = \mathrm{range}(g)$ a *subprism of a prism P*. Thus, when we say that *Q is a subprism of a prism $P \in \mathrm{perf}(X)$*, we mean that $Q = f[E]$, where f is a witness function for P, $E \in \mathbb{P}_{\omega_1}$, and $E \subset \mathrm{dom}(f)$. A family $\mathcal{E} \subset \mathrm{perf}(X)$ is $\mathcal{F}_{\mathrm{prism}}$-*dense* provided

$$\forall f \in \mathcal{F}_{\mathrm{prism}}\ \exists g \in \mathcal{F}_{\mathrm{prism}}\ (g \subset f \ \&\ \mathrm{range}(g) \in \mathcal{E}).$$

Using (4) it is easy to show that

Fact 2.3 *$\mathcal{E} \subset \mathrm{perf}(X)$ is $\mathcal{F}_{\mathrm{prism}}$-dense if and only if*

$$\forall \alpha < \omega_1\ \forall f \in \mathcal{F}_{\mathrm{prism}},\ f \colon \mathfrak{C}^\alpha \to X,\ \exists g \in \mathcal{F}_{\mathrm{prism}}\ (g \subset f \ \&\ \mathrm{range}(g) \in \mathcal{E}).$$

[1] In the language of forcing a coordinate function f is simply a nice name for an element from X.

Thus, to establish $\mathcal{F}_{\text{prism}}$-density we can always assume that the witness function f for the prism P is in a *standard form*, that is, defined on the entire set $\mathfrak{C}^\alpha$.

Now we are ready to state the axiom.

CPA$_{\text{prism}}$: $\mathfrak{c} = \omega_2$ and for every Polish space X and every $\mathcal{F}_{\text{prism}}$-dense family $\mathcal{E} \subset \text{perf}(X)$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \le \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$.

The proof of the consistency of CPA$_{\text{prism}}$ can be found in [12, Prop. 4.2]. (See also [13].) We finish this section with yet another lemma which will be used in our applications.

Lemma 2.4 *For every $0 < \alpha < \omega_1$, $E \in \mathbb{P}_\alpha$, a Polish space X, and a continuous function $f\colon E \to X$ there exist $0 < \beta \le \alpha$ and $P \in \mathbb{P}_\alpha$, $P \subset E$, such that $f \circ \pi_\beta^{-1}$ is a function on $\pi_\beta[P] \in \mathbb{P}_\beta$ which is either one-to-one or constant.*

Lemma 2.4 is a particular case of [19, Thm. 20]. It can be also easily deduced from Lemma 2.1. (See also [13, Lemma 3.2.5].)

3 Covering Results and Their Discussion

The main consequence of CPA$_{\text{prism}}$ we discuss in this paper is the following theorem.

Theorem 3.1 *These facts follow from* CPA$_{\text{prism}}$*:*

(a) *For every Borel measurable function $g\colon \mathbb{R} \to \mathbb{R}$ there exists a family of functions $\{f_\xi \in \text{“}\mathcal{C}^\infty_{\text{perf}}\text{”}\colon \xi < \omega_1\}$ such that*

$$g = \bigcup_{\xi<\omega_1} f_\xi.$$

Moreover for each $\xi < \omega_1$ there exists an extension $\bar{f}_\xi\colon \mathbb{R} \to \mathbb{R}$ of f_ξ such that

(i) $\bar{f}_\xi \in \text{“}\mathcal{C}^1\text{”}$ *and*

(ii) *either $\bar{f}_\xi \in \mathcal{C}^1$ or $\bar{f}_\xi$ is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ such that $\bar{f}_\xi^{-1} \in \mathcal{C}^1$.*

(b) *There exists a sequence $\{f_\xi \in \mathbb{R}^{\mathbb{R}}\colon \xi < \omega_1\}$ of $\mathcal{C}^1$ functions such that*

$$\mathbb{R}^2 = \bigcup_{\xi<\omega_1} (f_\xi \cup f_\xi^{-1}).$$

The essence of Theorem 3.1 lies in the following real analysis fact. Its proof is combinatorial in nature and uses no extra set-theoretical assumptions.

Proposition 3.2 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be Borel and $0 < \alpha < \omega_1$.*

(a) *For every continuous injection $h\colon \mathfrak{C}^\alpha \to \mathbb{R}$ there exists an $E \in \mathbb{P}_\alpha$ such that $g \restriction h[E] \in \text{“}\mathcal{C}^\infty_{\text{perf}}\text{”}$ and there is an extension $f\colon \mathbb{R} \to \mathbb{R}$ of $g \restriction h[E]$ such that $f \in \text{“}\mathcal{C}^1\text{”}$ and either $f \in \mathcal{C}^1$ or f is a self-homeomorphism of $\mathbb{R}$ with $f^{-1} \in \mathcal{C}^1$.*

(b) *For every continuous injection* $h\colon \mathfrak{C}^\alpha \to \mathbb{R}^2$ *there exists an* $E \in \mathbb{P}_\alpha$ *such that either* $F = h[E] \subset \mathbb{R}^2$ *or its inverse,* F^{-1}*, is a function which can be extended to a* $\mathcal{C}^1$ *function* $f\colon \mathbb{R} \to \mathbb{R}$.

With Proposition 3.2 in hand, the proof of Theorem 3.1 becomes an easy exercise.

Proof of Theorem 3.1 (a) Let $g\colon \mathbb{R} \to \mathbb{R}$ be a Borel function and let $\mathcal{E}$ be the family of all $P \in \text{perf}(\mathbb{R})$ such that

$g \restriction P \in$ "$\mathcal{C}^\infty_{\text{perf}}$" and there is an extension $f\colon \mathbb{R} \to \mathbb{R}$ of $g \restriction P$ such that $f \in$ "$\mathcal{C}^1$" and either $f \in \mathcal{C}^1$ or f is a self-homeomorphism of $\mathbb{R}$ with $f^{-1} \in \mathcal{C}^1$.

By Proposition 3.2(a), family $\mathcal{E}$ is $\mathcal{F}_{\text{prism}}$-dense: if $P \in \text{perf}(\mathbb{R})$ is a prism witnessed by $h\colon \mathfrak{C}^\alpha \to \mathbb{R}$ from $\mathcal{F}_{\text{prism}}$, then $Q = h[E]$, as in the proposition, is a subprism of P with $Q \in \mathcal{E}$. So, by $\text{CPA}_{\text{prism}}$, there exists an $\mathcal{E}_0 \in [\mathcal{E}]^{\le\omega_1}$ such that $|\mathbb{R} \setminus \bigcup \mathcal{E}_0| \le \omega_1$. Let $\mathcal{E}_1 = \mathcal{E}_0 \cup \{\{r\}\colon r \in \mathbb{R} \setminus \bigcup \mathcal{E}_0\}$. Then the family $\{g \restriction P\colon P \in \mathcal{E}_1\}$ satisfies the theorem.

(b) Let $\mathcal{E}$ be the family of all $P \in \text{perf}(\mathbb{R}^2)$ such that either P or P^{-1} is a function which can be extended to a $\mathcal{C}^1$ function $f\colon \mathbb{R} \to \mathbb{R}$. By Proposition 3.2(b) family $\mathcal{E}$ is $\mathcal{F}_{\text{prism}}$-dense, so there exists an $\mathcal{E}_0 \in [\mathcal{E}]^{\le\omega_1}$ such that $|\mathbb{R} \setminus \bigcup \mathcal{E}_0| \le \omega_1$. Let $\mathcal{E}_1 = \mathcal{E}_0 \cup \{\{x\} : x \in \mathbb{R}^2 \setminus \bigcup \mathcal{E}_0\}$. For every $P \in \mathcal{E}_1$ let $f_P\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function which extends either P or P^{-1}. Then family $\{f_P : P \in \mathcal{E}_1\}$ is as desired. ■

The proof of Proposition 3.2 will be left to section 4. Meanwhile we would like to present a discussion of Theorem 3.1.

First we want to reformulate Theorem 3.1 in a language of a *covering number* cov defined below, where X is an infinite set (in our case $X \subset \mathbb{R}^2$ with $|X| = \mathfrak{c}$) and $\mathcal{A}, \mathcal{F} \subset \mathcal{P}(X)$:

$$\text{cov}(\mathcal{A}, \mathcal{F}) = \min(\{\kappa\colon (\forall A \in \mathcal{A})(\exists \mathcal{G} \in [\mathcal{F}]^{\le\kappa})\ A \subset \bigcup \mathcal{G}\} \cup \{|X|^+\}).$$

If $A \subset X$ we will write $\text{cov}(A, \mathcal{F})$ for $\text{cov}(\{A\}, \mathcal{F})$. Notice the following monotonicity of the cov operator: for every $A \subset B \subset X$, $\mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(X)$, and $\mathcal{F} \subset \mathcal{G} \subset \mathcal{P}(X)$

$$\text{cov}(\mathcal{A}, \mathcal{G}) \le \text{cov}(\mathcal{B}, \mathcal{G}) \le \text{cov}(\mathcal{B}, \mathcal{F}) \quad \text{and} \quad \text{cov}(A, \mathcal{G}) \le \text{cov}(B, \mathcal{G}) \le \text{cov}(B, \mathcal{F}).$$

In terms of the cov operator, Theorem 3.1 can be expressed in the following form, where Borel stands for the class of all Borel functions $f\colon \mathbb{R} \to \mathbb{R}$:

Corollary 3.3 $\text{CPA}_{\text{prism}}$ *implies that*

(a) $\text{cov}\big(\text{Borel}, \text{“}\mathcal{C}^\infty_{\text{perf}}\text{”}\big) = \omega_1 < \mathfrak{c}$;
(b) $\text{cov}(\text{Borel}, \text{“}\mathcal{C}^1\text{”}) = \omega_1 < \mathfrak{c}$;
(c) $\text{cov}(\text{Borel}, \mathcal{C}^1 \cup (\mathcal{C}^1)^{-1}) = \omega_1 < \mathfrak{c}$;
(d) $\text{cov}(\mathbb{R}^2, \mathcal{C}^1 \cup (\mathcal{C}^1)^{-1}) = \omega_1 < \mathfrak{c}$.

Proof The fact that all numbers $\text{cov}(\mathcal{A}, \mathcal{G})$ listed above are $\leq \omega_1$ follows directly from Theorem 3.1. The other inequalities follow from Examples 7.6 and 7.8. ■

Theorem 3.1(b) and Corollary 3.3(d) can be treated as generalizations of a result of Steprāns [26] who proved that in the iterated perfect set model we have

$$\text{cov}(\mathbb{R}^2, (\text{“}D^1\text{”}) \cup (\text{“}D^1\text{”})^{-1}) \leq \omega_1.$$

This clearly follows from Corollary 3.3(d) since $\mathcal{C}^1 \subsetneq \text{“}D^1\text{”}$. (See survey article [4]. See also [11, Cor. 9] for more information how to “locate” Steprāns' result in [26].)

The following proposition shows that Theorem 3.1 is, in a way, the best possible. (Parts (i), (ii), and (iii) relate, respectively, to items (b), (c) and (d), and (a) from Corollary 3.3.)

Proposition 3.4 *The following is true in ZFC.*

(i) $\text{cov}(\text{Borel}, \mathcal{C}^1) = \text{cov}(\text{“}\mathcal{C}^1\text{”}, \mathcal{C}^1) = \text{cov}(\text{“}\mathcal{C}^1\text{”}, D^1_{\text{perf}}) = \mathfrak{c}$. *Moreover,* $\text{cov}(\text{“}\mathcal{C}^n\text{”}, \mathcal{C}^n) = \text{cov}(\text{“}\mathcal{C}^n\text{”}, D^n_{\text{perf}}) = \mathfrak{c}$ *for every* $0 < n < \omega$.

(ii) $\text{cov}(\text{Borel}, \mathcal{C}^2 \cup (\mathcal{C}^2)^{-1}) = \text{cov}(\text{“}\mathcal{C}^2\text{”}, D^2_{\text{perf}} \cup (D^2_{\text{perf}})^{-1}) = \mathfrak{c}$, *and* $\text{cov}(\mathbb{R}^2, \mathcal{C}^2 \cup (\mathcal{C}^2)^{-1}) = \text{cov}(\text{“}\mathcal{C}^2\text{”}, D^2_{\text{perf}} \cup (D^2_{\text{perf}})^{-1}) = \mathfrak{c}$.

(iii) $\text{cov}(\text{Borel}, \mathcal{C}^\infty_{\text{perf}}) = \text{cov}(\text{“}\mathcal{C}^1\text{”}, \mathcal{C}^\infty_{\text{perf}}) = \text{cov}(\text{“}\mathcal{C}^1\text{”}, D^1_{\text{perf}}) = \mathfrak{c}$, *and* $\text{cov}(\text{Borel}, \text{“}\mathcal{C}^\infty\text{”}) = \text{cov}(\mathcal{C}^1, \text{“}\mathcal{C}^\infty\text{”}) = \text{cov}(\mathcal{C}^1, \text{“}D^2\text{”}) = \mathfrak{c}$. *Moreover,* $\text{cov}(\mathcal{C}^n, \text{“}D^{n+1}\text{”}) = \mathfrak{c}$ *for every* $0 < n < \omega$.

Proof (i) follows immediately from Examples 7.2 and 7.3.

(ii) follows from monotonicity of cov operator and Example 7.1.

The first part of (iii) follows from (i). The remaining two parts follow, respectively, from Examples 7.4 and 7.5. ■

Corollary 3.3 and Proposition 3.4 establish the values of the cov operator for all classes in Chart 1 except for $\text{cov}(D^n, \mathcal{C}^n)$ and $\text{cov}(\text{“}D^n\text{”}, \text{“}\mathcal{C}^n\text{”})$. These are established in the following theorem, whose proof will be left to Section 6.

Theorem 3.5 *If* $\text{CPA}_{\text{prism}}$ *holds, then for every* $0 < n < \omega$,

$$\text{cov}(D^n, \mathcal{C}^n) = \text{cov}(\text{“}D^n\text{”}, \text{“}\mathcal{C}^n\text{”}) = \omega_1 < \mathfrak{c}.$$

With this theorem in hand we can summarize the values of the cov operator between the classes from Chart 1 in the following graphical form. Here the mark “$\mathfrak{c}$” next to the arrow means that the covering of the larger class by the functions from the smaller class is equal to $\mathfrak{c}$ and that this can be proved in ZFC. The mark “$< \mathfrak{c}$” next to the arrow means that it is consistent with ZFC (and it follows from $\text{CPA}_{\text{prism}}$) that the appropriate cov number is $< \mathfrak{c}$. (From Examples 7.6, 7.7, and 7.8 it follows that all these numbers are greater than or equal to $\min\{\text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})\} > \omega$. So under the continuum hypothesis CH or Martin's Axiom MA all these numbers are equal to $\mathfrak{c}$.)

$$
\begin{array}{ccccc}
 & & D^1 & \xleftarrow[<\mathfrak{c}]{} & \mathcal{C}^1 \\
 & & \Big\downarrow{\scriptstyle \mathfrak{c}} & & \Big\downarrow{\scriptstyle \mathfrak{c}} \\
\mathcal{C}^0 & \xleftarrow[<\mathfrak{c}]{} & \text{“}D^1\text{”} & \xleftarrow[<\mathfrak{c}]{} & \text{“}\mathcal{C}^1\text{”}
\end{array}
\qquad
\begin{array}{ccccc}
 & & D^{n+1} & \xleftarrow[<\mathfrak{c}]{} & \mathcal{C}^{n+1} \\
 & & \Big\downarrow{\scriptstyle \mathfrak{c}} & & \Big\downarrow{\scriptstyle \mathfrak{c}} \\
\mathcal{C}^n & \xleftarrow[\mathfrak{c}]{} & \text{“}D^{n+1}\text{”} & \xleftarrow[<\mathfrak{c}]{} & \text{“}\mathcal{C}^{n+1}\text{”}
\end{array}
$$

Chart 2. Values of cov operator: for $n = 0$ (left) and $n > 0$ (right).

The values of cov next to the vertical arrows are justified by $\text{cov}(\text{“}\mathcal{C}^n\text{”}, D^n) = \mathfrak{c}$ (Proposition 3.4(i)), while the symbols “$<\mathfrak{c}$” below the horizontal arrows follow from Theorem 3.5. The remaining arrow of the right part of the chart is the restatement of the last part of Proposition 3.4(iii), while its counterpart in the left part of the chart follows from Corollary 3.3(b): $\text{cov}(\mathcal{C}, \text{“}\mathcal{C}^1\text{”}) = \text{cov}(\text{Borel}, \text{“}\mathcal{C}^1\text{”}) < \mathfrak{c}$ is a consequence of $\text{CPA}_{\text{prism}}$. Finally let us mention that in Corollary 3.3(b) there is no chance to increase family Borel in any essential way and keep the result. This follows from the following fact:

$$\text{cov}(\text{Sc}, \mathcal{C}) = \text{cov}(\mathbb{R}^{\mathbb{R}}, \mathcal{C}) \geq \text{cof}(\mathfrak{c}), \tag{8}$$

where Sc stands for the family of all symmetrically continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ which are, in particular, continuous outside of some set of measure zero and first category. (See [10, Cor. 1.1] and the remarks below on the operator dec.)

Number $\text{cov}(\mathcal{A}, \mathcal{F})$ is very closely related to the decomposition number,

$$\text{dec}(\mathcal{A}, \mathcal{F}) = \min\left(\{\kappa \geq \omega\colon (\forall A \in \mathcal{A})(\exists \text{ a partition } \mathcal{G} \in [\mathcal{F}]^\kappa \text{ of } A)\} \cup \{|X|^+\}\right),$$

which was first studied by Cichoń, Morayne, Pawlikowski, and Solecki [7] for the Baire class α functions. (More information on $\text{dec}(\mathcal{F}, \mathcal{G})$ can be found in a survey article [8, sec. 4].) It is easy to see that if $\mathcal{A}$ and $\mathcal{F}$ are some classes of partial functions and $\mathcal{F}_r$ denotes all possible restrictions of functions from $\mathcal{F}$, then $\text{cov}(\mathcal{A}, \mathcal{F}) = \text{dec}(\mathcal{A}, \mathcal{F}_r)$. In particular, for all situations relevant to our discussion above, the operators cov and dec have the same values.

Our number cov is also related to the following general class of problems. We say that the families $\mathcal{A}, \mathcal{F} \subset \mathcal{P}(X)$ satisfy the *Intersection Theorem,* which we denote by

$$\text{IntTh}(\mathcal{A}, \mathcal{F}),$$

if for every $A \in \mathcal{A}$ there exists an $F \in \mathcal{G}$ such that $|A \cap F| = |X|$. If $\mathcal{A} = \{A\}$ we will write $\text{IntTh}(A, \mathcal{F})$ in place of $\text{IntTh}(\mathcal{A}, \mathcal{F})$. This kind of theorem has been studied for a big part of the last century. In particular, in the early 1940's Ulam asked in the *Scottish Book* [21, Problem 17.1] if $\text{IntTh}(\mathcal{C}, \text{Analytic})$ holds, that is, whether for every $f \in \mathcal{C}$ there exists a real analytic function $g\colon \mathbb{R} \to \mathbb{R}$ which agrees with f on a perfect set, (see [27].) In 1947 Zahorski [29] gave a negative answer to this question by proving that the proposition $\text{IntTh}(\mathcal{C}^\infty, \text{Analytic})$ is false. In the same paper, he also raised a natural question which has become known as the Ulam–Zahorski Problem: *Does* $\mathrm{IntTh}(\mathcal{C}, \mathcal{G})$ *hold for* $\mathcal{G} = \mathcal{C}^\infty$ *(or* $\mathcal{G} = \mathcal{C}^n$ *or* $\mathcal{G} = D^n$*)?* Here is a quick summary of what is known on this problem, (see [4].)

Zahorski [29]:	$\neg\, \mathrm{IntTh}(\mathcal{C}^\infty, \text{Analytic})$.
Agronsky, Bruckner, Laczkovich, Preiss [1]:	$\mathrm{IntTh}(\mathcal{C}, \mathcal{C}^1)$.
Olevskiĭ [24]:	$\mathrm{IntTh}(\mathcal{C}^1, \mathcal{C}^2)$.
Olevskiĭ [24]:	$\neg\, \mathrm{IntTh}(\mathcal{C}, \mathcal{C}^2)$ and $\neg\, \mathrm{IntTh}(\mathcal{C}^n, \mathcal{C}^{n+1})$ for $n \geq 2$.

We are interested in these problems since for the families $\mathcal{A}, \mathcal{F} \in \mathcal{P}(\mathbb{R}^n)$ of uncountable Borel sets

$$\neg\, \mathrm{IntTh}(\mathcal{A}, \mathcal{F}) \implies \mathrm{cov}(\mathcal{A}, \mathcal{F}) = \mathfrak{c} \tag{9}$$

as, in this situation, if $\neg\, \mathrm{IntTh}(\mathcal{A}, \mathcal{F})$ then there exists an $A \in \mathcal{A}$, $|A| = \mathfrak{c}$, such that $|A \cap F| \leq \omega$ for every $F \in \mathcal{F}$. Thus in the examples relevant to Proposition 3.4 instead of proving $\mathrm{cov}(\mathcal{A}, \mathcal{F}) = \mathfrak{c}$, in fact we will be showing a stronger fact that $\neg\, \mathrm{IntTh}(A_0, \mathcal{F})$ for appropriate $A_0 \subset A \in \mathcal{A}$.

4 Proof of Proposition 3.2

Proposition 3.2 will be deduced from the following fact, which is a generalization of a theorem of Morayne [23]. (Morayne proved his results for E and E_1 being perfect sets, that is, for $\alpha = 1$.) For a set X, we will use symbol Δ_X to denote the diagonal in $X \times X$, that is, $\Delta_X = \{\langle x, x\rangle : x \in X\}$. We will usually write simply Δ in place of Δ_X, since X is always clear from the context.

Proposition 4.1 *Let* $0 < \alpha < \omega_1$, $E \in \mathbb{P}_\alpha$, $h\colon E \to \mathbb{R}$ *be a continuous injection, and* G *be a function from* $(h[E])^2 \setminus \Delta$ *into* $[0, 1]$ *which is continuous and symmetric, that is, such that* $G(x, y) = G(y, x)$ *for all* $x, y \in (h[E])^2 \setminus \Delta$*. Then there exists an* $E_1 \in \mathbb{P}_\alpha$, $E_1 \subset E$*, such that* G *is uniformly continuous on* $(h[E_1])^2 \setminus \Delta$.

The proof of Proposition 4.1 will be presented in the next section. In the proof of Proposition 3.2 we will also use the following lemma.

Lemma 4.2 *Let* $g\colon \mathbb{R} \to \mathbb{R}$ *be Borel,* $0 < \alpha < \omega_1$*, and* $E \in \mathbb{P}_\alpha$*. For every continuous injection* $h\colon E \to \mathbb{R}$ *there exist subset* $E_1 \in \mathbb{P}_\alpha$ *of* E *and a "*$\mathcal{C}^1$*" function* $f\colon \mathbb{R} \to \mathbb{R}$ *such that* f *extends* $g \restriction h[E_1]$.

In addition we can require that either $f \in \mathcal{C}^1$ *or*

($\star$) $f' \restriction h[E_1]$ *is constant equal to* ∞ *or* $-\infty$ *and* f *is a self-homeomorphism of* $\mathbb{R}$ *such that* $f^{-1} \in \mathcal{C}^1$.

Proof First note that there exists an $E' \in \mathbb{P}_\alpha$, $E' \subset E$, such that

$$g \restriction h[E'] \text{ is continuous.} \tag{10}$$

Indeed, let $h_0 \in \Phi_{\text{prism}}$ be such that $E = h_0[\mathfrak{C}^\alpha]$ and let U be a comeager subset of $h[E] = (h\circ h_0)[\mathfrak{C}^\alpha]$ such that the restriction $g \restriction U$ is continuous. Then $(h\circ h_0)^{-1}(U)$ is comeager in $\mathfrak{C}^\alpha$ and, by Claim 2.2, there is a perfect cube $Q \subset (h \circ h_0)^{-1}(U)$. The set $E' = h_0[Q] \in \mathbb{P}_\alpha$ has the desired property since $h[E'] = h[h_0[Q]] \subset U$.

Now let $k\colon [-\infty, \infty] \to [0, 1]$ be a homeomorphism and let G be defined on $(h[E'])^2 \setminus \Delta$ by

$$G(x, y) = k\left(\frac{g(x) - g(y)}{x - y}\right).$$

Then, by Proposition 4.1, there exists an $E'_1 \in \mathbb{P}_\alpha$, $E'_1 \subset E'$, such that G is uniformly continuous on $(h[E'_1])^2 \setminus \Delta$. So, there exists a uniformly continuous extension of $G \restriction (h[E'_1])^2 \setminus \Delta$ to $\hat{G} \restriction (h[E'_1])^2$. Clearly $k^{-1}(\hat{G}(x, x))$ is the derivative (possibly infinite) of $g_0 = g \restriction h[E'_1]$ for every $x \in h[E'_1]$, so $g_0 \in$ "$\mathcal{C}^1(h[E'_1])$".

Now, if $(g'_0)^{-1}(\mathbb{R})$ is non-empty then, as in the argument for (10) we can find an $E_1 \in \mathbb{P}_\alpha$, $E_1 \subset E'_1$, such that $h[E_1] \subset (g'_0)^{-1}(\mathbb{R})$. This obviously implies $g \restriction h[E_1] \in \mathcal{C}^1_{\text{perf}}$. But we also know that the difference quotient function $\frac{g(x)-g(y)}{x-y}$ is uniformly continuous on $(h[E_1])^2 \setminus \Delta$. So, by Whitney's extension theorem [28] (see also Lemma 6.1), we can find a $\mathcal{C}^1$ extension $f\colon \mathbb{R} \to \mathbb{R}$ of $g \restriction h[E_1]$.

So, assume that $(g'_0)^{-1}(\mathbb{R}) = \varnothing$. Then either $(g'_0)^{-1}(\infty)$ or $(g'_0)^{-1}(-\infty)$ is non-empty and open in $h[E'_1]$. Assume the former case. Similarly as above we can find an $E''_1 \in \mathbb{P}_\alpha$, $E''_1 \subset E'_1$, such that $g'_0[h[E''_1]] = \{\infty\}$. Then, by a version of Whitney's extension theorem from [3, Thm. 2.1], we can find a "$\mathcal{C}^1$" extension $f_0\colon \mathbb{R} \to \mathbb{R}$ of $g \restriction h[E''_1]$.

But then there exists an open interval J in $\mathbb{R}$ intersecting $h[E''_1]$ on the closure of which f'_0 is positive. So $f_1 = f_0 \restriction \text{cl}(J)$ is strictly increasing and the derivative of f_1^{-1} is continuous, non-negative, and bounded. Thus there exists a homeomorphism $f_2\colon \mathbb{R} \to \mathbb{R}$ extending f_1^{-1} with $f_2 \in \mathcal{C}^1$. Now put $f = f_2^{-1}$ and take an $E_1 \in \mathbb{P}_\alpha$ with $E_1 \subset E''_1 \cap h^{-1}(J)$. It is easy to see that E_1 and f are as required. ∎

Proof of Proposition 3.2(a) By Lemma 4.2 we can find an $E_0 \in \mathbb{P}_\alpha$ for which there is an extension $f\colon \mathbb{R} \to \mathbb{R}$ of $g \restriction h[E_0]$ such that $f \in$ "$\mathcal{C}^1$" and either $f \in \mathcal{C}^1$ or f is a self-homeomorphism of $\mathbb{R}$ with $f^{-1} \in \mathcal{C}^1$. Thus, it is enough to find a subset $E \in \mathbb{P}_\alpha$ of E_0 for which $g \restriction h[E] \in$ "$\mathcal{C}^\infty_{\text{perf}}$".

If there exists a subset $E \in \mathbb{P}_\alpha$ of E_0 and $n < \omega$ such that

(11) $$f = g \restriction h[E] \in \text{"}\mathcal{C}^n_{\text{perf}}\text{" and } f^{(n)} \text{ has a constant value } \infty \text{ or } -\infty,$$

then this E is as desired. So assume that there is no such E. We will use Fusion Lemma 2.1 with $\mathcal{A} = \mathbb{P}_\alpha$ to find a subprism E of E_0 for which $g \restriction h[E] \in \mathcal{C}^\infty_{\text{perf}}$.

First notice that we can assume that $E_0 = \mathfrak{C}^\alpha$, since we can replace h with $h \circ h_0$, where $h_0 \in \Phi_{\text{prism}}$ is such that $E_0 = h_0[\mathfrak{C}^\alpha]$. For $k < \omega$ let $\mathcal{D}_k \subset [\mathbb{P}_\alpha]^{<\omega}$ be the collection of all finite families $\mathcal{E}$ of pairwise disjoint sets each of a diameter less than 2^{-k} such that

(12) $$g \restriction \bigcup\{h[E]\colon E \in \mathcal{E}\} \in \mathcal{C}^k_{\text{perf}}.$$

We need to show that $\mathcal{D}_k$'s satisfy the assumptions of Lemma 2.1.

It is obvious that the conditions (P1) and (P2) are satisfied. To see that (P3) holds for $k < \omega$ fix $\bar{E} \in \mathbb{P}_\alpha$ and $\gamma < \alpha$. Applying Lemma 4.2 k-times and using the fact that (11) is false, we can find a sequence $\bar{E} = P_0 \supset \cdots \supset P_k$ from $\mathbb{P}_\alpha$ such that $g \restriction h[P_i] \in \mathcal{C}^i_{\rm perf}$ for each $i \leq k$. Take disjoint $E_0, E_1 \in \mathbb{P}_\alpha$ subsets of P_k, each of diameter less than 2^{-k}, such that $\pi_\gamma[E_0] = \pi_\gamma[E_1]$. It is easy to see that E_0 and E_1 satisfy the requirements of the condition (P3).

Now, by Lemma 2.1, there exist $\mathcal{E}_k \in \mathcal{D}_k$ such that $E = \bigcap_{k<\omega} \bigcup \mathcal{E}_k \in \mathbb{P}_\alpha$. Clearly $g \restriction h[E] \in \mathcal{C}^\infty_{\rm perf}$ for such an E. ∎

Proof of Proposition 3.2(b) Let π_x and π_y be the projections of $\mathbb{R}^2$ onto x-axis and y-axis, respectively, and consider functions $h_x = \pi_x \circ h$ and $h_y = \pi_y \circ h$. Applying Lemma 2.4 two times we can find $\beta_x, \beta_y \leq \alpha$ and $E = P_y \subset P_x$ from $\mathbb{P}_\alpha$ such that $h_x \circ \pi^{-1}_{\beta_x}$ is a function on $\pi_{\beta_x}[P_x] \in \mathbb{P}_{\beta_x}$, $h_y \circ \pi^{-1}_{\beta_y}$ is a function on $\pi_{\beta_y}[P_y] \in \mathbb{P}_{\beta_y}$, and each of these functions is either one-to-one or constant. Notice that

$$\text{either } h_x \text{ or } h_y \text{ is one-to-one on } E. \tag{13}$$

To see this, first note that for every $z \in E$ we have

$$h(z) = \langle \pi_x \circ h(z), \pi_y \circ h(z) \rangle = \langle (h_x \circ \pi^{-1}_{\beta_x})(\pi_{\beta_x}(z)), (h_y \circ \pi^{-1}_{\beta_y})(\pi_{\beta_y}(z)) \rangle.$$

Since h is one-to-one this implies that $\max\{\beta_x, \beta_y\} = \alpha$. By symmetry, we can assume that $\alpha = \beta_x$. Thus, $h_x = h_x \circ \pi^{-1}_{\beta_x}$ is either one-to-one or constant on $P_x = \pi_{\beta_x}[P_x]$. If h_x is one-to-one on P_x, then (13) holds. So, assume that h_x is constant on P_x. Then $\pi_x \circ h = h_x$ is constant on $E \subset P_x$, and so $h_y = \pi_y \circ h$ must be one-to-one on E, since h is one-to-one. Thus, (13) holds.

By symmetry, we can assume that h_x is one-to-one on E. So $\pi_x \circ h$ is a one-to-one function from E onto $\pi_x[h[E]] \subset \mathbb{R}$. In particular, $F_0 = h[E] \subset \mathbb{R}^2$ is a function from $\pi_x[h[E]]$ into $\mathbb{R}$. Then, by Lemma 4.2 used with $g = F_0$ and $h = \pi_x \circ h \restriction E$, we can find a subset $E_1 \in \mathbb{P}_\alpha$ of E and a function $f\colon \mathbb{R} \to \mathbb{R}$ extending $h[E_1] = g \restriction h[E_1]$ such that either f or f^{-1} belongs to $\mathcal{C}^1$. ∎

5 Proposition 4.1: A Generalization of a Theorem of Morayne

Our proof of Proposition 4.1 is based on the following lemmas, the first of which is a version of a theorem of Galvin [16, 17]. (For the proof see [20, Thm. 19.7] or [6]. Galvin proved his results for $\alpha = 1$.)

Lemma 5.1 *For every $0 < \alpha < \omega_1$ and every continuous symmetric function h from $(\mathfrak{C}^\alpha)^2 \setminus \Delta$ into $2 = \{0, 1\}$ there exists a $P \in \mathbb{P}_\alpha$ such that h is constant on $P^2 \setminus \Delta$.*

Proof For $j < 2$ let G_j be the set of all $s \in \mathfrak{C}^\alpha$ such that

$$(\forall \beta < \alpha)(\forall \varepsilon > 0)(\exists t \in \mathfrak{C}^\alpha)\ 0 < \rho(s, t) < \varepsilon \ \&\ s \restriction \beta = t \restriction \beta \ \&\ h(s, t) = j$$

and notice that

$$\text{each } G_j \text{ is a } G_\delta\text{-set} \quad \text{and} \quad \mathfrak{C}^\alpha = G_0 \cup G_1. \tag{14}$$

Indeed, to see that G_j is a G_δ-set it is enough to note that for every $\beta < \alpha$ and $\varepsilon > 0$ the set

$$G_j^{\beta,\varepsilon} = \{s \in \mathfrak{C}^\alpha \colon (\exists t \in \mathfrak{C}^\alpha)\ 0 < \rho(s,t) < \varepsilon \ \&\ s \restriction \beta = t \restriction \beta \ \&\ h(s,t) = j\}$$

is open in $\mathfrak{C}^\alpha$. So let $s \in G_j^{\beta,\varepsilon}$ and take $t \in \mathfrak{C}^\alpha$ witnessing it, that is, such that $0 < \rho(s,t) < \varepsilon$, $s \restriction \beta = t \restriction \beta$, and $h(s,t) = j$. We can choose basic open neighborhoods U and V of s and t, respectively, such that $U \times V \setminus \Delta \subset h^{-1}(j)$. In addition we can assume that $\pi_\beta[U] = \pi_\beta[V]$ and that each of the sets U and V has diameter less than $\delta = (\varepsilon - \rho(s,t))/3$. Then $s \in U \subset G_i^{\beta,\varepsilon}$ since for every $s' \in U$ there exists a $t' \in V$, $t' \neq s'$, with $s' \restriction \beta = t' \restriction \beta$ (since $\pi_\beta[U] = \pi_\beta[V]$), $h(s',t') \in h[U \times V \setminus \Delta] = \{j\}$ and

$$0 < \rho(s',t') \leq \rho(s',s) + \rho(s,t) + \rho(t,t') \leq \delta + \rho(s,t) + \delta < \varepsilon.$$

Thus each $G_j^{\beta,\varepsilon}$ is open and G_j is a G_δ-set.

To see the second part of (14) assume, by way of contradiction, that there exists an $s \in \mathfrak{C}^\alpha \setminus (G_0 \cup G_1)$. Let β_0, ε_0 and β_1, ε_1 witness that $s \notin G_0$ and $s \notin G_1$, respectively. Put $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\} > 0$ and $\beta = \max\{\beta_0, \beta_1\} < \alpha$ and find $t \in \mathfrak{C}^\alpha$ such that $t \restriction \beta = s \restriction \beta$, $\rho(s,t) < \varepsilon$, and $t(\beta) \neq s(\beta)$. Then there exists a $j < 2$ such that $h(s,t) = j$ and this, together with $t \restriction \beta_j = s \restriction \beta_j$ and $\rho(s,t) < \varepsilon_j$, contradicts the choice of β_j and ε_j. This finishes the proof of (14).

Next, find a $j < 2$ and a basic clopen set U in $\mathfrak{C}^\alpha$ such that G_j is residual in U. Replacing $\mathfrak{C}^\alpha$ with U, if necessary, we can assume that G_j is residual in $\mathfrak{C}^\alpha$. Using Fusion Lemma 2.1 with $\mathcal{A} = \mathcal{B}_\alpha$, we will find a $P \in \mathbb{P}_\alpha$ for which $P^2 \setminus \Delta \subset h^{-1}(j)$.

For each $k < \omega$ let $\mathcal{D}_k \subset [\mathcal{B}_\alpha]^{<\omega}$ be the collection of all families $\{P_i \colon i < m\}$ of sets of diameter less than 2^{-k} such that

$$P_i \times P_n \subset h^{-1}(j) \ \text{ for all } \ i < n < m. \tag{15}$$

It is obvious that the $\mathcal{D}_k$'s satisfy conditions (P1) and (P2) from Lemma 2.1. Thus, we need only to check (P3).

So, take $E \in \mathcal{B}_\alpha$ and $\gamma < \alpha$. It is enough to find disjoint $E_0, E_1 \in \mathcal{B}_\alpha$, subsets of E, such that $\pi_\gamma[E_0] = \pi_\gamma[E_1]$ and

$$E_0 \times E_1 \subset h^{-1}(j). \tag{16}$$

For this choose an $s \in E \cap G_j$ and let $\varepsilon_0 > 0$ be such that $B_\alpha(s,\varepsilon_0) \subset E$. By the definition of G_j we can find a $t \in \mathfrak{C}^\alpha$ for which $0 < \rho(s,t) < \varepsilon_0$, $s \restriction \gamma = t \restriction \gamma$, and $h(s,t) = j$. In particular $s,t \in E$ and $\langle s,t \rangle \in h^{-1}(j)$. Since h is continuous we can find an $\varepsilon > 0$ small enough that $E_0 = B_\alpha(s,\varepsilon)$ and $E_1 = B_\alpha(t,\varepsilon)$ are disjoint subsets of E for which (16) holds.

Now, by Lemma 2.1, there exist $\mathcal{E}_k = \{P_i^k : i < m_k\} \in \mathcal{D}_k$ such that

$$P = \bigcap_{k<\omega} \bigcup_{i<m_k} P_i^k \in \mathbb{P}_\alpha.$$

It is enough to show that $P^2 \setminus \Delta \subset h^{-1}(j)$. To see this, take different $s, t \in P$ and let $k < \omega$ be such that the distance between s and t is greater than 2^{-k}. Then they must belong to different P_i^k's from $\mathcal{E}_k$ and so, by (15), $\langle s, t\rangle \in h^{-1}(j)$. ■

We will also need the following simple fact, which must be well known.

Lemma 5.2 *There exists a continuous function $h\colon \mathfrak{C} \to [0,1]$ with the following property. If X is a zero-dimensional Polish space, then for every continuous function $f\colon X \to [0,1]$ there exists a continuous $g\colon X \to \mathfrak{C}$ such that $f = h \circ g$.*

Proof Let $\{U_\sigma : \sigma \in 2^{<\omega}\}$ be an open basis for $[0,1]$ such that $U_\varnothing = [0,1]$ and, for every $\sigma \in 2^k$, $U_\sigma = U_{\sigma^\frown 0} \cup U_{\sigma^\frown 1}$ and $\operatorname{diam}(U_\sigma) \le 2^{1-k}$. For every $s \in 2^\omega$ let $h(s) \in [0,1]$ be such that $\{h(s)\} = \bigcap_{n<\omega} \operatorname{cl}(U_{s\restriction n})$. It is clear that h is continuous.

To see that h is as required take X and f as in the lemma. For every $\sigma \in 2^{<\omega}$ choose an open set $V_\sigma \subset f^{-1}(U_\sigma)$ such that $V_\varnothing = X$, $V_{\sigma^\frown 0}$ and $V_{\sigma^\frown 1}$ are disjoint, and $V_{\sigma^\frown 0} \cup V_{\sigma^\frown 1} = V_\sigma$. This can be easily done by induction on the length of σ using zero-dimensionality of X.[2] Thus for every $n < \omega$ the sets $\{V_\sigma : \sigma \in 2^n\}$ form a clopen partition of X.

Define $g(x)$ as the unique $s \in \mathfrak{C}$ for which $x \in \bigcap_{n<\omega} V_{s\restriction n}$. Clearly g is continuous. Moreover, if $g(x) = s$ then

$$x \in \bigcap_{n<\omega} V_{s\restriction n} \subset f^{-1}\Big(\bigcap_{n<\omega} \operatorname{cl}(U_{s\restriction n})\Big) = f^{-1}(\{h(s)\}) = f^{-1}(\{h(g(x))\})$$

so that $f(x) \in \{h(g(x))\}$. Hence $f = h \circ g$. ■

The next lemma is already a very close approximation of Proposition 4.1.

Lemma 5.3 *If $\alpha < \omega_1$ and H is a continuous symmetric function from a set $(\mathfrak{C}^\alpha)^2 \setminus \Delta$ into $\mathfrak{C}$ then there exists an $E \in \mathbb{P}_\alpha$ such that H is uniformly continuous on $E^2 \setminus \Delta$.*

Proof For $n < \omega$ define $h_n\colon (\mathfrak{C}^\alpha)^2 \setminus \Delta \to 2$ by $h_n(s,t) = H(s,t)(n)$. Thus each h_n satisfies the assumptions of Lemma 5.1.

Using Fusion Lemma 2.1 with $\mathcal{A} = \mathbb{P}_\alpha$ we will find an $E \in \mathbb{P}_\alpha$ for which each h_n is uniformly continuous on $E^2 \setminus \Delta$. Then clearly $H = \langle h_n : n < \omega\rangle$ is also uniformly continuous on this set.

[2] Recall that every second countable zero-dimensional space X is strongly zero-dimensional, see *e.g.*, [18, Thm. 6.2.7]. In particular, for every open cover $\{W_0, W_1\}$ of X there are disjoint clopen sets $V_0 \subset W_0$ and $V_1 \subset W_1$ such that $V_0 \cup V_1 = X$.

For $k<\omega$ let $\mathcal{D}_k\subset[\mathbb{P}_\alpha]^{<\omega}$ be the collection of all families $\{P_i\colon i<m\}$ of pairwise disjoint sets such that

$$h_k \text{ is constant on } P_i\times P_i\setminus\Delta \text{ for each } i<m. \tag{17}$$

Clearly sets $\mathcal{D}_k$'s satisfy conditions (P1) and (P2) from Lemma 2.1. Thus, we need to verify only (P3).

So, fix $k<\omega$, $E\in\mathbb{P}_\alpha$, and $\gamma<\alpha$. It is enough to find disjoint subprisms E_0,E_1 of E such that $\pi_\gamma[E_0]=\pi_\gamma[E_1]$ and

$$h_k \text{ is constant on } E_j\times E_j\setminus\Delta \text{ for each } j<2.$$

Let $f\in\Phi_{\rm prism}(\alpha)$ be such that $E=f[\mathfrak{C}^\alpha]$ and let $h\colon(\mathfrak{C}^\alpha)^2\setminus\Delta\to 2$ be defined by $h(s,t)=h_k(f(s),f(t))$. Then h satisfies the assumptions of Lemma 5.1 so there exists a $P\in\mathbb{P}_\alpha$ such that h is constant on $P^2\setminus\Delta$. Choose disjoint subsets $E_0,E_1\in\mathbb{P}_\alpha$ of P such that $\pi_\gamma[E_0]=\pi_\gamma[E_1]$. (If $g\in\Phi_{\rm prism}(\alpha)$ is such that $P=f[\mathfrak{C}^\alpha]$ and $B_i=\{x\in\mathfrak{C}^\alpha\colon x(\gamma)(0)=i\}$ then we can put $E_i=g[B_i]$.) Then E_0 and E_1 satisfy (P3).

Now, by Lemma 2.1, there exist $\mathcal{E}_k=\left\{P^k_i\colon i<m_k\right\}\in\mathcal{D}_k$ such that

$$E=\bigcap_{k<\omega}\bigcup_{i<m_k}P^k_i\in\mathbb{P}_\alpha.$$

Notice that if $\left\{P^k_i\colon i<m_k\right\}$ belongs to $\mathcal{D}_k$ then h_k is uniformly continuous on

$$\left(\bigcup_{i<m_k}P^k_i\right)^2\setminus\Delta=\left(\bigcup_{i\neq n}P^k_i\times P^k_n\right)\cup\bigcup_{i<m_k}\left(P^k_i\times P^k_i\setminus\Delta\right).$$

So each h_k is uniformly continuous $E^2\setminus\Delta\subset\left(\bigcup_{i<m_k}P^k_i\right)^2\setminus\Delta.$ ∎

Proof of Proposition 4.1 Let $f_0\in\Phi_{\rm prism}(\alpha)$ be such that $E=f_0[\mathfrak{C}^\alpha]$ and put

$$F=G\circ\langle h\circ f_0,h\circ f_0\rangle\colon(\mathfrak{C}^\alpha)^2\setminus\Delta\to[0,1]. \tag{18}$$

Note also that

$$F=h_0\circ H \tag{19}$$

for some continuous symmetric function from $H\colon(\mathfrak{C}^\alpha)^2\setminus\Delta\to\mathfrak{C}$ and continuous $h_0\colon\mathfrak{C}\to[0,1]$. This follows immediately from Lemma 5.2 used with $f=F\restriction\{\langle x,y\rangle\in\mathfrak{C}^\alpha\times\mathfrak{C}^\alpha\colon x<y\}$, where $<$ is the lexicographical order on $\mathfrak{C}^\alpha$. (We use the lexicographical order in which $\mathfrak{C}^\alpha$ is identified with $2^{\alpha\times\omega}$ and $\alpha\times\omega$ is ordered in type ω. Then the set $\{\langle x,y\rangle\in\mathfrak{C}^\alpha\times\mathfrak{C}^\alpha\colon x<y\}$ is open in $\mathfrak{C}^\alpha\times\mathfrak{C}^\alpha$.)

Then by Lemma 5.3, there exists an $E_0\in\mathbb{P}_\alpha$ such that H is uniformly continuous on $(E_0)^2\setminus\Delta$. So H can be extended to a uniformly continuous function $\hat{H}$ on $(E_0)^2$. Then the function

$$\hat{G}=h_0\circ\hat{H}\circ\langle h\circ f_0,h\circ f_0\rangle^{-1}=h_0\circ\hat{H}\circ\langle(f_0)^{-1}\circ h^{-1},(f_0)^{-1}\circ h^{-1}\rangle$$

is also uniformly continuous on $(h[f_0[E_0]])^2$. Put $E_1 = f_0[E_0]$ and notice that it is as desired.

Indeed, clearly $E_1 \in \mathbb{P}_\alpha$ and $E_1 \subset E$. Moreover, it is not difficult to see that $G \restriction (h[E_1])^2 \setminus \Delta = \hat{G} \restriction (h[E_1])^2 \setminus \Delta$. So G is uniformly continuous on $(h[E_1])^2 \setminus \Delta$. ■

6 Theorem 3.5: on $\mathrm{cov}\,(D^n, \mathcal{C}^n) < \mathfrak{c}$

In the proof we will use the following lemma.

Lemma 6.1 *For $n < \omega$ let $f \in \mathcal{C}^n$ and let $P \subset \mathbb{R}$ be a perfect set for which the function $F\colon P^2 \setminus \Delta \to \mathbb{R}$ defined by*

$$F(x, y) = \frac{f^{(n)}(x) - f^{(n)}(y)}{x - y}$$

is uniformly continuous and bounded. Then $f \restriction P$ can be extended to a $\mathcal{C}^{n+1}$ function.

Proof This follows from the fact that $f \restriction P$ satisfies the assumptions of Whitney's extension theorem. To see this, notice first that F naturally extends to a continuous function on P^2 with $F(a, a) = f^{(n+1)}(a)$. Next, for $q = 1, 2, 3, \ldots$ and $a \in P$ let

$$\eta_q(a) = \sup\left\{\left|\frac{f^{(n)}(x) - f^{(n)}(a)}{x - a} - f^{(n+1)}(a)\right| : 0 < |x - a| < \frac{1}{q}\right\}.$$

In the second part of the proof of [15, Thm. 3.1.15] it is shown that if

$$\lim_{q\to\infty} \sup\{\eta_q(a)\colon a \in P\} = 0, \tag{20}$$

then $f \restriction P$ satisfies the assumptions of Whitney's extension theorem. However we have

$$\frac{f^{(n)}(x) - f^{(n)}(a)}{x - a} - f^{(n+1)}(a) = F(x, a) - F(a, a),$$

so uniform continuity of F clearly implies (20). ■

Proof of Theorem 3.5 The lower bound inequalities

$$\mathrm{cov}\,(D^n, \mathcal{C}^n) > \omega \quad \text{and} \quad \mathrm{cov}\,(\text{“}D^n\text{”}, \text{“}\mathcal{C}^n\text{”}) > \omega$$

follow from Example 7.7. So it is enough to prove only that these numbers are $\leq \omega_1$.

To prove $\mathrm{cov}\,(D^n, \mathcal{C}^n) \leq \omega_1$, take an $f \in D^n$ and note that, by $\mathrm{CPA}_{\mathrm{prism}}$, it is enough to show that the set

$$\mathcal{E} = \{E \in \mathrm{perf}(\mathbb{R})\colon (\exists h \in \mathcal{C}^n(\mathbb{R}))\ h \restriction E = f \restriction E\}$$

is $\mathcal{F}_{\mathrm{prism}}$-dense. So fix a prism P in $\mathbb{R}$ and let $k\colon [-\infty, \infty] \to [0, 1]$ be a homeomorphism. Applying Proposition 4.1 n times in the same way as in the proof of Lemma 4.2, we find a subprism E of P such that for each $i < n$ the function $k \circ F_i \colon E^2 \setminus \Delta \to [0,1]$ is uniformly continuous, where $F_i \colon E^2 \setminus \Delta \to \mathbb{R}$ is defined by

$$F_i(x, y) = \frac{f^{(i)}(x) - f^{(i)}(y)}{x - y}.$$

So each F_i can be extended to a continuous function $\bar{F}_i \colon E^2 \to [-\infty, \infty]$. Note also that since $\bar{F}_i(x,x) = f^{(i+1)}(x) \in \mathbb{R}$, as $f \in D^n$, we in fact have $\bar{F}_i[E^2] \subset \mathbb{R}$.

Next, starting with $f_0 = f$ we use Lemma 6.1 to prove by induction that for every $i < n$ there exists an $f_{i+1} \in \mathcal{C}^{i+1}(\mathbb{R})$ extending $f_i \restriction E$. Then the function $h = f_n \in \mathcal{C}^n(\mathbb{R})$ witnesses that $E \in \mathcal{E}$.

To prove $\operatorname{cov}(\text{“}D^n\text{”}, \text{“}\mathcal{C}^n\text{”}) \le \omega_1$, take an $f \in \text{“}D^n\text{”}$. As before, it is enough to show that

$$\mathcal{E}' = \{E' \in \operatorname{perf}(\mathbb{R}) \colon (\exists h \in \text{“}\mathcal{C}^n(\mathbb{R})\text{”})\ h \restriction E' = f \restriction E'\}$$

is $\mathcal{F}_{\text{prism}}$-dense. So fix a prism P in $\mathbb{R}$ and find E, F_i's, and $\bar{F}_i$'s as above. Note that F_i's are well defined since $f \in \text{“}D^n\text{”} \subset \mathcal{C}^{n-1}$. By the same reason we have that $\bar{F}_i[P^2] \subset \mathbb{R}$ for all $i < n-1$. However, $\bar{F}_{n-1}$ can have infinite values.

Proceeding as in the proof of Lemma 4.2, decreasing E if necessary, we can assume that either the range of $\bar{F}_{n-1}$ is bounded or $\bar{F}_{n-1} \restriction P^2 \cap \Delta$ is a constant equal to ∞ or $-\infty$. If $\bar{F}_{n-1}$ is bounded, then taking $E' = E$ we are done as in the previous case. So assume that $\bar{F}_{n-1}[P^2 \cap \Delta] = \{\infty\}$. (The case of $-\infty$ is handled by replacing f with $-f$.) Then $f^{(n-1)}$ and E satisfy the assumptions of Brown's version of Whitney's extension theorem [3, Thm. 2.1]. So, we can find a “$\mathcal{C}^1$” extension $g \colon \mathbb{R} \to \mathbb{R}$ of $f^{(n-1)} \restriction E$ such that $g'[E] = (f^{(n-1)})'[E] = \{\infty\}$ and $g'[\mathbb{R} \setminus E] \subset \mathbb{R}$. By $(n-1)$-times integrating g we can find a $G \colon \mathbb{R} \to \mathbb{R}$ such that $G^{(n-1)} = g$. Then $G \in \text{“}\mathcal{C}^n\text{”}$. Next notice that $G - f \in \mathcal{C}^n(E)$, since $(G - f)^{(n-1)} = g - f^{(n-1)} \equiv 0$ on E. Now, proceeding as above for the case of $f \in \mathcal{C}^n$, we can find a subprism E' of E and a function $\hat{h} \in \mathcal{C}^n(\mathbb{R})$ extending $G - f \restriction E'$. Then function $h = G - \hat{h}$ belongs to “$\mathcal{C}^n$” as a difference of functions from “$\mathcal{C}^n$” and $\mathcal{C}^n$. Moreover, h extends $f \restriction E'$ since $h = G - \hat{h} = G - (G - f) = f$ on E'. So, h witnesses $E' \in \mathcal{E}'$. ■

7 Examples Related to cov Operator

We will start with the examples needed for the proof of Proposition 3.4 which give $\mathfrak{c}$ as a lower bound for the appropriate numbers $\operatorname{cov}(\mathcal{A}, \mathcal{F})$.

Example 7.1 *There exist a homeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ and a perfect set $P \subset \mathbb{R}$ such that $h, h^{-1} \in$ “$\mathcal{C}^2$”, $h'' \restriction P \equiv \infty$, and $(h^{-1})'' \restriction h[P] \equiv -\infty$. In particular* $\neg \operatorname{IntTh}\left(h \restriction P, D^2_{\text{perf}} \cup (D^2_{\text{perf}})^{-1}\right)$ *and*

$$\operatorname{cov}\left(\text{“}\mathcal{C}^2\text{”}, D^2_{\text{perf}} \cup (D^2_{\text{perf}})^{-1}\right) = \operatorname{cov}\left(h, D^2_{\text{perf}} \cup (D^2_{\text{perf}})^{-1}\right) = \mathfrak{c}.$$

Proof First notice that there exist a strictly increasing homeomorphism h_0 from $\mathbb{R}$ onto $(0, \infty)$ and a perfect set $P \subset \mathbb{R}$ such that

$$h_0 \in \text{“}\mathcal{C}^1\text{” and } h_0' \restriction P \equiv \infty. \tag{21}$$

Indeed, let C be an arbitrary nowhere dense perfect subset of $[2, 3)$ with $2 \in C$ and let $d(x)$ denote the distance between $x \in \mathbb{R}$ and C. Let $f_0 \colon (0, \infty) \to [0, \infty)$ be defined by $f_0(x) = x^{-2}$ for $x \in (0, 1]$ and $f_0(x) = d(x)$ for $x \in [1, \infty)$. Then f_0 is continuous and $f_0(x) = 0$ precisely when $x \in C$. Define a strictly increasing function f from $(0, \infty)$ onto $\mathbb{R}$ by a formula $f(x) = \int_1^x f_0(t)\, dt$. Then $f' = f_0$ and $f(x) = 1 - \frac{1}{x}$ on $(0, 1)$. It is easy to see that $h_0 = f^{-1}$ and $P = f[C] \subset (0, \infty)$ satisfy (21).

Now put $h(x) = \int_0^x h_0(t)\, dt$. Then clearly h is strictly increasing since h_0 is positive. Also, h is onto $\mathbb{R}$, as on $(-\infty, 0)$ we have $h_0(x) = \frac{1}{1-x}$ and so $h(x) = -\ln(1-x)$. It is easy to see that $h' = h_0$, so by (21), $h \in$ "$\mathcal{C}^2$" and $h'' \restriction P \equiv \infty$. Also, if $g = h^{-1}$ then $g'(x) = 1/h'(g(x)) = 1/h_0(g(x)) > 0$ is strictly decreasing and $h^{-1} = g \in \mathcal{C}^1$. Thus, to see that $h^{-1} = g \in$ "$\mathcal{C}^2$" and that $(h^{-1})'' \equiv -\infty$ on $h[P] = h[f[C]]$ it is enough to differentiate $g'(x)$ (note that the differentiation formulas are valid if just one of the terms is infinite) to get $g''(x) = -[h'(g(x))]^{-2} h''(g(x)) g'(x) = -h''(g(x))(g'(x))^3$. Thus, h and P have the desired properties.

To see the additional part, note first that for every $f \in D^2_{\text{perf}}$ functions f and $h \restriction P$ may agree on at most countable set S, since at any point x of a perfect subset Q of S we would have

$$(h \restriction Q)''(x) = \infty \neq (f \restriction Q)''(x).$$

Similarly, $|f \cap (h \restriction P)| \leq \omega$ for every $f \in (D^2_{\text{perf}})^{-1}$. This clearly implies the additional part. ■

Example 7.2 *There exists a perfect set $P \subset \mathbb{R}$ and a function $f \in$ "$\mathcal{C}^1$" such that $f'(x) = \infty$ for every $x \in P$. In particular $\neg\,\mathrm{IntTh}\left(f \restriction P, D^1_{\text{perf}}\right)$ and*

$$\mathrm{cov}\left(\mathrm{Borel}, \mathcal{C}^1\right) = \mathrm{cov}\left(\text{“}\mathcal{C}^1\text{”}, \mathcal{C}^1\right) = \mathrm{cov}\left(\text{“}\mathcal{C}^1\text{”}, D^1_{\text{perf}}\right) = \mathrm{cov}\left(f, D^1_{\text{perf}}\right) = \mathfrak{c}.$$

Proof If f is a function h_0 from (21) then it has the desired properties.

For such an f and any function $g \in D^1_{\text{perf}}$ the intersection $f \cap g$ must be finite. So

$$\mathfrak{c} \geq \mathrm{cov}\left(\mathrm{Borel}, \mathcal{C}^1\right) \geq \mathrm{cov}\left(\text{“}\mathcal{C}^1\text{”}, D^1_{\text{perf}}\right) \geq \mathrm{cov}\left(f, D^1_{\text{perf}}\right) \geq \mathfrak{c}.$$

Monotonicity of cov operator gives the other equations. ■

Example 7.3 *For every $0 < n < \omega$ there exists an $f \in$ "$\mathcal{C}^n$" and a perfect set $P \subset \mathbb{R}$ such that $\neg\,\mathrm{IntTh}(f \restriction P, D^n_{\text{perf}})$ so that*

$$\mathrm{cov}(\text{“}\mathcal{C}^n\text{”}, \mathcal{C}^n) = \mathrm{cov}(\text{“}\mathcal{C}^n\text{”}, D^n_{\text{perf}}) = \mathrm{cov}(f, D^n_{\text{perf}}) = \mathfrak{c}.$$

Proof For $n = 1$ this is a restatement of Example 7.2. The general case can be done by induction: If f is good for some n and F is a definite integral of f then $F \in$ "$\mathcal{C}^{n+1}$" and $\neg\,\mathrm{IntTh}(F \restriction P, D^{n+1}_{\text{perf}}) = \mathfrak{c}$. ■

Example 7.4 *There exists an $f \in \mathcal{C}^1$ and a perfect set $P \subset \mathbb{R}$ such that $|(f \restriction P) \cap g| \leq \omega$ for every $g \in$ "D^2". In particular $\neg\,\mathrm{IntTh}(f \restriction P,$ "D^2"$)$ and*

$$\mathrm{cov}\left(\mathcal{C}^1, \text{“}D^2\text{”}\right) = \mathrm{cov}\left(f, \text{“}D^2\text{”}\right) = \mathfrak{c}.$$

Proof In [1, Thm. 22] the authors construct a perfect set $P \subset [0,1]$ and a function $f \in \mathcal{C}^1$ which have the desired properties. The argument for this is implicitly included in the proof of [1, Thm. 22].

Function f has the property that $f'(x) = 0$ for all $x \in P$. Now, assume that some $g \in$ "D^2" agrees with f on a perfect set $Q \subset P$. Then clearly we would have $(g \restriction Q)'' \equiv [(g \restriction Q)']' \equiv [(f \restriction Q)']' \equiv [0]' \equiv 0$. On the other hand, in [1, Thm. 22] it is shown that for such a g we would have $g''(x) \in \{\pm\infty\}$ for every $x \in Q$, a contradiction.[3] ■

Example 7.5 *For every* $0 < n < \omega$ *there exist an* $f \in \mathcal{C}^n$ *and a perfect set* $P \subset \mathbb{R}$ *such that* $\neg\,\mathrm{IntTh}(f \restriction P, \text{“}D^{n+1}\text{”})$ *and*

$$\mathrm{cov}\left(\mathcal{C}^n, \text{“}D^{n+1}\text{”}\right) = \mathrm{cov}\left(f, \text{“}D^{n+1}\text{”}\right) = \mathfrak{c}.$$

Proof For $n = 1$ this is a restatement of Example 7.4. The general case can be done by induction: If f is good for some n and F is a definite integral of f then $F \in \mathcal{C}^{n+1}$ and $\neg\,\mathrm{IntTh}(F \restriction P, D^{n+1}_{\mathrm{perf}}) = \mathfrak{c}$. ■

Next we will describe the examples showing that the $\mathrm{cov}(\mathcal{A}, \mathcal{F})$ numbers considered in Corollary 3.3 and Theorem 3.5 have values greater than ω. In what follows $\mathrm{cov}(\mathcal{M})$ ($\mathrm{cov}(\mathcal{N})$, respectively) will stand for the smallest cardinality of a family $\mathcal{F} \subset \mathcal{P}(\mathbb{R})$ of measure zero sets (nowhere dense, respectively) such that $\mathbb{R} = \bigcup \mathcal{F}$.

Example 7.6 *There exists a function* $f \in D^1$ *such that*

$$\mathrm{cov}\left(f, \text{“}\mathcal{C}^1\text{”} \cup (D^1)^{-1}\right) \geq \mathrm{cov}(\mathcal{M}) > \omega.$$

In particular

$$\mathrm{cov}\left(\mathrm{Borel}, \text{“}\mathcal{C}^1\text{”}\right) \geq \mathrm{cov}\left(D^1, \text{“}\mathcal{C}^1\text{”}\right) \geq \mathrm{cov}(\mathcal{M}) > \omega$$

and

$$\mathrm{cov}\left(\mathrm{Borel}, \mathcal{C}^1 \cup (\mathcal{C}^1)^{-1}\right) \geq \mathrm{cov}\left(D^1, \mathcal{C}^1 \cup (\mathcal{C}^1)^{-1}\right) \geq \mathrm{cov}(\mathcal{M}) > \omega.$$

Proof We will construct the function f only on $[0,1]$. It can be easily modified to a function defined on $\mathbb{R}$.

Let $E \subset [0,1]$ be an F_σ-set of measure 1 such that $E^c = [0,1] \setminus E$ is dense in $[0,1]$. It is well known that there exists a derivative $g\colon [0,1] \to [0,1]$ such that $g[E] \subset (0,1]$ and $g[E^c] = \{0\}$. (See *e.g.*, [5, p. 24].) Let $f\colon [0,1] \to \mathbb{R}$ be such that $f' = g$. We claim that this f is as desired.

Indeed, by way of contradiction assume that for some $\kappa < \mathrm{cov}(\mathcal{M})$ there exists a family $\{h_\xi \in \mathbb{R}^\mathbb{R} \colon \xi < \kappa\} \subset$ "$\mathcal{C}^1$" $\cup (D^1)^{-1}$ such that $f \subset \bigcup_{\xi<\kappa} h_\xi$. Since h_ξ are closed subsets of $\mathbb{R}^2$ and the graph of f is compact, we see that the x-coordinate

[3] Actually, the calculation in [1, Thm. 22] is done under the assumption that $g \in C^2$, but it works also under our weaker assumption that $g \in$ "D^2".

projections $P_\xi = \pi_x[f \cap h_\xi]$ are closed. So, $[0,1]$ is covered by less than $\text{cov}(\mathcal{M})$ closed sets P_ξ. Thus, there exists an $\eta < \kappa$ such that P_η has non-empty interior $U = \text{int}(P_\eta)$.

Now, if $h_\eta \in$ "$\mathcal{C}^1$" then $h'_\eta = f' = g$ on U, which is impossible, since h'_η is continuous, while g is not continuous on any non-empty open set. So assume that $h_\eta \in (D^1)^{-1}$. Note that f is strictly increasing as an integral of function g which is strictly positive a.e. So f^{-1} is strictly increasing and agrees with $h = h_\eta^{-1} \in D^1$ on an open set $f[U]$. But then if $x \in U \setminus E$ then $h'(f(x)) = (f^{-1})'(f(x)) = \frac{1}{f'(x)} = \infty$, which contradicts $h \in D^1$. ■

Note also that if f from Example 7.6 is replaced by its $(n-1)$-st antiderivative then we get also the following example.

Example 7.7 *For any* $0 < n < \omega_1$ *there exists an* $f \in D^n$ *such that*

$$\text{cov}\left(D^n, \text{“}\mathcal{C}^n\text{”}\right) \geq \text{cov}\left(f, \text{“}\mathcal{C}^n\text{”}\right) \geq \text{cov}(\mathcal{M}) > \omega.$$

Example 7.8 *There exists an* $f \in \mathcal{C}^0$ *such that*

$$\text{cov}(\mathcal{C}^0, \text{“}D^1_{\text{perf}}\text{”}) \geq \text{cov}(f, \text{“}D^1_{\text{perf}}\text{”}) \geq \text{cov}(\mathcal{N}) > \omega.$$

Moreover, for every $n < \omega$ *if* $F \in \mathcal{C}^n$ *is such that* $F^{(n)} = f$ *then*

$$\text{cov}(\mathcal{C}^n, \text{“}D^{n+1}_{\text{perf}}\text{”}) \geq \text{cov}(F, \text{“}D^{n+1}_{\text{perf}}\text{”}) \geq \text{cov}(\mathcal{N}) > \omega$$

and

$$\text{cov}\left(\text{Borel}, \text{“}\mathcal{C}^\infty_{\text{perf}}\text{”}\right) \geq \text{cov}\left(\mathcal{C}^n, \text{“}\mathcal{C}^\infty_{\text{perf}}\text{”}\right) \geq \text{cov}\left(F, \text{“}\mathcal{C}^\infty_{\text{perf}}\text{”}\right) \geq \text{cov}(\mathcal{N}) > \omega.$$

Proof A continuous function f justifying $\text{cov}(f, \text{“}D^1_{\text{perf}}\text{”}) \geq \text{cov}(\mathcal{N})$ was pointed out by Morayne: just take any $f \in \mathcal{C}$ for which there is a set $A \subset \mathbb{R}$ of positive measure for which $|f^{-1}(a)| = \mathfrak{c}$ for all $a \in A$. (See [26, Thm. 6.1].)

To see the additional part, let $\mathcal{G} = \{g_\xi \colon \xi < \kappa\}$ be an infinite subset of "D^{n+1}_{perf}" $\cup$ "$\mathcal{C}^\infty_{\text{perf}}$" such that $F \subset \bigcup \mathcal{G}$. We need to show that $\kappa \geq \text{cov}(\mathcal{N})$. For this, first note that for every $\xi < \kappa$ the domain of $F \cap g_\xi$ can be represented as a union of a perfect set P_ξ (which can be empty) and a countable (scattered) set S_ξ. Let $S = \bigcup_{\xi<\kappa} S_\xi$ and note that it has cardinality at most κ. Since $F \restriction P_\xi = g_\xi \restriction P_\xi$, by an easy induction on $i \leq n$ we can prove that

$$F^{(i)} \restriction P_\xi = (g_\xi \restriction P_\xi)^{(i)} \text{ provided } g_\xi \in \text{“}D^i_{\text{perf}}\text{” and } P_\xi \neq \varnothing. \tag{22}$$

Thus, if $g_\xi \in$ "D^{n+1}_{perf}" and $P_\xi \neq \varnothing$, then $f \restriction P_\xi = F^{(n)} \restriction P_\xi = (g_\xi \restriction P_\xi)^{(n)} \in$ "D^1_{perf}". On the other hand, if $g_\xi \in$ "$\mathcal{C}^\infty_{\text{perf}}$" $\setminus$ "D^{n+1}_{perf}" then $P_\xi = \varnothing$. Indeed, otherwise there is an $i \leq n$ such that $g_\xi \in$ "D^i_{perf}" and $g_\xi^{(i)}$ is constant equal to ∞ or $-\infty$. So, by (22), for any $x \in P_\xi$ a real number $F^i(x)$ belongs to $\{-\infty, \infty\}$, a contradiction.

Thus $\mathcal{F} = \{f \restriction P_\xi \colon \xi < \kappa \;\&\; P_\xi \neq \varnothing\} \cup \{f \restriction \{x\} \colon x \in S\} \subset$ "D^1_{perf}" has cardinality at most κ and it covers f. So, by the first part, $\kappa \geq \text{cov}(\mathcal{N})$. ■

8 Proof of Fusion Lemma 2.1

Notice that if $P \in \mathbb{P}_\alpha$ and $0 < \beta < \alpha$ then

(23) $$P \cap \pi_\beta^{-1}(P') \in \mathbb{P}_\alpha \quad \text{for every } P' \in \mathbb{P}_\beta \text{ with } P' \subset \pi_\beta[P].$$

Indeed, let $f \in \Phi_{\text{prism}}(\beta)$ and $g \in \Phi_{\text{prism}}(\alpha)$ be such that $f[\mathfrak{C}^\beta] = P'$ and $g[\mathfrak{C}^\alpha] = P$. Let $Q = (g \restriction\restriction \beta)^{-1}[P'] = (g \restriction\restriction \beta)^{-1} \circ f[\mathfrak{C}^\beta]$. Then, $Q \in \mathbb{P}_\beta$ since, by (5), $(g \restriction\restriction \beta)^{-1} \circ f \in \Phi_{\text{prism}}(\beta)$. Thus $\pi_\beta^{-1}(Q)$ belongs to $\mathbb{P}_\alpha$ and $P \cap \pi_\beta^{-1}(P') = g[\pi_\beta^{-1}(Q)] \in \mathbb{P}_\alpha$.

For a fixed $0 < \alpha < \omega_1$ let $\{\langle \beta_k, n_k \rangle : k < \omega\}$ be an enumeration of $\alpha \times \omega$ used in the definition (1) of the metric ρ and let

(24) $$A_k = \{\langle \beta_i, n_i \rangle : i < k\} \quad \text{for every } k < \omega.$$

Lemma 8.1 (Master Fusion Lemma) *Let $0 < \alpha < \omega_1$ and for every $k < \omega$ let $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha : s \in 2^{A_k}\right\}$. Assume that for every $k < \omega$, $s, t \in 2^{A_k}$, and $\beta < \alpha$ we have:*

(i) *the diameter of E_s is less than or equal to 2^{-k},*
(ii) *if $r \in \bigcup_{i<\omega} 2^{A_i}$ and $r \subset s$ then $E_s \subset E_r$,*
(iii) *if $s \restriction (\beta \times \omega) = t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] = \pi_\beta[E_t]$,*
(iv) *if $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] \cap \pi_\beta[E_t] = \varnothing$.*

Then $Q = \bigcap_{k<\omega} \bigcup \mathcal{E}_k$ belongs to $\mathbb{P}_\alpha$.

Proof For $x \in \mathfrak{C}^\alpha$ let $\bar{x} \in 2^{\alpha \times \omega}$ be defined by $\bar{x}(\beta, n) = x(\beta)(n)$.

First note that, by conditions (i) and (iv), for every $k < \omega$ the sets in $\mathcal{E}_k$ are pairwise disjoint and each of the diameter at most 2^{-k}. Thus, taking into account (ii), function $h \colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ defined by

$$h(x) = r \iff \{r\} = \bigcap_{k<\omega} E_{\bar{x} \restriction A_k}$$

is well defined and is one-to-one. It is also easy to see that h is continuous and that $Q = h[\mathfrak{C}^\alpha]$. Thus, we need to prove only that $h \in \Phi_{\text{prism}}(\alpha)$, that is, that h is projection-keeping.

To show this fix $\beta < \alpha$, put $S = \bigcup_{i<\omega} 2^{A_i}$, and notice that, by (i) and (iii), for every $x \in \mathfrak{C}^\alpha$ we have

$$\begin{aligned}
\{h(x) \restriction \beta\} &= \pi_\beta\left[\bigcap\{E_{\bar{x}\restriction A_k} : k < \omega\}\right] \\
&= \bigcap\{\pi_\beta[E_{\bar{x}\restriction A_k}] : k < \omega\} \\
&= \bigcap\{\pi_\beta[E_s] : s \in S \;\&\; s \subset \bar{x}\} \\
&= \bigcap\{\pi_\beta[E_s] : s \in S \;\&\; s \restriction (\beta \times \omega) \subset \bar{x}\}.
\end{aligned}$$

Now, if $x \restriction \beta = y \restriction \beta$ then for every $s \in S$

$$s \restriction (\beta \times \omega) \subset \bar{x} \ \Leftrightarrow \ s \restriction (\beta \times \omega) \subset \bar{y},$$

so $h(x) \restriction \beta = h(y) \restriction \beta$.

On the other hand, if $x \restriction \beta \neq y \restriction \beta$ then there exists $k < \omega$ big enough such that for $s = \bar{x} \restriction A_k$ and $t = \bar{y} \restriction A_k$ we have $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$. But then $\{h(x) \restriction \beta\}$ and $\{h(y) \restriction \beta\}$ are subsets of $\pi_\beta[E_s]$ and $\pi_\beta[E_t]$, respectively, which, by (iv), are disjoint. So, $h(x) \restriction \beta \neq h(y) \restriction \beta$. ■

Proof of Lemma 2.1 Let us define $\mathcal{D}_{-1} = \{\{\mathfrak{C}^\alpha\}\}$. It is enough to construct a sequence $\langle \mathcal{E}_k \in \mathcal{D}_{k-1} \colon k < \omega \rangle$ satisfying conditions (i)–(iv) from Lemma 8.1. This will be done by induction on $k < \omega$.

We start with $\mathcal{E}_0 = \{\mathfrak{C}^\alpha\}$. Clearly at this stage (i)–(iv) are satisfied. So, assume that for some $k < \omega$ a sequence $\langle \mathcal{E}_j \colon j \leq k \rangle$ satisfying (i)–(iv) is already defined. We will construct $\mathcal{E}_{k+1}$.

Let $\{s_i \colon i < 2^k\}$ be an enumeration of 2^{A_k}. Thus $\mathcal{E}_k = \{E_{s_i} \colon i < 2^k\}$. Also, let $\gamma = \max\{\beta_0, \ldots, \beta_k\} < \alpha$, and for every $i, m < 2^k$ put

$$\beta_i^m = \max\{\beta \leq \gamma \colon s_i \restriction (\beta \times \omega) = s_m \restriction (\beta \times \omega)\}.$$

As a first step of the proof we will construct, by induction on $m \leq 2^k$, the sequences $\langle \{E_{s_i}^m \in \mathcal{A} \colon i < 2^k\} \colon m \leq 2^k \rangle$ and $\langle P_m^j \in \mathcal{A} \colon j < 2 \;\&\; m < 2^k \rangle$ such that for every $n < m \leq 2^k$ and $i < 2^k$,

(a) $\mathcal{E}^m \stackrel{\text{def}}{=} \{E_{s_i}^m \in \mathcal{A} \colon i < 2^k\}$ satisfies (iii),
(b) $E_{s_i}^n \supset E_{s_i}^m$,
(c) P_n^0 and P_n^1 are disjoint subsets of $E_{s_n}^n$ such that $\{P_n^0, P_n^1\} \in \mathcal{D}_k$ and $\pi_\gamma[P_n^0] = \pi_\gamma[P_n^1] = \pi_\gamma[E_{s_n}^{n+1}]$.

We start with putting $E_{s_i}^0 = E_{s_i}$ for every $i < 2^k$. So, (a)–(c) clearly hold. Next, if for an $m < 2^k$ family $\mathcal{E}^m$ satisfying (iii) is already constructed, apply (P3) to find disjoint $P_m^0, P_m^1 \in \mathcal{D}_k$ subsets of $E_{s_m}^m$ for which $\pi_\gamma[P_m^0] = \pi_\gamma[P_m^1]$. Then for $i < 2^k$ we put

$$E_{s_i}^{m+1} = E_{s_i}^m \cap \pi_{\beta_i^m}^{-1}(\pi_{\beta_i^m}[P_m^0]) = \left\{x \in E_{s_i}^m : x \restriction \beta_i^m \in \pi_{\beta_i^m}[P_m^0]\right\}. \quad (25)$$

Notice that $\pi_{\beta_i^m}[P_m^0] \subset \pi_{\beta_i^m}[E_{s_m}^m] = \pi_{\beta_i^m}[E_{s_i}^m]$, so by (23), $E_{s_i}^{m+1} \in \mathcal{A}$. Also, by the inductive assumption (a),

$$\pi_{\beta_i^m}[E_{s_i}^{m+1}] = \pi_{\beta_i^m}[E_{s_i}^m] \cap \pi_{\beta_i^m}[P_m^0] = \pi_{\beta_i^m}[E_{s_m}^m] \cap \pi_{\beta_i^m}[P_m^0] = \pi_{\beta_i^m}[P_m^0].$$

Since $\beta_m^m = \gamma$, this implies immediately (c). It is clear that (b) holds. Thus, it is enough to show that $\mathcal{E}^{m+1}$ satisfies (iii). So, pick $\beta < \alpha$ and different $i < j < 2^k$ such that $s_i \restriction (\beta \times \omega) = s_j \restriction (\beta \times \omega)$. If $\beta \leq \beta_i^m$ then also $\beta \leq \beta_j^m$ and $\pi_\beta[E_{s_i}^{m+1}] =$

$\pi_\beta[P^0_m] = \pi_\beta[E^{m+1}_{s_j}]$. So, assume that $\beta > \beta^m_i$ and $\beta > \beta^m_j$. Then $\beta^m_i = \beta^m_j$ and

$$\begin{aligned}\pi_\beta[E^{m+1}_{s_i}] &= \left\{\pi_\beta(x) \colon x \in E^m_{s_i} \ \&\ \pi_\beta(x) \restriction \beta^m_i \in \pi_{\beta^m_i}[P^0_m]\right\} \\ &= \left\{\pi_\beta(x) \colon x \in E^m_{s_j} \ \&\ \pi_\beta(x) \restriction \beta^m_j \in \pi_{\beta^m_j}[P^0_m]\right\} \\ &= \pi_\beta[E^{m+1}_{s_j}].\end{aligned}$$

So $\mathcal{E}^{m+1}$ satisfies (iii). This finishes the construction.

Next for $i < 2^k$ put $E'_{s_i} = E^{2^k}_{s_i} \subset E_{s_i}$ and notice that

(P3′) for every $n < 2^k$ there are disjoint $F^0_{s_n}, F^1_{s_n} \in \mathcal{A}$ such that $F^0_{s_n} \cup F^1_{s_n} \subset E'_{s_n}$, $\{F^0_{s_n}, F^1_{s_n}\} \in \mathcal{D}_k$, and $\pi_\gamma[F^0_{s_n}] = \pi_\gamma[F^1_{s_n}] = \pi_\gamma[E'_{s_n}]$.

Indeed, for $j < 2$ define $F^j_{s_n} = P^j_n \cap \pi^{-1}_\gamma(\pi_\gamma[E'_{s_n}])$ and note that $F^j_{s_n} \in \mathcal{A}$ by (23), since $\pi_\gamma[E'_{s_n}] \subset \pi_\gamma[E^{n+1}_{s_n}] = \pi_\gamma[P^j_n]$. So $\{F^0_{s_n}, F^1_{s_n}\} \in \mathcal{D}_k$ by (P1). The equations hold since, by (c),

$$\begin{aligned}\pi_\gamma[F^j_{s_n}] &= \left\{\pi_\gamma(x) \colon x \in P^j_n \ \&\ x \restriction \gamma \in \pi_\gamma[E'_{s_n}]\right\} \\ &= \left\{\pi_\gamma(x) \colon x \in E^{n+1}_{s_n} \ \&\ x \restriction \gamma \in \pi_\gamma[E'_{s_n}]\right\} \\ &= \pi_\gamma[E'_{s_n}].\end{aligned}$$

Finally, $F^j_{s_n} = \{x \in P^j_n \colon x \restriction \gamma \in \pi_\gamma[E'_{s_n}]\}$ is a subset of E'_{s_n} since $P^j_n \subset E^{n+1}_{s_n}$ and, by (25), if $x \in E^{n+1}_{s_n} \setminus E'_{s_n}$ then $x \restriction \gamma \notin \pi_\gamma[E'_{s_n}]$. So, (P3′) holds.

Next, by induction on $i < 2^k$, choose a sequence $\langle x^j_i \in F^j_{s_i} : j < 2 \ \&\ i < 2^k\rangle$ such that for every $j < 2$ and $m \le i < 2^k$,

$$x^0_i \restriction \beta_k = x^1_i \restriction \beta_k, \quad x^0_i(\beta_k) \ne x^1_i(\beta_k), \quad \text{and} \quad x^j_i \restriction \beta^m_i = x^j_m \restriction \beta^m_i. \tag{26}$$

By (P3′) it is easy to find x^0_0 and x^1_0 satisfying (26). So, assume that for some $0 < i < 2^k$ we already have defined $\langle x^j_m : j < 2 \ \&\ m < i\rangle$. To find x^0_i and x^1_i let $\beta = \max\{\beta^m_i \colon m < i\}$ and choose an $n < i$ which witnesses it, that is, such that $\beta = \beta^n_i$. Since, by (a) and (P3′), $\pi_\beta[F^j_{s_i}] = \pi_\beta[F^j_{s_n}]$ for $j < 2$, we can find an $x^0_i \in F^0_{s_i}$ extending $x^0_n \restriction \beta$. Then $x^0_i \restriction \beta^m_i = x^0_n \restriction \beta^m_i = x^0_m \restriction \beta^m_i$ for all $m < i$.

Next, if $\beta_k < \beta$, as above we choose an $x^1_i \in F^1_{s_i}$ extending $x^1_n \restriction \beta$ and note that (26) is satisfied since this was the case for $i = n$. So, assume that $\beta \le \beta_k \le \gamma$. Then, by (P3′), we can find an $x^1_i \in F^1_{s_i}$ extending $x^0_i \restriction \beta_k$ such that $x^1_i(\beta_k) \ne x^0_i(\beta_k)$. Then (26) holds as well.

Finally, for $s \in 2^{A_k}$ and $j < 2$ let $s\hat{\ }j$ stand for $s \cup \{\langle\langle\beta_k, n_k\rangle, j\rangle\} \in 2^{A_{k+1}}$, and for $i < 2^k$ define

$$E_{s_i\hat{\ }j} = F^j_{s_i} \cap B_\alpha(x^j_i, 2^{-k}).$$

Let $\mathcal{E}_{k+1} = \{E_s \colon s \in 2^{A_{k+1}}\}$. To finish the proof it is enough to show that $\mathcal{E}_{k+1}$ satisfies (i)–(iv) from Lemma 8.1. Thus, (i) follows from the fact that $E_{s_i\hat{\ }j} \subset B_\alpha(x^j_i, 2^{-k})$; (ii) is justified by $E_{s_i\hat{\ }j} \subset F^j_{s_i} \subset E'_{s_i} \subset E_{s_i}$; and (iii), (iv) can be easily deduced from (P3′), (26), and (2). ∎

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