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CROWDED AND SELECTIVE ULTRAFILTERS UNDER THE COVERING PROPERTY AXIOM

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Abstract. In the paper we formulate an axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, which is the most prominent version of the Covering Property Axiom CPA, and discuss several of its implications. In particular, we show that it implies that the following cardinal characteristics of continuum are equal to ω_1, while $\mathfrak{c} = \omega_2$: the independence number $\mathfrak{i}$, the reaping number $\mathfrak{r}$, the almost disjoint number $\mathfrak{a}$, and the ultrafilter base number $\mathfrak{u}$. We will also show that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies the existence of crowded and selective ultrafilters as well as nonselective P-points. In addition we prove that under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ every selective ultrafilter is ω_1-generated. The paper finishes with the proof that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ holds in the iterated perfect set model.

1. Introduction and preliminaries

The Covering Property Axiom, CPA, constitutes an attempt to axiomatize the iterated perfect set (Sacks) model. In this paper we will consider its

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prominent version, $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, as well its three weaker variations: $\mathrm{CPA}_{\mathrm{prism}}$, $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$, and $\mathrm{CPA}_{\mathrm{cube}}$. They are related to each other by the following implications.

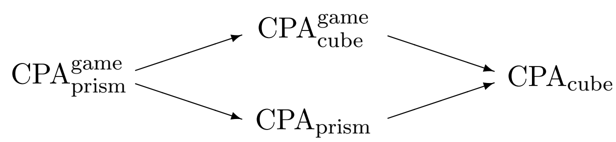

Chart 1.

Although in some cases the stronger versions of CPA are useful (see e.g. [8, Chapter 6]), it is known that the axiom $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ captures the essence of the Sacks model concerning the standard cardinal characteristics of continuum. This follows from a resent result of J. Zapletal [27] who proved that for a "nice" cardinal invariant κ if $\kappa < \mathfrak{c}$ holds in any forcing extension then $\kappa < \mathfrak{c}$ follows already from $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$.

The Covering Property Axiom is quite simple in formulation and use, nevertheless it requires some new concepts. To facilitate the absorption of these concepts we decided to introduce the axiom in three steps, beginning with its simplest form. (More on CPA can be found in [8].) Thus, in Section 2 we formulate the simplest version of the axiom, $\mathrm{CPA}_{\mathrm{cube}}$, and show that it implies that every selective ultrafilter is generated by ω_1 sets and that the reaping number $\mathfrak{r}$ is equal to ω_1. In Section 3 we will formulate axiom $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ and show that it implies $\mathrm{CPA}_{\mathrm{cube}}$ as well as the existence of a family $\mathcal{F} \subset [\omega]^\omega$ of cardinality ω_1 which is simultaneously maximal almost disjoint, MAD, and reaping. In particular, $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ implies that $\mathfrak{a} = \omega_1 < \mathfrak{c}$. In Section 4 we will formulate axioms $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ and $\mathrm{CPA}_{\mathrm{prism}}$ and show that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies all other versions of the axiom. We will also show there that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies the existence of selective and crowded ultrafilters as well as nonselective P-points. In addition we prove there that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ implies the existence of a family $\mathcal{F} \subset [\omega]^\omega$ of cardinality ω_1 which is simultaneously independent and splitting. In particular, under $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ we have $\mathfrak{s} = \mathfrak{i} = \mathfrak{u} = \omega_1 < \mathfrak{c}$. In the last section of the paper we will prove the prism fusion lemma, which has been used in Section 4, and show that $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ holds in the iterated perfect set model.

Our set theoretic terminology is standard and follows that of [6]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. A Cantor set 2^ω will be denoted by a symbol $\mathfrak{C}$. We use the term *Polish space* for a complete separable metric space `without isolated points`. For a Polish space X, the symbol $\mathrm{Perf}(X)$ will stand for the collection of all subsets of

X homeomorphic to a Cantor set $\mathfrak{C}$. For a function $f\colon X \to \mathbb{R}$ and $A \subset X$ an image of A under f is denoted by $f[A]$, that is, $f[A] = \{f(x)\colon x \in A\}$.

For an ideal $\mathcal{I}$ on ω containing all finite subsets of ω we will use the following generalized selectivity terminology. We say (see Farah [14]) that an ideal $\mathcal{I}$ is *selective* provided for every sequence $F_0 \supset F_1 \supset \cdots$ of sets from $\mathcal{I}^+ \stackrel{\text{def}}{=} \mathcal{P}(\omega) \setminus \mathcal{I}$ there exists an $F_\infty \in \mathcal{I}^+$ (called a *diagonalization* of this sequence) such that $F_\infty \setminus \{0, \ldots, n\} \subset F_n$ for all $n \in F_\infty$. Notice that this definition agrees with the definition of selectivity given by Grigorieff in [15, p. 365]. (The ideals selective in the above sense Grigorieff calls *inductive* but he also proves [15, Corollary 1.15] that the inductive ideals and the ideals selective in his sense are the same notions.)

For $A, B \subset \omega$ we will write $A \subseteq^* B$ when $|A \setminus B| < \omega$. A set $\mathcal{D} \subset \mathcal{I}^+$ is *dense* in $\mathcal{I}^+$ provided for every $B \in \mathcal{I}^+$ there exists an $A \in \mathcal{D}$ such that $A \subseteq^* B$; set $\mathcal{D}$ is *open* in $\mathcal{I}^+$ if $B \in \mathcal{D}$ provided there is an $A \in \mathcal{D}$ such that $B \subseteq^* A$. For $\bar{\mathcal{D}} = \langle \mathcal{D}_n \subset \mathcal{I}^+ \colon n < \omega \rangle$ we say that $F_\infty \in \mathcal{I}^+$ is a *diagonalization* of $\bar{\mathcal{D}}$ provided $F_\infty \setminus \{0, \ldots, n\} \in \mathcal{D}_n$ for every $n < \omega$. Following Farah [14] we say that an ideal $\mathcal{I}$ on ω is *semiselective* provided for every sequence $\bar{\mathcal{D}} = \langle \mathcal{D}_n \subset \mathcal{I}^+ \colon n < \omega \rangle$ of dense and open subsets of $\mathcal{I}^+$ the family of all diagonalizations of $\bar{\mathcal{D}}$ is dense in $\mathcal{I}^+$.

Following Grigorieff [15, p. 390] we say that $\mathcal{I}$ is *weakly selective* (or *weak selective*) provided for every $A \in \mathcal{I}^+$ and $f\colon A \to \omega$ there exists a $B \in \mathcal{I}^+$ such that $f \restriction B$ is either one-to-one or constant. (Farah in [14, Section 2] terms such ideals as *having the Q^+-property*. Note also that Baumgartner and Laver in [2] call such ideals selective, despite the fact that they claim to use Grigorieff's terminology from [15].)

We have the following implications between these notions. (See Farah [14, Section 2].)

$$\mathcal{I} \text{ is selective} \implies \mathcal{I} \text{ is semiselective} \implies \mathcal{I} \text{ is weakly selective}$$

All these notions represent different generalizations of the properties of the ideal $[\omega]^{<\omega}$. In particular, it is easy to see that $[\omega]^{<\omega}$ is selective.

We say that an ideal $\mathcal{I}$ on a countable set X is selective (weakly selective) provided it is such upon an identification of X with ω via an arbitrary bijection. A filter $\mathcal{F}$ on a countable set X is selective (semiselective, weakly selective) provided so is its dual ideal $\mathcal{I} = \{X \setminus F \colon F \in \mathcal{F}\}$.

It is important to note that a maximal ideal (or an ultrafilter) is selective if and only if it is weakly selective. This follows, for example, directly from the definitions of these notions as in Grigorieff [15]. Recall also that the existence of selective ultrafilters cannot be proved in ZFC. (Kunen [21] proved that there are no selective ultrafilters in the random real model. This also follows from the fact that every selective ultrafilter is a P-point, while Shelah proved that there are models with no P-points, see e.g. [1, Theorem 4.4.7].)

2. Axiom $\mathrm{CPA}_{\mathrm{cube}}$ and its consequences

For a Polish space X we will consider $\mathrm{Perf}(X)$ as ordered by inclusion. Thus, a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is *dense in* $\mathrm{Perf}(X)$ provided for every $P \in \mathrm{Perf}(X)$ there exists a $Q \in \mathcal{E}$ such that $Q \subset P$.

Axiom $\mathrm{CPA}_{\mathrm{cube}}$ will be of the form

> if $\mathcal{E} \subset \mathrm{Perf}(X)$ is *appropriately* dense in $\mathrm{Perf}(X)$ then some portion $\mathcal{E}_0$ of $\mathcal{E}$ covers almost all of X in a sense that $|X \setminus \bigcup \mathcal{E}_0| < \mathfrak{c}$.

If the word "appropriately" in the above is ignored, then it implies the following statement.

Naive-CPA**:** If $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$ then $|X \setminus \bigcup \mathcal{E}| < \mathfrak{c}$.

It is a very good candidate for our axiom in the sense that it implies all the properties we are interested in. It has, however, one major flaw — *it is false!* This is the case since $S \subset X \setminus \bigcup \mathcal{E}$ for some dense set $\mathcal{E}$ in $\mathrm{Perf}(X)$ provided

> for each $P \in \mathrm{Perf}(X)$ there is a $Q \in \mathrm{Perf}(X)$ such that $Q \subset P \setminus S$.

This means that the family $\mathcal{G}$ of all sets of the form $X \setminus \bigcup \mathcal{E}$, where $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$, coincides with the σ-ideal s_0 of Marczewski's sets, since $\mathcal{G}$ is clearly hereditary. Thus we have

$$s_0 = \left\{ X \setminus \bigcup \mathcal{E} \colon \ \mathcal{E} \text{ is dense in } \mathrm{Perf}(X) \right\}. \tag{2.1}$$

However, it is well known (see e.g. [24, Theorem 5.10]) that there are s_0-sets of cardinality $\mathfrak{c}$. Thus, our Naïve-CPA "axiom" cannot be consistent with ZFC.

In order to formulate the real axiom $\mathrm{CPA}_{\mathrm{cube}}$ we need the following terminology and notation. A subset C of a product $\mathfrak{C}^\eta$ of the Cantor set is said to be a *perfect cube* if $C = \prod_{n\in\eta} C_n$, where $C_n \in \mathrm{Perf}(\mathfrak{C})$ for each n. For a fixed Polish space X let $\mathcal{F}_{\mathrm{cube}}$ stand for the family of all continuous injections from a perfect cube $C \subset \mathfrak{C}^\omega$ onto a set P from $\mathrm{Perf}(X)$. We consider each function $f \in \mathcal{F}_{\mathrm{cube}}$ from C onto P as a coordinate system imposed on P.[1] We say that $P \in \mathrm{Perf}(X)$ is a *cube* if it is determined by an (implicitly given) witness function $f \in \mathcal{F}_{\mathrm{cube}}$ onto P, and Q is a *subcube of a cube* $P \in \mathrm{Perf}(X)$ provided $Q = f[C]$, where $f \in \mathcal{F}_{\mathrm{cube}}$ is the witness function for P and C is a subcube of the domain of f.

We say that a family $\mathcal{E} \subset \mathrm{Perf}(X)$ is $\mathcal{F}_{\mathrm{cube}}$*-dense* (or *cube-dense*) in $\mathrm{Perf}(X)$ provided every cube $P \in \mathrm{Perf}(X)$ contains a subcube $Q \in \mathcal{E}$. More formally, $\mathcal{E} \subset \mathrm{Perf}(X)$ is $\mathcal{F}_{\mathrm{cube}}$-dense provided

$$\forall f \in \mathcal{F}_{\mathrm{cube}}\ \exists g \in \mathcal{F}_{\mathrm{cube}}\ (g \subset f\ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{2.2}$$

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[1] In a language of forcing a coordinate function f is simply a nice name for an element from X.

It is easy to see that the notion of $\mathcal{F}_{\rm cube}$-density is a generalization of the notion of density as defined in the first paragraph of this section:

$$\text{if } \mathcal{E} \text{ is } \mathcal{F}_{\rm cube}\text{-dense in } \mathrm{Perf}(X) \text{ then } \mathcal{E} \text{ is dense in } \mathrm{Perf}(X).$$

On the other hand, the converse implication is not true, as shown by the following simple example.

Example 2.1. Let $X = \mathfrak{C} \times \mathfrak{C}$ and let $\mathcal{E}$ be the family of all $P \in \mathrm{Perf}(X)$ such that either

- all vertical sections of P are countable, or
- all horizontal sections of P are countable.

Then $\mathcal{E}$ is dense in $\mathrm{Perf}(X)$, but it is not $\mathcal{F}_{\rm cube}$-dense in $\mathrm{Perf}(X)$.

With these notions in hand we are ready to formulate our axiom[2] $\mathrm{CPA}_{\rm cube}$.

$\mathrm{CPA}_{\rm cube}$**:** $\mathfrak{c} = \omega_2$ and for every Polish space X and every $\mathcal{F}_{\rm cube}$-dense family $\mathcal{E} \subset \mathrm{Perf}(X)$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \le \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$.

It is also worth noticing that in order to check that $\mathcal{E}$ is $\mathcal{F}_{\rm cube}$-dense it is enough to consider in condition (2.2) only functions f defined on the entire space $\mathfrak{C}^\omega$, that is

Fact 2.2. *$\mathcal{E} \subset \mathrm{Perf}(X)$ is $\mathcal{F}_{\rm cube}$-dense if and only if*

$$\forall f \in \mathcal{F}_{\rm cube},\ \mathrm{dom}(f) = \mathfrak{C}^\omega,\ \exists g \in \mathcal{F}_{\rm cube}\ (g \subset f\ \&\ \mathrm{range}(g) \in \mathcal{E}). \tag{2.3}$$

Proof. To see this, let Φ be the family of all bijections $h = \langle h_n \rangle_{n<\omega}$ between the perfect cubes $\prod_{n\in\omega} D_n$ and $\prod_{n\in\omega} C_n$ in $\mathfrak{C}^\omega$ such that each h_n is a homeomorphism between D_n and C_n. Then

$$f \circ h \in \mathcal{F}_{\rm cube} \ \text{ for every } f \in \mathcal{F}_{\rm cube} \text{ and } h \in \Phi \text{ with } \mathrm{range}(h) \subset \mathrm{dom}(f).$$

Now take an arbitrary $f \colon C \to X$ from $\mathcal{F}_{\rm cube}$ and choose an $h \in \Phi$ mapping $\mathfrak{C}^\omega$ onto C. Then $\hat{f} = f \circ h \in \mathcal{F}_{\rm cube}$ maps $\mathfrak{C}^\omega$ into X and, using (2.3), we can find a $\hat{g} \in \mathcal{F}_{\rm cube}$ such that $\hat{g} \subset \hat{f}$ and $\mathrm{range}(\hat{g}) \in \mathcal{E}$. Then $g = f \restriction h[\mathrm{dom}(\hat{g})]$ satisfies (2.2). □

One of the most convenient tools for proving $\mathcal{F}_{\rm cube}$-density is the following fact.

[2]This version of the axiom, as well as its prism version $\mathrm{CPA}_{\rm prism}$, can be also formulated replacing the inequalities "$\le \omega_1$" with "$< \mathfrak{c}$" and removing the condition "$\mathfrak{c} = \omega_2$." Such a version of $\mathrm{CPA}_{\rm cube}$ implies $\mathfrak{c} \ge \omega_2$. Also all consequences of the axioms $\mathrm{CPA}_{\rm cube}$ and $\mathrm{CPA}_{\rm prism}$ presented in this paper follow also from the modified versions of these axioms. However, we do not know if the modified axioms are consistent with $\mathfrak{c} > \omega_2$. We know only that the modified $\mathrm{CPA}_{\rm prism}$ implies that $\mathfrak{c}$ is a successor cardinal. (See [7] or [8].)

Claim 2.3. *Consider $\mathfrak{C}^\omega$ with its usual topology and its usual product measure. If $G \subset \mathfrak{C}^\omega$ is either comeager or of full measure in $\mathfrak{C}^\omega$ then it contains a perfect cube $\prod_{i<\omega} P_i$.*

Proof. It follows easily, by induction on coordinates, from the following well known fact.

> For every comeager (full measure) subset H of $\mathfrak{C} \times \mathfrak{C}$ there are a perfect set $P \subset \mathfrak{C}$ and a comeager (full measure) subset $\hat{H}$ of $\mathfrak{C}$ such that $P \times \hat{H} \subset H$.

The category version is easy and can be found in [20, Exercise 19.3]. (Its version for $\mathbb{R}^2$ is also proved, for example, in [9, condition ($\star$), p. 416].) The measure version follows easily from the fact that

> for every full measure subset H of $[0,1] \times [0,1]$ there are a perfect set $P \subset \mathfrak{C}$ and a positive inner measure subset $\hat{H}$ of $[0,1]$ such that $P \times \hat{H} \subset H$

which is proved by Eggleston [13] and, independently, by Brodskiĭ [4]. □

Next we will proceed to demonstrate some consequences of $\mathrm{CPA}_{\mathrm{cube}}$. The most important combinatorial fact for us concerning semiselective ideals is the following property. (See Theorem 2.1 and Remark 4.1 in [14].) This is a generalization of a theorem of Laver [22] who proved this fact for the ideal $\mathcal{I} = [\omega]^{<\omega}$.

Proposition 2.4 (Farah [14]). *Let $\mathcal{I}$ be a semiselective ideal on ω. For every analytic set $S \subset \mathfrak{C}^\omega \times [\omega]^\omega$ and every $A \in \mathcal{I}^+$ there exist a $B \in \mathcal{I}^+ \cap \mathcal{P}(A)$ and a perfect cube C in $\mathfrak{C}^\omega$ such that $C \times [B]^\omega$ is either contained in or disjoint with S.*

With this fact in hand we can prove the following theorem.

Theorem 2.5. *Assume that $\mathrm{CPA}_{\mathrm{cube}}$ holds. If $\mathcal{I}$ is a semiselective ideal then there is a family $\mathcal{W} \subset \mathcal{I}^+$, $|\mathcal{W}| \le \omega_1$, such that for every analytic set $A \subset [\omega]^\omega$ there is a $W \in \mathcal{W}$ for which either $[W]^\omega \subset A$ or $[W]^\omega \cap A = \emptyset$.*

Proof. Let $S \subset \mathfrak{C} \times [\omega]^\omega$ be a universal analytic set, that is such that the family $\{S_x \colon x \in \mathfrak{C}\}$ (where $S_x = \{y \in [\omega]^\omega \colon \langle x, y\rangle \in S\}$) contains all analytic subsets of $[\omega]^\omega$. (See e.g. [18, Lemma 39.4].) In fact, we will take S such that for any analytic set A in $[\omega]^\omega$

$$|\{x \in \mathfrak{C} \colon S_x = A\}| = \mathfrak{c}. \tag{2.4}$$

(If $U \subset \mathfrak{C} \times [\omega]^\omega$ is a universal analytic set then $S = \mathfrak{C} \times U \subset \mathfrak{C} \times \mathfrak{C} \times [\omega]^\omega$ satisfies (2.4), where we identify $\mathfrak{C} \times \mathfrak{C}$ with $\mathfrak{C}$.) For this particular set S

consider the family $\mathcal{E}$ of all $Q \in \mathrm{Perf}(\mathfrak{C})$ for which there exists a $W_Q \in \mathcal{I}^+$ such that

$$Q \times [W_Q]^\omega \text{ is either contained in or disjoint from } S. \tag{2.5}$$

Note that, by Proposition 2.4, the family $\mathcal{E}$ is $\mathcal{F}_{\mathrm{cube}}$-dense in $\mathrm{Perf}(\mathfrak{C})$. So, by $\mathrm{CPA}_{\mathrm{cube}}$, there exists an $\mathcal{E}_0 \subset \mathcal{E}$, $|\mathcal{E}_0| \leq \omega_1$, such that $|\mathfrak{C} \setminus \bigcup \mathcal{E}_0| < \mathfrak{c}$. Let

$$\mathcal{W} = \{W_Q \colon Q \in \mathcal{E}_0\}.$$

It is enough to see that this $\mathcal{W}$ is as required.

Clearly $|\mathcal{W}| \leq \omega_1$. Also, by (2.4), for an analytic set $A \subset [\omega]^\omega$ there exist a $Q \in \mathcal{E}_0$ and an $x \in Q$ such that $A = S_x$. So, by (2.5), $\{x\} \times [W_Q]^\omega$ is either contained in or disjoint from $\{x\} \times S_x = \{x\} \times A$. □

Recall (see e.g. [1] or [26]) that a family $\mathcal{W} \subset [\omega]^\omega$ is a *reaping family* provided

$$\forall A \in [\omega]^\omega \ \exists W \in \mathcal{W} \ (W \subset A \ \text{ or } \ W \subset \omega \setminus A).$$

The *reaping* (or *refinement*) number $\mathfrak{r}$ is defined as the minimum cardinality of a reaping family. Also, a number $\mathfrak{r}_\sigma$ is defined as the smallest cardinality of a family $\mathcal{W} \subset [\omega]^\omega$ such that for every sequence $\langle A_n \in [\omega]^\omega \colon n < \omega\rangle$ there exists a $W \in \mathcal{W}$ such that for every $n < \omega$ either $W \subseteq^* A_n$ or $W \subseteq^* \omega \setminus A_n$. (See [5] or [26].) Clearly $\mathfrak{r} \leq \mathfrak{r}_\sigma$.

Corollary 2.6. *If* $\mathrm{CPA}_{\mathrm{cube}}$ *holds then for every semiselective ideal* $\mathcal{I}$ *there exists a family* $\mathcal{W} \subset \mathcal{I}^+$, $|\mathcal{W}| \leq \omega_1$, *such that for every* $A \in [\omega]^\omega$ *there is a* $W \in \mathcal{W}$ *for which either* $W \subseteq^* A$ *or* $W \subseteq^* \omega \setminus A$.

In particular, $\mathrm{CPA}_{\mathrm{cube}}$ *implies that* $\mathfrak{r} = \omega_1 < \mathfrak{c}$.

Proof. The family $\mathcal{W}$ from Theorem 2.5 works: since $[A]^\omega$ is analytic in $[\omega]^\omega$ there exists a $W \in \mathcal{W}$ such that either $[W]^\omega \subset [A]^\omega$ or $[W]^\omega \cap [A]^\omega = \emptyset$. □

Corollary 2.7. *If* $\mathrm{CPA}_{\mathrm{cube}}$ *holds then every selective ultrafilter* $\mathcal{F}$ *on* ω *is generated by a family of size* $\omega_1 < \mathfrak{c}$.

Proof. If $\mathcal{F}$ is a selective ultrafilter on ω then $\mathcal{I} = \mathcal{P}(\omega) \setminus \mathcal{F}$ is a selective ideal and $\mathcal{I}^+ = \mathcal{F}$. Let $\mathcal{W} \subset \mathcal{I}^+ = \mathcal{F}$ be as in Corollary 2.6. Then $\mathcal{W}$ generates $\mathcal{F}$.

Indeed, if $A \in \mathcal{F}$ then there exists a $W \in \mathcal{W}$ such that either $W \subset A$ or $W \subset \omega \setminus A$. But it is impossible that $W \subset \omega \setminus A$ since then we would have $\emptyset = A \cap W \in \mathcal{F}$. □

As mentioned above, in Theorem 4.8 we will prove that some version of our axiom implies that there exists a selective ultrafilter on ω. In particular, the assumptions of the next corollary are implied by such a version of our axiom.

Corollary 2.8. *If* $\mathrm{CPA}_{\mathrm{cube}}$ *holds and there exists a selective ultrafilter* $\mathcal{F}$ *on* ω *then* $\mathfrak{r}_\sigma = \omega_1 < \mathfrak{c}$.

Proof. Let $\mathcal{W} \in [\mathcal{F}]^{\leq\omega_1}$ be a generating family for $\mathcal{F}$. We will show that it justifies $\mathfrak{r}_\sigma = \omega_1$. Indeed, take a sequence $\langle A_n \in [\omega]^\omega \colon n < \omega\rangle$. For every $n < \omega$ let A_n^* belong to $\mathcal{F} \cap \{A_n, \omega \setminus A_n\}$. Since $\mathcal{F}$ is selective, there exists an $A \in \mathcal{F}$ such that $A \subseteq^* A_n^*$ for every $n < \omega$. Let $W \in \mathcal{W}$ be such that $W \subset A$. Then for every $n < \omega$ either $W \subseteq^* A_n$ or $W \subseteq^* \omega \setminus A_n$. □

We are particularly interested in the number $\mathfrak{r}_\sigma$ since it is related to different variants of sets of uniqueness coming from harmonic analysis, as described in the survey paper [5]. In particular, from [5, Theorem 12.6] it follows that an appropriate version of our axiom implies that all covering numbers described in the paper are equal to ω_1.

3. $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ and numbers $\mathfrak{a}$ and $\mathfrak{r}$

Before we get to the formulation of our next version of the axiom it is good to note that in many applications we would prefer to have a full covering of a Polish space X rather that the almost covering as claimed by $\mathrm{CPA}_{\mathrm{cube}}$. To get better access to the missing singletons we will extend the notion of a cube by allowing also the *constant cubes*: a family $\mathcal{C}_{\mathrm{cube}}(X)$ of constant "cubes" is defined as the family of all constant functions from a perfect cube $C \subset \mathfrak{C}^\omega$ into X. We define also $\mathcal{F}^*_{\mathrm{cube}}(X)$ as

$$\mathcal{F}^*_{\mathrm{cube}} = \mathcal{F}_{\mathrm{cube}} \cup \mathcal{C}_{\mathrm{cube}}. \tag{3.1}$$

Thus, $\mathcal{F}^*_{\mathrm{cube}}$ is the family of all continuous functions from a perfect cube $C \subset \mathfrak{C}^\omega$ into X which are either one-to-one or constant. Now the range of every $f \in \mathcal{F}^*_{\mathrm{cube}}$ belongs to the family $\mathrm{Perf}^*(X)$ of all sets P such that either $P \in \mathrm{Perf}(X)$ or P is a singleton. The terms "$P \in \mathrm{Perf}^*(X)$ is a cube" and "Q is a subcube of a cube $P \in \mathrm{Perf}^*(X)$" are defined in a natural way.

Consider also the following game $\mathrm{GAME}_{\mathrm{cube}}(X)$ of length ω_1. The game has two players, Player I and Player II. At each stage $\xi < \omega_1$ of the game Player I can play an arbitrary cube $P_\xi \in \mathrm{Perf}^*(X)$ and Player II must respond with a subcube Q_ξ of P_ξ. The game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ is won by Player I provided

$$\bigcup_{\xi<\omega_1} Q_\xi = X;$$

otherwise the game is won by Player II.

Recall also that a strategy for Player II is any function S with the property that $S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle, P_\xi)$ is a subcube of P_ξ, where $\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle$ is any partial game. A game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi<\omega_1\rangle$ is played according to a strategy S for Player II provided $Q_\xi = S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle, P_\xi)$ for every $\xi<\omega_1$. A strategy S for Player II is a *winning strategy* for Player II provided Player II wins any game played according to the strategy S.

Here is our new version of the axiom.[3]

$\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$: $\mathfrak{c}=\omega_2$ and for any Polish space X Player II has no winning strategy in the game $\mathrm{GAME}_{\mathrm{cube}}(X)$.

Notice that

Proposition 3.1. *Axiom* $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ *implies* $\mathrm{CPA}_{\mathrm{cube}}$.

Proof. Let $\mathcal{E}\subset \mathrm{Perf}(X)$ be $\mathcal{F}_{\mathrm{cube}}$-dense. Thus for every cube $P\in\mathrm{Perf}(X)$ there exists a subcube $s(P)\in\mathcal{E}$ of P. Now, for a singleton $P\in\mathrm{Perf}^*(X)$ put $s(P)=P$ and consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle, P_\xi) = s(P_\xi).$$

By $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ it is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi<\omega_1\rangle$ in which $Q_\xi = s(P_\xi)$ for every $\xi<\omega_1$ and Player II loses, that is, $X=\bigcup_{\xi<\omega_1} Q_\xi$. Now, let $\mathcal{E}_0=\{Q_\xi\colon \xi<\omega_1 \ \&\ Q_\xi\in\mathrm{Perf}(X)\}$. Then $|X\setminus\bigcup\mathcal{E}_0|\le\omega_1$, so $\mathrm{CPA}_{\mathrm{cube}}$ is justified. □

Recall that a family $\mathcal{A}\subset[\omega]^\omega$ is *almost disjoint* provided $|A\cap B|<\omega$ and it is *maximal almost disjoint, MAD,* provided it is not a proper subfamily of any other almost disjoint family. The cardinal $\mathfrak{a}$ is defined as follows:

$$\mathfrak{a}=\min\{|\mathcal{A}|\colon \mathcal{A} \text{ is infinite and MAD}\}.$$

The fact that $\mathfrak{a}=\omega_1$ holds in the iterated perfect set model was apparently first noticed by Spinas (see Andreas Blass [3, Section 11.5]) though it seems that the proof of this result was never provided.

Theorem 3.2. $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ *implies that* $\mathfrak{a}=\omega_1$.

Our proof of Theorem 3.2 is based on the following lemma.

[3]Note that if we remove the assumption $\mathfrak{c}=\omega_2$ from the axiom then the remaining part follows from the continuum hypothesis. Thus, for most of our applications the assumption that $\mathfrak{c}\ge\omega_2$ is essential. On the other hand, the main body of the axiom and $\mathfrak{c}\ge\omega_2$ imply that $\mathfrak{c}=\omega_2$.

Lemma 3.3. *For every countably infinite almost disjoint family $\mathcal{W} \subset [\omega]^\omega$ and a cube $P \in \mathrm{Perf}([\omega]^\omega)$ there exist a $W \in [\omega]^\omega$ and a subcube Q of P such that $\mathcal{W} \cup \{W\}$ is almost disjoint but $\mathcal{W} \cup \{W, x\}$ is not almost disjoint for every $x \in Q$.*

Proof. Let $\mathcal{W} = \{W_i \colon i < \omega\}$. For every $i < \omega$ choose sets $V_i \subset W_i$ such that: the V_i's are pairwise disjoint, each $W_i \setminus V_i$ is finite, but $V_\omega = \omega \setminus \bigcup_{i<\omega} V_i$ is infinite. Let

$$B = \{x \in P \colon (\forall i \le \omega)\ |x \cap V_i| < \omega\}$$

and notice that B is a Borel subset of P. (In fact, B is an $F_{\sigma\delta}$-set.) So, by Claim 2.3, there is a subcube P^* of P such that either $P^* \subset B$ or $P^* \cap B = \emptyset$.

If $P^* \cap B = \emptyset$ then $W = V_\omega$ and $Q = P^*$ satisfy the conclusion of the lemma. So, suppose that $P^* \subset B$. Let $h \colon \mathfrak{C}^\omega \to P^*$, $h \in \mathcal{F}_{\mathrm{cube}}$, be a coordinate function making P^* a cube, let λ be the standard product probability measure on $\mathfrak{C}^\omega$, and define a Borel measure μ on P^* by the formula $\mu(B) = \lambda(h^{-1}(B))$.

For $i, n < \omega$ let

$$P_i^n = \{x \in P^* \colon x \cap V_i \subset n\}.$$

Then all the sets P_i^n are Borel (in fact, they are closed) and $P^* = \bigcup_{n<\omega} P_i^n$ for every $i < \omega$. Thus for each $i < \omega$ there exists an $n(i) < \omega$ such that

$$\mu\left(P_i^{n(i)}\right) > 1 - 2^{-i}.$$

Then the set $T = \bigcup_{j<\omega} \bigcap_{j<i<\omega} P_i^{n(i)}$ has a μ-measure 1 so, by Claim 2.3, there is a subcube Q of P^* which is a subset of T. Let

$$W = \bigcup_{i<\omega} \left[V_i \cap n(i)\right].$$

We claim that W and Q satisfy the lemma.

It is obvious that W is almost disjoint with each W_i. So, fix an $x \in Q$. To finish the proof it is enough to show that

$$x \subseteq^* W.$$

But $x \in Q \subset \bigcup_{j<\omega} \bigcap_{j<i<\omega} P_i^{n(i)}$. Thus, there exists a $j < \omega$ such that $x \in \bigcap_{j<i<\omega} P_i^{n(i)}$. So, $x \cap \bigcup_{j<i<\omega} V_i = \bigcup_{j<i<\omega} (x \cap V_i) \subset \bigcup_{j<i<\omega} (V_i \cap n(i)) \subset W$ and the set

$$x \setminus W \subset x \cap \left(V_\omega \cup \bigcup_{i \le j} V_i\right) = (x \cap V_\omega) \cup \bigcup_{i \le j} (x \cap V_i)$$

is finite. □

Proof of Theorem 3.2. For a countably infinite almost disjoint family $\mathcal{W} \subset [\omega]^\omega$ and a cube $P \in \mathrm{Perf}([\omega]^\omega)$ let $W(\mathcal{W}, P) \in [\omega]^\omega$ and a subcube $Q(\mathcal{W}, P)$ of P be as in Lemma 3.3. For $P = \{x\} \in \mathrm{Perf}^*([\omega]^\omega)$ we put $Q(\mathcal{W}, P) = P$ and define $W(\mathcal{W}, P)$ as some arbitrary W almost disjoint with each set from $\mathcal{W}$ and such that $A \cap x$ is infinite for some $A \in \mathcal{W} \cup \{W\}$.

Let $\mathcal{A}_0 \subset [\omega]^\omega$ be an arbitrary infinite almost disjoint family and consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(\mathcal{A}_0 \cup \{W_\eta \colon \eta < \xi\}, P_\xi),$$

where W_η's are defined inductively by $W_\eta = W(\mathcal{A}_0 \cup \{W_\zeta \colon \zeta < \eta\}, P_\eta)$. In other words, Player II remembers (recovers) the sets W_η associated with the sets P_η played so far, and he uses them (and Lemma 3.3) to get the next answer $Q_\xi = Q(\mathcal{A}_0 \cup \{W_\eta \colon \eta < \xi\}, P_\xi)$, while remembering (or recovering each time) the set $W_\xi = W(\mathcal{A}_0 \cup \{W_\eta \colon \eta < \xi\}, P_\xi)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S in which Player II loses, that is, $[\omega]^\omega = \bigcup_{\xi<\omega_1} Q_\xi$.

Now, notice that the family $\mathcal{A} = \mathcal{A}_0 \cup \{W_\xi \colon \xi < \omega_1\}$ is a MAD family. It is clear that $\mathcal{A}$ is almost disjoint, since every set W_ξ was chosen as almost disjoint with every set from $\mathcal{A}_0 \cup \{W_\zeta \colon \zeta < \xi\}$. To see that $\mathcal{A}$ is maximal it is enough to note that every $x \in [\omega]^\omega$ belongs to a Q_ξ for some $\xi < \omega_1$, and so there is an $A \in \mathcal{A}_0 \cup \{W_\eta \colon \eta \le \xi\}$ such that $A \cap x$ is infinite. □

By Theorem 3.2 we see that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ implies the existence of MAD family of size ω_1. Next we will show that such a family can be simultaneously a reaping family. This result is similar in flavor to that from Theorem 4.20.

Theorem 3.4. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ *implies that there exists a family* $\mathcal{F} \subset [\omega]^\omega$ *of cardinality* ω_1 *which is simultaneously MAD and reaping.*

Proof. The proof is just a slight modification of that for Theorem 3.2.

For a countably infinite almost disjoint family $\mathcal{W} \subset [\omega]^\omega$ and a cube $P \in \mathrm{Perf}([\omega]^\omega)$ let $W_0 \in [\omega]^\omega$ and a subcube Q_0 of P be as in Lemma 3.3. Let $A \in [\omega]^\omega$ be almost disjoint with every set from $\mathcal{W} \cup \{W_0\}$. By Laver's theorem [22] we can also find a subcube Q_1 of Q_0 and a $W_1 \in [A]^\omega$ such that

- either $W_1 \cap x = \emptyset$ for every $x \in Q_1$,
- or else $W_1 \subset x$ for every $x \in Q_1$.

Let $Q(\mathcal{W}, P) = Q_1$ and $\mathcal{W}(\mathcal{W}, P) = \{W_0, W_1\}$. If $P \in \mathrm{Perf}^*([\omega]^\omega)$ is a singleton then we put $Q(\mathcal{W}, P) = P$ and we can easy find W_0 and W_1 satisfying the above conditions.

Let $\mathcal{A}_0 \subset [\omega]^\omega$ be an arbitrary infinite almost disjoint family and consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle\colon \eta<\xi\rangle, P_\xi) = Q\left(\mathcal{A}_0 \cup \bigcup\{\mathcal{W}_\eta\colon \eta<\xi\}, P_\xi\right),$$

where $\mathcal{W}_\eta$'s are defined inductively by $\mathcal{W}_\eta = \mathcal{W}(\mathcal{A}_0 \cup \bigcup\{\mathcal{W}_\eta\colon \eta<\xi\}, P_\eta)$.

By $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle\colon \xi<\omega_1\rangle$ played according to S in which and Player II loses, that is, $[\omega]^\omega = \bigcup_{\xi<\omega_1} Q_\xi$. Then the family $\mathcal{F} = \mathcal{A}_0 \cup \bigcup\{\mathcal{W}_\xi\colon \xi<\omega_1\}$ is MAD and reaping. $\square$

4. On $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$, selective and crowded ultrafilters, nonselective P-points, and numbers $\mathfrak{i}$ and $\mathfrak{u}$

The axioms $\mathrm{CPA}_{\mathrm{cube}}^{\mathrm{game}}$ and $\mathrm{CPA}_{\mathrm{cube}}$ dealt with the notion of $\mathcal{F}_{\mathrm{cube}}$-density, where $\mathcal{F}_{\mathrm{cube}}$ is the family of all injections $f\colon C\to X$ with C being a perfect cube in $\mathfrak{C}^\omega$. In the applications of these axioms we were using the facts that different subfamilies of $\mathrm{Perf}(X)$ were $\mathcal{F}_{\mathrm{cube}}$-dense. Unfortunately, in many cases the notion of $\mathcal{F}_{\mathrm{cube}}$-density is too weak to do the job — in the applications that follow the families $\mathcal{E}\subset \mathrm{Perf}(X)$ will not be $\mathcal{F}_{\mathrm{cube}}$-dense, but they will be dense in a weaker sense defined below. Luckily, this weaker notion of density still leads to consistent axioms.

4.1. Prisms and $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$.

To define this weaker notion of density, let us first take another look at the notion of cube. Let A be a non-empty countable set of ordinal numbers. The notion of a perfect cube in $\mathfrak{C}^A$ can be defined the same way as it was done for $\mathfrak{C}^\omega$. However, it will be more convenient for us to define it as follows. Let Φ_{cube} be the family of all continuous injection $f\colon \mathfrak{C}^A\to\mathfrak{C}^A$ such that

$$f(x)(\alpha) = f(y)(\alpha) \text{ for all } \alpha\in A \text{ and } x,y\in\mathfrak{C}^A \text{ with } x(\alpha)=y(\alpha).$$

In other words Φ_{cube} is the family of all functions of the form $f=\langle f_\alpha\rangle_{\alpha\in A}$, where each f_α is an injection from $\mathfrak{C}$ into $\mathfrak{C}$. Then the family of all perfect cubes in $\mathfrak{C}^A$ for an appropriate A is equal to

$$\mathrm{CUBE} = \{\mathrm{range}(f)\colon f\in\Phi_{\mathrm{cube}}\}$$

and $\mathcal{F}_{\mathrm{cube}}$ is the family all continuous injections $f\colon C\to X$ with $C\subset\mathfrak{C}^\omega$ and $C\in\mathrm{CUBE}$.

In the definitions that follow the notion of "cube" will be replaced by that of a "prism." So, let $\Phi_{\rm prism}(A)$ be the family of all continuous injection $f\colon \mathfrak{C}^A \to \mathfrak{C}^A$ with the property that

$$f(x)\restriction\alpha = f(y)\restriction\alpha \iff x\restriction\alpha = y\restriction\alpha \text{ for all } \alpha\in A \text{ and } x,y\in\mathfrak{C}^A \tag{4.1}$$

or, equivalently, such that for every $\alpha\in A$

$$f\restriction\restriction\alpha \stackrel{\rm def}{=} \{\langle x\restriction\alpha, y\restriction\alpha\rangle\colon \langle x,y\rangle\in f\}$$

is a one-to-one function from $\mathfrak{C}^{A\cap\alpha}$ into $\mathfrak{C}^{A\cap\alpha}$. For example, if $A=\{0,1,2\}$ then $f\in\Phi_{\rm prism}(A)$ provided there exist continuous functions $f_0\colon\mathfrak{C}\to\mathfrak{C}$, $f_1\colon\mathfrak{C}^2\to\mathfrak{C}$, and $f_2\colon\mathfrak{C}^3\to\mathfrak{C}$ such that

$$f(x_0,x_1,x_2)=\langle f_0(x_0), f_1(x_0,x_1), f_2(x_0,x_1,x_2)\rangle$$

for all $x_0,x_1,x_2\in\mathfrak{C}$ and maps f_0, $\langle f_0,f_1\rangle$, and f are one-to-one. Functions f from $\Phi_{\rm prism}(A)$ were first introduced, in more general setting, in [19] where they are called *projection-keeping homeomorphisms*. Note that

$$\Phi_{\rm prism}(A) \text{ is closed under compositions} \tag{4.2}$$

and that for every ordinal number $\alpha>0$

$$\text{if } f\in\Phi_{\rm prism}(A) \text{ then } f\restriction\restriction\alpha\in\Phi_{\rm prism}(A\cap\alpha). \tag{4.3}$$

Let

$$\mathbb{P}_A=\{\mathrm{range}(f)\colon f\in\Phi_{\rm prism}(A)\}.$$

We will write $\Phi_{\rm prism}$ for $\bigcup_{0<\alpha<\omega_1}\Phi_{\rm prism}(\alpha)$ and define

$$\mathbb{P}_{\omega_1}\stackrel{\rm def}{=}\bigcup_{0<\alpha<\omega_1}\mathbb{P}_\alpha=\{\mathrm{range}(f)\colon f\in\Phi_{\rm prism}\}.$$

Following [19] we will refer to elements of $\mathbb{P}_{\omega_1}$ as *iterated perfect sets*. Also let $\mathcal{F}_{\rm prism}(X)$ (or just $\mathcal{F}_{\rm prism}$, if X is clear from the context) be the family of all continuous injections $f\colon P\to X$ where $P\in\mathbb{P}_{\omega_1}$ and X is a fixed Polish space.

We say that a family $\mathcal{E}\subset\mathrm{Perf}(X)$ is $\mathcal{F}_{\rm prism}$-*dense* provided

$$\forall f\in\mathcal{F}_{\rm prism}\ \exists g\in\mathcal{F}_{\rm prism}\ (g\subset f\ \&\ \mathrm{range}(g)\in\mathcal{E}).$$

Similarly as in Fact 2.2, using (4.2) we can also prove that

Fact 4.1. *$\mathcal{E}\subset\mathrm{Perf}(X)$ is $\mathcal{F}_{\rm prism}$-dense if and only if*

$$\forall\alpha<\omega_1\ \ \forall f\in\mathcal{F}_{\rm prism},\ \mathrm{dom}(f)=\mathfrak{C}^\alpha\quad\exists g\in\mathcal{F}_{\rm prism}\quad (g\subset f\ \&\ \mathrm{range}(g)\in\mathcal{E}) \tag{4.4}$$

Notice also that $\Phi_{\rm cube} \subset \Phi_{\rm prism}$, so every cube is also a prism. From this and Fact 4.1 it also easy to see that

$$\text{if } \mathcal{E} \subset \mathrm{Perf}(X) \text{ is } \mathcal{F}_{\rm cube}\text{-dense then } \mathcal{E} \text{ is also } \mathcal{F}_{\rm prism}\text{-dense.} \tag{4.5}$$

The converse of (4.5), however, is false. (See [8, Remark 3.3.2].)

We also adopt the shortcuts similar to that for cubes. Thus, we say that $P \in \mathrm{Perf}(X)$ is a *prism* if we consider it with an (implicitly given) witness function $f \in \mathcal{F}_{\rm prism}$ onto P. By Fact 4.1 to establish $\mathcal{F}_{\rm prism}$-density we can always assume that the witness function f is in a *standard form*, that is, defined on the entire set $\mathfrak{C}^\alpha$ for an appropriate $\alpha < \omega_1$. Then Q is a *subprism of a prism* $P \in \mathrm{Perf}(X)$ provided $Q = f[E]$, where $f \in \mathcal{F}_{\rm prism}$ is as above and $E \in \mathbb{P}_\alpha$. Also singletons $\{x\}$ in X will be identified with constant functions from $E \in \mathbb{P}_{\omega_1}$ to $\{x\}$, and these functions will be considered as elements of $\mathcal{F}^*_{\rm prism}$, similarly as in (3.1).

Now we are ready to state the next version of our axiom, in which the game $\mathrm{GAME}_{\rm prism}(X)$ is an obvious generalization of $\mathrm{GAME}_{\rm cube}(X)$.

$\mathrm{CPA}^{\rm game}_{\rm prism}$**:** $\mathfrak{c} = \omega_2$ and for any Polish space X Player II has no winning strategy in the game $\mathrm{GAME}_{\rm prism}(X)$.

Notice that if a prism $P \in \mathrm{Perf}(X)$ is considered with a witness function $f \in \mathcal{F}_{\rm prism}$ in a standard form (i.e., f is from $\mathfrak{C}^\alpha$ onto P) then P is also a cube and any subcube of P is also a subprism of P. Thus, any Player II strategy in a game $\mathrm{GAME}_{\rm cube}(X)$ can be translated to a strategy in a game $\mathrm{GAME}_{\rm prism}(X)$. (You need to identify appropriately $\mathfrak{C}^\alpha$ with $\mathfrak{C}^\omega$: first you identify $\mathfrak{C}^\alpha$ with $\mathfrak{C}^\omega \times \mathfrak{C}^{\alpha\setminus\{0\}}$, which is important for a finite α, and then this second space identify with $\mathfrak{C}^\omega$ coordinatewise.) In particular, $\mathrm{CPA}^{\rm game}_{\rm prism}$ implies $\mathrm{CPA}^{\rm game}_{\rm cube}$. Also, essentially the same argument as used for Proposition 3.1 gives also the following.

Proposition 4.2. *Axiom* $\mathrm{CPA}^{\rm game}_{\rm prism}$ *implies the following prism version of the axiom* $\mathrm{CPA}_{\rm cube}$*:*

$\mathrm{CPA}_{\rm prism}$**:** $\mathfrak{c} = \omega_2$ *and for every Polish space* X *and every* $\mathcal{F}_{\rm prism}$*-dense family* $\mathcal{E} \subset \mathrm{Perf}(X)$ *there exists an* $\mathcal{E}_0 \subset \mathcal{E}$ *such that* $|\mathcal{E}_0| \le \omega_1$ *and* $|X \setminus \bigcup \mathcal{E}_0| \le \omega_1$.

By (4.5) it is also obvious that $\mathrm{CPA}_{\rm prism}$ implies $\mathrm{CPA}_{\rm cube}$. All these implications are summarized in Chart 1.

In what follows for a fixed $0 < \alpha < \omega_1$ and $0 < \beta \le \alpha$ the symbol π_β will stand for the projection from $\mathfrak{C}^\alpha$ onto $\mathfrak{C}^\beta$. We will always consider $\mathfrak{C}^\alpha$ with the following standard metric ρ: fix an enumeration $\{\langle \beta_k, n_k\rangle \colon k < \omega\}$ of $\alpha \times \omega$ and for distinct $x, y \in \mathfrak{C}^\alpha$ define

$$\rho(x,y) = 2^{-\min\{k<\omega\colon x(\beta_k)(n_k) \neq y(\beta_k)(n_k)\}}. \tag{4.6}$$

The open ball in $\mathfrak{C}^\alpha$ with a center at $z \in \mathfrak{C}^\alpha$ and radius $\varepsilon > 0$ will be denoted by $B_\alpha(z, \varepsilon)$. Notice that in this metric any two open balls are either disjoint or one is a subset of another. Also for every $\gamma < \alpha$ and $\varepsilon > 0$

$$\pi_\gamma[B_\alpha(s,\varepsilon)] = \pi_\gamma[B_\alpha(t,\varepsilon)] \quad \text{for every } s,t \in \mathfrak{C}^\alpha \text{ with } s \restriction \gamma = t \restriction \gamma. \tag{4.7}$$

It is also easy to see that any $B_\alpha(z, \varepsilon)$ is a clopen set and, in fact, it is a perfect cube in $\mathfrak{C}^\alpha$, so it belongs to $\mathbb{P}_\alpha$. In fact, more can be said:

$$\text{if } \mathcal{B}_\alpha \stackrel{\text{def}}{=} \{B \subset \mathfrak{C}^\alpha \colon B \text{ is clopen in } \mathfrak{C}^\alpha\} \text{ then } \mathcal{B}_\alpha \subset \mathbb{P}_\alpha. \tag{4.8}$$

This is the case, since any clopen E in $\mathfrak{C}^\alpha$ is a finite union of disjoint open balls, each of which belongs to $\mathbb{P}_\alpha$, and it is easy to see that $\mathbb{P}_\alpha$ is closed under finite unions of disjoint sets.

From this we conclude immediately that

$$\text{a clopen subset of } E \in \mathbb{P}_\alpha \text{ belongs to } \mathbb{P}_\alpha \tag{4.9}$$

and

$$\text{a clopen subset of a prism is its subprism.} \tag{4.10}$$

Notice also that if $P \in \mathbb{P}_\alpha$ then

$$P \cap \pi_\beta^{-1}(P') \in \mathbb{P}_\alpha \quad \text{for every } P' \in \mathbb{P}_\beta \text{ with } P' \subset \pi_\beta[P]. \tag{4.11}$$

Indeed, let $f \in \Phi_{\text{prism}}(\beta)$ and $g \in \Phi_{\text{prism}}(\alpha)$ be such that $f[\mathfrak{C}^\beta] = P'$ and $g[\mathfrak{C}^\alpha] = P$. Let $Q = (g \restriction \beta)^{-1}[P'] = (g \restriction \beta)^{-1} \circ f[\mathfrak{C}^\beta] \in \mathbb{P}_\beta$. Then $\pi_\beta^{-1}(Q)$ belongs to $\mathbb{P}_\alpha$ and $P \cap \pi_\beta^{-1}(P') = g[\pi_\beta^{-1}(Q)] \in \mathbb{P}_\alpha$.

4.2. Fusion Lemmas.

One of the main technical tools used to prove that a family of perfect sets is dense is the so called fusion lemma. It says that for an appropriately chosen decreasing sequence $\{P_n \colon n < \omega\}$ of perfect sets its intersection $P = \bigcap_{n<\omega} P_n$, called the *fusion*, is still a perfect set. The simple structure of perfect cubes makes it quite easy to formulate a "cube fusion lemma" in which the fusion set P is also a cube. However, so far we did not have any need for such a lemma (at least in an explicit form), since its use was always hidden in the proofs of the results we quoted, like Claim 2.3 or Proposition 2.4. On the other hand, the new and more complicated structure of prisms does not leave us the option of avoiding fusion arguments any longer — we have to face it up front.

For a fixed $0 < \alpha < \omega_1$ let $\{\langle \beta_k, n_k \rangle \colon k < \omega\}$ be an enumeration of $\alpha \times \omega$ used in the definition (4.6) of the metric ρ and let

$$A_k = \{\langle \beta_i, n_i \rangle \colon i < k\} \quad \text{for every } k < \omega. \tag{4.12}$$

Lemma 4.3 (Fusion Sequence). *Let $0 < \alpha < \omega_1$ and for every $k < \omega$ let $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha \colon s \in 2^{A_k}\right\}$. Assume that for every $k < \omega$, $s,t \in 2^{A_k}$, $r \in \bigcup_{i<\omega} 2^{A_i}$, and $\beta < \alpha$ we have:*

(i) *the diameter of E_s is less than or equal to 2^{-k},*
(ii) *if $r \subset s$ then $E_s \subset E_r$,*
(ag) *(agreement) if $s \restriction (\beta \times \omega) = t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] = \pi_\beta[E_t]$,*
(sp) *(split) if $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$ then $\pi_\beta[E_s] \cap \pi_\beta[E_t] = \emptyset$.*

Then $Q = \bigcap_{k<\omega} \bigcup \mathcal{E}_k$ belongs to $\mathbb{P}_\alpha$.

Proof. For $x \in \mathfrak{C}^\alpha$ let $\bar{x} \in 2^{\alpha \times \omega}$ be defined by $\bar{x}(\beta, n) = x(\beta)(n)$.

First note that, by conditions (i) and (sp), for every $k < \omega$ the sets in $\mathcal{E}_k$ are pairwise disjoint and each of the diameter at most 2^{-k}. Thus, taking into account (ii), the function $h \colon \mathfrak{C}^\alpha \to \mathfrak{C}^\alpha$ defined by

$$h(x) = r \iff \{r\} = \bigcap_{k<\omega} E_{\bar{x} \restriction A_k}$$

is well defined and is one-to-one. It is also easy to see that h is continuous and that $Q = h\left[\mathfrak{C}^\alpha\right]$. Thus, we need to prove only that $h \in \Phi_{\text{prism}}(\alpha)$, that is, that h is projection-keeping.

To show this fix $\beta < \alpha$, put $S = \bigcup_{i<\omega} 2^{A_i}$, and notice that, by (i) and (ag), for every $x \in \mathfrak{C}^\alpha$ we have

$$\begin{aligned} \{h(x) \restriction \beta\} &= \pi_\beta\left[\bigcap\{E_{\bar{x} \restriction A_k} \colon k < \omega\}\right] \\ &= \bigcap\{\pi_\beta[E_{\bar{x} \restriction A_k}] \colon k < \omega\} \\ &= \bigcap\{\pi_\beta[E_s] \colon s \in S \ \&\ s \subset \bar{x}\} \\ &= \bigcap\{\pi_\beta[E_s] \colon s \in S \ \&\ s \restriction (\beta \times \omega) \subset \bar{x}\}. \end{aligned}$$

Now, if $x \restriction \beta = y \restriction \beta$ then for every $s \in S$

$$s \restriction (\beta \times \omega) \subset \bar{x} \quad \Leftrightarrow \quad s \restriction (\beta \times \omega) \subset \bar{y}$$

so $h(x) \restriction \beta = h(y) \restriction \beta$.

On the other hand, if $x \restriction \beta \neq y \restriction \beta$ then there exists a $k < \omega$ big enough such that for $s = \bar{x} \restriction A_k$ and $t = \bar{y} \restriction A_k$ we have $s \restriction (\beta \times \omega) \neq t \restriction (\beta \times \omega)$. But then $\{h(x) \restriction \beta\}$ and $\{h(y) \restriction \beta\}$ are subsets of $\pi_\beta[E_s]$ and $\pi_\beta[E_t]$, respectively, which, by (sp), are disjoint. So, $h(x) \restriction \beta \neq h(y) \restriction \beta$. □

In all of our applications the task of constructing sequences $\langle \mathcal{E}_k \colon k < \omega \rangle$ satisfying specific conditions (ag) and (sp) can be reduced to checking some simple density properties listed in our next lemma. In its statement we consider $\mathbb{P}_\alpha$ as ordered by inclusion and use the standard terminology from the theory of partially ordered sets: $D \subset \mathbb{P}_\alpha$ is *dense* provided for every

$E \in \mathbb{P}_\alpha$ there is an $E' \in D$ with $E' \subset E$; it is *open* provided for every $E \in D$ if $E' \in \mathbb{P}_\alpha$ and $E' \subset E$ then $E' \in D$. Moreover, for a family $\mathcal{E}$ of pairwise disjoint subsets of $\mathbb{P}_\alpha$ we say that $\mathcal{E}' \subset \mathbb{P}_\alpha$ is a *refinement of* $\mathcal{E}$ provided $\mathcal{E}' = \{P_E \colon E \in \mathcal{E}\}$ where $P_E \subset E$ for all $E \in \mathcal{E}$.

Lemma 4.4. *Let $0 < \alpha < \omega_1$ and $k < \omega$. If $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha \colon s \in 2^{A_k}\right\}$ satisfies (ag) and (sp) then*

(A) *there exists an $\mathcal{E}_{k+1} = \left\{E_s \in \mathbb{P}_\alpha \colon s \in 2^{A_{k+1}}\right\}$ such that (i), (ii), (ag), and (sp) hold for all $s, t \in 2^{A_{k+1}}$ and $r \in 2^{A_k}$.*

Moreover, if $\mathcal{D} \subset [\mathbb{P}_\alpha]^{<\omega}$ is a family of pairwise disjoint sets such that $\emptyset \in \mathcal{D}$, $\mathcal{D}$ is closed under refinements, and

(†) *for every $\mathcal{E} \in \mathcal{D}$ and $E \in \mathbb{P}_\alpha$ which is disjoint with $\bigcup \mathcal{E}$ there exists an $E' \in \mathbb{P}_\alpha \cap \mathcal{P}(E)$ such that $\{E'\} \cup \mathcal{E} \in \mathcal{D}$*

then

(B) *there exists a refinement $\mathcal{E}'_k \in \mathcal{D}$ of $\mathcal{E}_k$ satisfying (ag) and (sp),*
(C) *there exists an $\mathcal{E}_{k+1}$ as in (A) such that $\mathcal{E}_{k+1} \in \mathcal{D}$.*

The proof of this lemma will be postponed to the last section of this paper. One of the most important consequences of Lemma 4.4 is the following.

Corollary 4.5. *Let $0 < \alpha < \omega_1$ and let $\{D_k \colon k < \omega\}$ be a collection of dense open subsets of $\mathbb{P}_\alpha$. If for every $k < \omega$*

$$D_k^* = \left\{\bigcup \mathcal{D} \colon \mathcal{D} \in [D_k]^{<\omega} \textit{ and the sets in } \mathcal{D} \textit{ are pairwise disjoint}\right\}$$

then $\bar{D} = \bigcap_{k<\omega} D_k^*$ is open and dense in $\mathbb{P}_\alpha$.

Proof. It is clear that $\bar{D}$ is open. To see its density notice that the families

$$\mathcal{D}_k = \left\{\mathcal{D} \in [D_k]^{<\omega} \text{ and sets in } \mathcal{D} \text{ are pairwise disjoint}\right\}$$

satisfy condition (†). Let $E \in \mathbb{P}_\alpha$, choose an $E_\emptyset \in D_0 \subset D_0^*$ below E, and put $\mathcal{E}_0 = \{E_\emptyset\}$. Applying (C) from Lemma 4.4 by induction we can define families $\mathcal{E}_k \in \mathcal{D}_k$, $k < \omega$, such that conditions (i), (ii), (ag), and (sp) from Lemma 4.3 are satisfied. But then $Q = \bigcap_{k<\omega} \bigcup \mathcal{E}_k \subset E$ belongs to $\bar{D}$. □

4.3. Selective ultrafilters and number $\mathfrak{u}$.

Recall that every weakly selective ultrafilter is selective and that the ideal $\mathcal{I} = [\omega]^{<\omega}$ is selective. Another example of a weakly selective ideal which we will use in what follows is given below.

Fact 4.6. *The ideal $\mathcal{I}$ of nowhere dense subset of rationals $\mathbb{Q}$ is weakly selective.*

Proof. Let $A \in \mathcal{I}^+$ and take an $f\colon A \to \omega$. If there is a $B \in \mathcal{I}^+ \cap \mathcal{P}(A)$ such that $f \restriction B$ constant then we are done. So, assume that it is not the case and let $A_0 \subset A$ be dense on some interval. By induction on $n < \omega$ define a sequence $\{b_n \in A_0\colon n < \omega\}$ dense in A_0 such that f restricted to $B = \{b_n\colon n < \omega\}$ is one-to-one. Then B is as desired. □

In what follows we will also need the following fact about weakly selective ideals, which can be found in Grigorieff [15, Proposition 14].

Proposition 4.7. *Let $\mathcal{I}$ be a weakly selective ideal on ω and $A \in \mathcal{I}^+$. If $T \subset A^{<\omega}$ is a tree such that*

$$A \setminus \{j < \omega\colon s\hat{\ }j \in T\} \in \mathcal{I} \quad \textit{for every } s \in T$$

then there exists a branch b of T such that $b[\omega] \in \mathcal{I}^+$.

Theorem 4.8. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that for every selective ideal $\mathcal{I}$ on ω there exists a selective ultrafilter $\mathcal{F}$ on ω such that $\mathcal{F} \subset \mathcal{I}^+$.*

In particular if $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *holds then there is a selective ultrafilter on ω.*

The proof is based on the following lemma.

Lemma 4.9. *Let $\mathcal{I}$ be a weakly selective ideal on ω.*

(a) *For every $A \in \mathcal{I}^+$ and every prism P in ω^ω there exist a $B \in \mathcal{I}^+$, $B \subset A$, and a subprism Q of P such that either*
  (i) *$g \restriction B$ is one-to-one for every $g \in Q$, or else*
  (ii) *there exists an $n < \omega$ such that $g \restriction B$ is constant equal to n for every $g \in Q$.*

(b) *For every $A \in \mathcal{I}^+$ and every prism P in $[\omega]^\omega$ there exist a $B \in \mathcal{I}^+$, $B \subset A$, and a subprism Q of P such that either*
  – *$x \cap B = \emptyset$ for every $x \in Q$, or else*
  – *$B \subset x$ for every $x \in Q$.*

Proof. (a) Fix an $A \in \mathcal{I}^+$, an $f \in \mathcal{F}_{\mathrm{prism}}(\omega^\omega)$ from $\mathfrak{C}^\alpha$ onto P, and assume that for no subprism Q of P and $B \in \mathcal{I}^+ \cap \mathcal{P}(A)$ condition (ii) holds. We will find Q satisfying (i).

For $i, n < \omega$ let $D(i,n) = \{E_0 \in \mathbb{P}_\alpha\colon (\forall g \in E_0)\ f(g)(i) \neq n\}$ and for $\gamma \leq \alpha$, $E \in \mathbb{P}_\alpha$, and $A' \subset \omega$ put $D_\gamma(E,i,n) = \{\pi_\gamma[E_0]\colon E_0 \in D(i,n) \cap \mathcal{P}(E)\}$ and

$$D_\gamma(E,A',n) = \bigcap_{i \in A'} D_\gamma(E,i,n).$$

Notice that the sets $D_\gamma(E,i,n)$ and $D_\gamma(E,A',n)$ are open in $\mathbb{P}_\gamma$. By induction on $0 < \beta \leq \alpha$ we are going to prove the following property.

ψ_β: For all $0<\gamma\leq\beta$, $E\in\mathbb{P}_\alpha$, $n<\omega$, and $\hat{A}\in\mathcal{I}^+\cap\mathcal{P}(A)$ there exists an $A'\in\mathcal{I}^+\cap\mathcal{P}(\hat{A})$ such that $D_\gamma(E,A',n)\neq\emptyset$.

In what follows for $k<\omega$ and $\mathcal{E}_k=\left\{E_s\in\mathbb{P}_\beta\colon s\in 2^{A_k}\right\}$ satisfying (i), (ag), and (sp) from Lemma 4.3 and $\mathcal{E}_{k+1}=\left\{E_s\in\mathbb{P}_\beta\colon s\in 2^{A_{k+1}}\right\}$ we will write

$$\mathcal{E}_{k+1}\prec\mathcal{E}_k$$

provided (i), (ii), (ag), and (sp) hold for all $s,t\in 2^{A_{k+1}}$ and $r\in 2^{A_k}$. One of the main facts used in the proof of ψ_β is the following property.

(∗) If ψ_γ holds for all $\gamma<\beta$, $\bar{A}\in\mathcal{I}^+\cap\mathcal{P}(A)$, $n<\omega$, $E\in\mathbb{P}_\alpha$, $\mathcal{E}_k$ is as above and such that $\bigcup\mathcal{E}_k\subset\pi_\beta[E]$, and

$$Z(\bar{A},\mathcal{E}_k,n)=\left\{i\in\bar{A}\colon(\exists\mathcal{E}_{k+1}\prec\mathcal{E}_k)\ \bigcup\mathcal{E}_{k+1}\in D_\beta(E,i,n)\right\}$$

then $\bar{A}\setminus Z(\bar{A},\mathcal{E}_k,n)\in\mathcal{I}$.

In order to prove (∗) fix an $\hat{A}\in\mathcal{P}(\bar{A})\cap\mathcal{I}^+$ and note that it is enough to show that $\hat{A}\cap Z(\bar{A},\mathcal{E}_k,n)\neq\emptyset$. Fix an $\bar{\mathcal{E}}_{k+1}=\{\bar{E}_s\in\mathbb{P}_\beta\colon s\in 2^{A_{k+1}}\}$ such that $\bar{\mathcal{E}}_{k+1}\prec\mathcal{E}_k$. We can find such an $\bar{\mathcal{E}}_{k+1}$ by Lemma 4.4(A). Let $\gamma=\max\{\delta\colon\langle\delta,m\rangle\in A_{k+1}\}<\beta$.

First assume that $\gamma=0$. Then for every $s\in 2^{A_{k+1}}$ the set

$$Z_s=\left\{i\in\hat{A}\colon D_\beta(E,i,n)\cap\mathcal{P}(\bar{E}_s)=\emptyset\right\}$$

belongs to $\mathcal{I}$, since otherwise $Q=f\left[\pi_\beta^{-1}[\bar{E}_s]\cap E\right]$ and $B=Z_s\in\mathcal{I}^+\cap\mathcal{P}(A)$ would satisfy the condition (ii), contradicting our assumption. Let us define $A'=\hat{A}\setminus\bigcup\{Z_s\colon s\in 2^{A_{k+1}}\}\in\mathcal{I}^+$ and notice that $A'\subset Z(\bar{A},\mathcal{E}_k,n)$. Indeed, take an $i\in A'$ and for every $s\in 2^{A_{k+1}}$ choose $E_s\in D_\beta(E,i,n)\cap\mathcal{P}(\bar{E}_s)$. Then $\mathcal{E}_{k+1}\stackrel{\text{def}}{=}\{E_s\colon s\in 2^{A_{k+1}}\}\prec\mathcal{E}_k$, since (i), (ii), and (sp) hold for $\mathcal{E}_{k+1}$ as they were true for $\bar{\mathcal{E}}_{k+1}$, and (ag) is satisfied trivially, by the maximality of γ. Condition $\bigcup\mathcal{E}_{k+1}\in D_\beta(E,i,n)$ is guaranteed by the choice of E_s's, so indeed $i\in Z(\bar{A},\mathcal{E}_k,n)$.

Next assume that $\gamma>0$. Let $B=\{\langle\delta,m\rangle\in A_{k+1}\colon\delta<\gamma\}$, and define $\mathcal{E}^*_{k+1}=\{E^*_t\in\mathbb{P}_\gamma\colon t\in 2^B\}$, where $E^*_t=\pi_\gamma[\bar{E}_s]$ for any $s\in 2^{A_{k+1}}$ with $t\subset s$. Note that, by (ag), the definition of E^*_t is independent of the choice of s. It is easy to see that $\mathcal{E}^*_{k+1}$ satisfies (ag) and (sp), where α is replaced by γ. For $\hat{A}_0\in\mathcal{P}(\hat{A})\cap\mathcal{I}^+$ and $t\in 2^B$ define $D(\hat{A}_0,t)$ as the collection of all $E_0\in\mathbb{P}_\gamma$ for which there exists an $A'\in\mathcal{P}(\hat{A}_0)\cap\mathcal{I}^+$ such that

$$E_0\in\bigcap\left\{D_\gamma\left(\pi_\beta^{-1}[\bar{E}_s]\cap E,A',n\right)\colon t\subset s\in 2^{A_{k+1}}\right\}.$$

It is clear that each $D(\hat{A}_0,t)$ is open, since so is each $D_\gamma\left(\pi_\beta^{-1}[\bar{E}_s]\cap E,A',n\right)$. It is also important to notice that $D(\hat{A}_0,t)$ is dense below E^*_t. To see it, fix an $E_0\in\mathbb{P}_\gamma\cap\mathcal{P}(E^*_t)$ and let $\{s_1,\ldots,s_m\}$ be an enumeration of

$\{s \in 2^{A_{k+1}} \colon t \subset s\}$. By induction on $i \le m$ define two decreasing sequences $\{E_i \in \mathbb{P}_\gamma \colon i \le m\}$ and $\{\hat{A}_i \in \mathcal{I}^+ \colon i \le m\}$ such that $E_i \in D_\gamma\left(\pi_\gamma^{-1}(E_{i-1}) \cap (\pi_\beta^{-1}[\bar{E}_{s_i}] \cap E), \hat{A}_i, n\right)$ provided $0 < i \le m$. The inductive step can be made since condition ψ_γ holds. Then

$$E_m \in \bigcap\left\{D_\gamma\big(\pi_\beta^{-1}[\bar{E}_s] \cap E, \hat{A}_m, n\big) \colon t \subset s \in 2^{A_{k+1}}\right\}$$

and we have $E_m \in D(\hat{A}_0, t) \cap \mathcal{P}(E_0)$.

Let $\mathcal{D}$ be the collection of all pairwise disjoint families $\mathcal{E} \in [\mathbb{P}_\gamma]^{<\omega}$ for which there exists an $A' \in \mathcal{P}(\hat{A}) \cap \mathcal{I}^+$ working simultaneously for all $E_0 \in \mathcal{E}$, that is, such that for all $t \in 2^B$ and $E_0 \in \mathcal{E}$ if $E_0 \subset E_t^*$ then

$$E_0 \in \bigcap\left\{D_\gamma\left(\pi_\beta^{-1}[\bar{E}_s] \cap E, A', n\right) \colon t \subset s \in 2^{A_{k+1}}\right\}.$$

Notice that $\mathcal{D}$ satisfies condition (†) from Lemma 4.4 used with α replaced by γ. Indeed, if $\mathcal{E} \in \mathcal{D}$ is witnessed by $A' \in \mathcal{P}(\hat{A}) \cap \mathcal{I}^+$ and $E \in \mathbb{P}_\gamma$ is disjoint with $\bigcup \mathcal{E}$ choose $E' \in \mathbb{P}_\gamma$ below E which is either disjoint with $\bigcup \mathcal{E}_{k+1}^*$ or contained in some $E_t^* \in \mathcal{E}_{k+1}^*$. If $E' \cap \bigcup \mathcal{E}_{k+1}^* = \emptyset$ then $\{E'\} \cup \mathcal{E} \in \mathcal{D}$ is witnessed by A'. If $E' \subset E_t^* \in \mathcal{E}_{k+1}^*$ by the density of $D(A', t)$ below E_t^* we can find an $A'' \in \mathcal{P}(A') \cap \mathcal{I}^+$ and

$$E'' \in \mathcal{P}(E') \cap \bigcap\left\{D_\gamma\left(\pi_\beta^{-1}[\bar{E}_s] \cap E, A'', n\right) \colon t \subset s \in 2^{A_{k+1}}\right\}.$$

Then $\{E''\} \cup \mathcal{E} \in \mathcal{D}$ is witnessed by A''.

Now, by Lemma 4.4(B), there exists an $\hat{\mathcal{E}}_{k+1} = \left\{\hat{E}_t \in D \colon t \in 2^B\right\} \in \mathcal{E}$ satisfying (ag) and (sp) such that $\hat{E}_t \subset E_t^*$ for all $t \in 2^B$. Let $A' \in \mathcal{P}(\hat{A}) \cap \mathcal{I}^+$ witness $\hat{\mathcal{E}}_{k+1} \in \mathcal{E}$. We will show that $A' \subset Z(\bar{A}, \mathcal{E}_k, n)$. So fix an $i \in A'$. Since for every $t \in 2^B$ and $t \subset s \in 2^{A_{k+1}}$ we have $\hat{E}_t \in D_\gamma(\pi_\beta^{-1}[\bar{E}_s] \cap E, A', n)$ there exists an $\hat{E}_s \in D(\pi_\beta^{-1}[\bar{E}_s] \cap E, i, n)$ with $\pi_\gamma[\hat{E}_s] = \hat{E}_t$. Let $E_s = \pi_\beta[\hat{E}_s] \subset \bar{E}_s$ and notice that $\mathcal{E}_{k+1} \stackrel{\text{def}}{=} \{E_s \colon s \in 2^{A_{k+1}}\} \prec \mathcal{E}_k$. Indeed, $\mathcal{E}_{k+1}$ satisfies (i), (ii), and (sp) since they were true for $\bar{\mathcal{E}}_{k+1}$ and $\mathcal{E}_{k+1}$ is a refinement of $\bar{\mathcal{E}}_{k+1}$. Condition (ag) is satisfied by $\mathcal{E}_{k+1}$ since, by the maximality of γ, it is non-trivial for $\hat{\beta} \le \gamma$ and for such $\hat{\beta}$ it guaranteed by (ag) for $\hat{\mathcal{E}}_{k+1}$. Finally, $\bigcup \mathcal{E}_{k+1} \in D_\beta(E, i, n)$ is guaranteed by our definition, so indeed $i \in Z(\bar{A}, \mathcal{E}_k, n)$. This finishes the proof of $(*)$.

To prove ψ_β assume that ψ_γ holds for all $\gamma < \beta$. Fix $E \in \mathbb{P}_\alpha$, $n < \omega$, and $\hat{A} \in \mathcal{I}^+ \cap \mathcal{P}(A)$. We need to find an $A' \in \mathcal{I}^+ \cap \mathcal{P}(\hat{A})$ such that $D_\beta(E, A', n) \neq \emptyset$, that is, $\bigcap_{i \in A'} D_\beta(E, i, n) \neq \emptyset$. We will construct a tree $T \subset \hat{A}^{<\omega}$ and the mapping $T \ni s \mapsto \mathcal{E}_s \in [\mathbb{P}_\beta]^{<\omega}$ such that $\mathcal{E}_\emptyset = \{E\}$ and for every $r \in T$ and $s = r\hat{\ }i \in T$ we have $\mathcal{E}_s \prec \mathcal{E}_r$ and $\bigcup \mathcal{E}_s \in D_\beta(E, i, n)$. Notice that, by $(*)$, for every $r \in T$ we can define $\mathcal{E}_{r\hat{\ }i}$ for all $i \in Z(\hat{A}, \mathcal{E}_r, n)$. So we can ensure that T satisfies the assumptions of Proposition 4.7. Let b be a branch of T with $A' = b[\omega] \in \mathcal{I}^+$. By Lemma 4.3 $E_0 = \bigcap_{k<\omega} \bigcup \mathcal{E}_{b\restriction k}$ belongs to $\mathbb{P}_\beta$ and $E_0 \in \bigcap_{i \in A'} D_\beta(E,i,n)$. This concludes the proof of ψ_β.

For the conclusion of the proof we first need to refine the prism P. For every $i < \omega$ let $h_i \colon \mathfrak{C}^\alpha \to \omega \subset \mathbb{R}$ be defined by $h_i(g) = f(g)(i)$. Clearly each h_i is continuous. Hence each set $h_i^{-1}(n)$ is open in $\mathfrak{C}^\alpha$, so

$$D_i = \{E \in \mathbb{P}_\alpha \colon h_i \text{ is constant on } E\}$$

is dense and open in $\mathbb{P}_\alpha$. Thus, by Corollary 4.5, there exists an $E \in \bigcap_{i<\omega} D_i^*$, where

$$D_i^* = \left\{ \bigcup \mathcal{D} \colon \mathcal{D} \in [D_i]^{2^i} \text{ and the sets in } \mathcal{D} \text{ are pairwise disjoint} \right\}.$$

Let $P_0 = f[E]$. Then P_0 is a subprism of P. We will find a subprism Q of P_0. Notice also that, by our construction, for every $i < \omega$ there is a set $V_i \in [\omega]^{\le 2^i}$ such that

$$f(g)(i) \in V_i \text{ for all } g \in E \text{ and } i < \omega.$$

Notice also that since ψ_α holds, so is the conclusion of $(*)$ for $\beta = \alpha$. In particular, for every $n < \omega$ and $\mathcal{E}_k = \left\{E_s \in \mathbb{P}_\alpha \colon s \in 2^{A_k}\right\}$ satisfying (i), (ag), and (sp) from Lemma 4.3 and such that $\bigcup \mathcal{E}_k \subset E$ we have

$$Z(A, \mathcal{E}_k, n) = \left\{ i \in A \colon (\exists \mathcal{E}_{k+1} \prec \mathcal{E}_k) \ \bigcup \mathcal{E}_{k+1} \in D(i,n) \right\}$$

and $A \setminus Z(A, \mathcal{E}_k, n) \in \mathcal{I}$.

We will construct a tree $T \subset A^{<\omega}$ as in Proposition 4.7 and the mapping $T \ni s \mapsto \mathcal{E}_s \in [\mathbb{P}_\alpha]^{<\omega}$. The construction is done by induction on the levels of T. We start with $\mathcal{E}_\emptyset = \{E\}$ and, for every $r \in T$ and $s = r\hat{\ }i \in T$, we ensure that $\mathcal{E}_s \prec \mathcal{E}_r$ and

$$\bigcup \mathcal{E}_s \in \bigcap \{D(i,n) \colon n \in V_j \text{ for some } j \in \operatorname{range}(r)\}. \tag{4.13}$$

Notice that for every $r \in T$ if $Z_r = \bigcap \left\{ Z(A, \mathcal{E}_r, n) \colon n \in \bigcup_{j \in \operatorname{range}(r)} V_j \right\}$ then $A \setminus Z_r \in \mathcal{I}$. Moreover, for all $i \in Z_r$ we can find $\mathcal{E}_{r\hat{\ }i}$ as in (4.13). So, T as above can be constructed. Take a branch b of T with $B = b[\omega] \in \mathcal{I}^+$ and $E_0 = \bigcap_{k<\omega} \bigcup \mathcal{E}_{b\restriction k} \in \mathbb{P}_\beta$. Then B and $Q = f[E_0]$ satisfy (i). This finishes the proof of (a).

(b) Since the characteristic function χ gives an embedding from $[\omega]^\omega$ into $2^\omega \subset \omega^\omega$ the prism P can be identified with $\chi[P] = \{\chi_x \colon x \in P\}$. Applying part (a) to $\chi[P]$ we can find a subprism Q of P, $k < 2$, and $B \in \mathcal{I}^+$, $B \subset A$, such that $\chi_x \restriction B \equiv k$ for every $x \in Q$. If $k = 0$ this gives $x \cap B = \emptyset$ for every $x \in Q$. If $k = 1$ we have $B \subset x$ for every $x \in Q$. $\square$

Proof of Theorem 4.8. Let $\mathcal{I}$ be a selective ideal on ω. For a countable family $\mathcal{A} \subset \mathcal{I}^+$ linearly ordered by $\subset^*$ let $C(\mathcal{A}) \in \mathcal{I}^+$ be such that $C(\mathcal{A}) \subset^* A$ for every $A \in \mathcal{A}$.

For $A \in \mathcal{I}^+$ and $f \in \mathcal{F}_{\rm prism}(\omega^\omega)$ put $P = {\rm range}(f)$ and let $B(A,P) \in [A]^\omega$ and a subprism $Q(A,P)$ of P be as in Lemma 4.9(a). If $f \in \mathcal{C}_{\rm prism}(\omega^\omega)$ and $P = {\rm range}(f) = \{x\}$ then we put $Q(A,P) = P$ and take $B(A,P) \in [A]^\omega$ satisfying the conclusion of Lemma 4.9(a).

Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(C(\{B_\eta \colon \eta < \xi\}), P_\xi),$$

where the sets B_η are defined inductively by $B_\eta = B(C(\{B_\zeta \colon \zeta < \eta\}), P_\eta)$.

By ${\rm CPA}_{\rm prism}^{\rm game}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S in which and Player II loses, that is, $\omega^\omega = \bigcup_{\xi<\omega_1} Q_\xi$.

Now, let $\mathcal{F}$ be a filter generated by $\{B_\xi \colon \xi < \omega_1\}$ and notice that $\mathcal{F}$ is a selective ultrafilter. It is a filter, since $\{B_\xi \colon \xi < \omega_1\}$ is decreasing with respect to $\subset^*$. It also easy to see that

> for every $f \in \omega^\omega$ there exists a $B \in \mathcal{F}$ such that $f \restriction B$ is either one-to-one or constant.

Indeed, if $f \in \omega^\omega$ then there exists a $\xi < \omega_1$ such that $f \in Q_\xi$. Then $B = B_\xi$ is as desired.

Now, to see that $\mathcal{F}$ is an ultrafilter take an $A \subset \omega$ and let $f \in \omega^\omega$ be a characteristic function of A. Then $B \in \mathcal{F}$ as above is a subset of either A or its complement.

It is easy to see that the above two properties imply that $\mathcal{F}$ is a selective ultrafilter. $\square$

Notice that ${\rm CPA}_{\rm prism}^{\rm game}$ implies also that we have many different selective ultrafilters. The consistency of this fact, in a model obtained by adding many side-by-side Sacks reals, was first noticed by Hart in [16].

Remark 4.10. ${\rm CPA}_{\rm prism}^{\rm game}$ implies that there are ω_2 different selective ultrafilters.

Proof. This can be easily deduced by a simple transfinite induction from

(∗) for every family $\mathcal{U} = \{\mathcal{F}_\xi \colon \xi < \omega_1\}$ of ultrafilters on ω there is a selective ultrafilter $\mathcal{F} \notin \mathcal{U}$.

Property (∗) is proved as above, where we use $\mathcal{I} = [\omega]^{<\omega}$ and the operator $C(\{B_\eta \colon \eta < \xi\})$ is replaced with $C_\xi(\{B_\eta \colon \eta < \xi\}) \notin \mathcal{F}_\xi$. $\square$

Now, from the above we obtain that "$2^{\omega_1} = \omega_2$"+${\rm CPA}_{\rm prism}^{\rm game}$ (which is consistent) implies that there are 2^{ω_1} different selective ultrafilters. Since

CPA is also consistent with $2^{\omega_1} > \omega_2$ it is worth noticing that the existence of 2^{ω_1} different selective ultrafilters can be also deduced from a slightly stronger version of $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ also in this case. This can be found in [8].

Recall that the number $\mathfrak{u}$ is defined as the smallest cardinality of the base for a non-principal ultrafilter on ω. Thus Theorem 4.8 and Corollaries 2.7 and 2.8 imply that

Corollary 4.11. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that* $\mathfrak{u} = \mathfrak{r}_\sigma = \omega_1$.

4.4. Non-selective P-points and number $\mathfrak{i}$.

Recall that an ultrafilter $\mathcal{F}$ on ω is a *P-point* provided for every partition $\mathcal{P}$ of ω either $\mathcal{P} \cap \mathcal{F} \neq \emptyset$ or there is an $F \in \mathcal{F}$ such that $|F \cap P| < \omega$ for all $P \in \mathcal{P}$. Clearly every selective ultrafilter is a P-point. Thus, $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies the existence of a P-point. On the other hand, Shelah proved that there are models with no P-points. (See e.g. [1, thm. 4.4.7].) Hart in [16] proved that in a model obtained by adding many side-by-side Sacks reals there is a P-point which is not selective. Next, we will prove that this follows also from $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$. The main idea of the proof is the same as that used in [16].

For $m < \omega$ let $P_m = \{n < \omega \colon 2^m - 1 \leq n < 2^{m+1} - 1\}$ and define a partition $\mathcal{P}$ of ω by $\mathcal{P} = \{P_m \colon m < \omega\}$. Consider the following ideal $\bar{\mathcal{I}}$ on ω

$$\bar{\mathcal{I}} = \left\{ A \subset \omega \colon \limsup_{m\to\infty} |A \cap P_m| < \omega \right\} \tag{4.14}$$

and notice the following simple fact.

Fact 4.12. *If $\mathcal{A} \in [\bar{\mathcal{I}}^+]^{\leq\omega}$ is linearly ordered by $\subset^*$ then there is a $C(\mathcal{A}) \in \bar{\mathcal{I}}^+$ such that $C(\mathcal{A}) \subset^* A$ for all $A \in \mathcal{A}$.*

Proof. Let $\{A_n \colon n < \omega\} \subset \mathcal{A}$ be a $\subset^*$-decreasing sequence coinitial with $\mathcal{A}$. For every $i < \omega$ choose $m_i < \omega$ and $C_i \in [P_{m_i}]^i$ such that $C_i \subset \bigcap_{j\leq i} A_j$. Then $C(\mathcal{A}) = \bigcup_{i<\omega} C_i$ is as desired. □

To construct a nonselective P-point we are going to prove the following theorem.

Theorem 4.13. *If* $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *holds then there exists a $\subset^*$-decreasing sequence $\mathcal{B} = \{B_\xi \in \bar{\mathcal{I}}^+ \colon \xi < \omega_1\}$ such that the filter $\mathcal{F}$ generated by $\mathcal{B}$ is an ultrafilter on ω.*

Notice that from this we will immediately deduce the required result.

Corollary 4.14. *If* $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *holds then there exists a nonselective P-point.*

Proof. Let $\mathcal{F}$ be as in Theorem 4.13. Clearly $\mathcal{F}$ is nonselective, since $\mathcal{P}$ is disjoint with $\bar{\mathcal{I}}^+ \supset \mathcal{F}$ and every selector of $\mathcal{P}$ is in $\bar{\mathcal{I}} \subset \mathcal{P}(\omega) \setminus \mathcal{F}$. The fact that $\mathcal{F}$ is a P-point follows from the fact that $\mathcal{F}$ has a base linearly ordered by $\subset^*$. Indeed, if $\{S_n \colon n < \omega\} \subset \mathcal{P}(\omega) \setminus \mathcal{F}$ is a partition of ω then for every $m < \omega$ there is $\xi_m < \omega_1$ such that $B_{\xi_m} \subset^* \omega \setminus \bigcup_{n \leq m} S_n$. Let $\beta < \omega_1$ be such that $B_\beta \subset^* B_{\xi_m}$ for all $m < \omega$. Then $F = B_\beta \in \mathcal{F}$ is such that $|F \cap S_n| < \omega$ for all $n < \omega$. □

The proof of Theorem 4.13 will be based on the following lemma, which is analogous to Lemma 4.9. Note, that although the statement of this lemma is identical to that of Lemma 4.9(b), we cannot apply this lemma here, since the ideal $\bar{\mathcal{I}}$ is not weakly selective.

Lemma 4.15. *Let $\bar{\mathcal{I}}$ be as in* (4.14). *Then for every $A \in \bar{\mathcal{I}}^+$ and a prism P in 2^ω there exist a $B \in \bar{\mathcal{I}}^+$, $B \subset A$, a subprism Q of P, and a $j < 2$ such that*

(∘) *$g \restriction B$ is constant equal to j for every $g \in Q$.*

Proof. Fix an $A \in \bar{\mathcal{I}}^+$ and an $f \in \mathcal{F}_{\text{prism}}(2^\omega)$ from $\mathfrak{C}^\alpha$ onto P. Since $A \in \bar{\mathcal{I}}^+$, for every $k < \omega$ we can find an $m_k < \omega$ such that $|A \cap P_{m_k}| \geq k\, 2^{2^k}$. First we will construct a subprism Q_0 of P and a sequence $\langle A_k \in [A \cap P_{m_k}]^k \colon k < \omega \rangle$ such that for every $k < \omega$

$$g \restriction A_k \text{ is constant for every } g \in Q_0. \tag{4.15}$$

This will be done using Lemmas 4.4 and 4.3. So, for each $k < \omega$ let $\mathcal{D}_k$ be the collection of all pairwise disjoint families $\mathcal{E} \in [\mathbb{P}_\alpha]^{<\omega}$ such that for every $E \in \mathcal{E}$

$$f(h) \restriction P_{m_k} = f(h') \restriction P_{m_k} \text{ for all } h, h' \in E. \tag{4.16}$$

Clearly each $\mathcal{D}_k$ satisfies condition (†) from Lemma 4.4, so by an easy induction we can find a sequence $\langle \mathcal{E}_k \in \mathcal{D}_k \colon k < \omega \rangle$ satisfying the assumptions of Lemma 4.3. Let $E_0 = \bigcap_{k<\omega} \bigcup \mathcal{E}_k \in \mathbb{P}_\alpha$. We will show that $Q_0 = f[E_0]$ satisfies (4.15).

Indeed, fix a $k < \omega$ and notice that $\mathcal{E}_k = \{E_i \colon i < 2^k\}$. For each $i < 2^k$ choose an $h_i \in E_i$ and define $\varphi \colon A \cap P_{m_k} \to 2^{2^k}$ by $\varphi(p)(i) = f(h_i)(p)$. Since $|A \cap P_{m_k}| \geq k\, 2^{2^k}$, by the pigeon hole principle we can find an $s \in 2^{2^k}$ such that $|\varphi^{-1}(s)| \geq k$. Choose an $A_k \in [\varphi^{-1}(s)]^k$. Then for every $i < 2^k$ and $p \in A_k$ we have $f(h_i)(p) = \varphi(p)(i) = s(i)$. So, $f(h_i) \restriction A_k$ is constant equal to $s(i)$. Combining this with the inclusion $A_k \subset P_{m_k}$ and the condition (4.16) we obtain (4.15).

To finish the proof fix a selector $\bar{A}$ from the family $\{A_k \colon k < \omega\}$. Then $\bar{A} \in \mathcal{I}^+$, where $\mathcal{I}$ is the ideal of finite subsets of ω. Applying Lemma 4.9(a) to $\bar{A}$ and Q_0 we can find a $j < 2$, an $S \in [\bar{A}]^\omega$, and a subprism Q of Q_0 such that $g \restriction S$ is constant equal to j for every $g \in Q$. Put $B = \bigcup_{k \in S} A_k$. Then, by (4.15), $g \restriction B$ is constant equal to j for every $g \in Q$.

It is clear that $B \in \bar{\mathcal{I}}^+ \cap \mathcal{P}(A)$, since it is a union of infinitely many sets $A_k \in [A \cap P_{m_k}]^k$. □

Note also that, similarly as for Remark 4.10, the conclusion of the following fact holds in a model obtained by adding many side-by-side Sacks reals. This was first noticed by Hart in [16].

Remark 4.16. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there are ω_2 different nonselective P-points.

The existence of 2^{ω_1} different such ultrafilters follows also from a slightly stronger version of $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$. This can be found in [8].

Recall also that a family $\mathcal{J} \subset [\omega]^\omega$ is an *independent family* provided the set

$$\bigcap_{A \in \mathcal{A}} A \cap \bigcap_{B \in \mathcal{B}} (\omega \setminus B)$$

is infinite for every disjoint finite subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{J}$. It is often convenient to express this definition in a slightly different notation. Thus, for $W \subset \omega$ let $W^0 = W$ and $W^1 = \omega \setminus W$. A family $\mathcal{J} \subset [\omega]^\omega$ is independent provided the set

$$\bigcap_{W \in \mathcal{J}_0} W^{\tau(W)}$$

is infinite for every finite subset $\mathcal{J}_0$ of $\mathcal{J}$ and $\tau \colon \mathcal{J}_0 \to \{0,1\}$.

The *independence* cardinal $\mathfrak{i}$ is defined as follows:

$$\mathfrak{i} = \min\{|\mathcal{J}| \colon \mathcal{J} \text{ is infinite maximal independent family}\}.$$

According to Andreas Blass [3, Section 11.5] the fact that the equation $\mathfrak{i} = \omega_1$ holds in the iterated perfect set model was first proved by Eisworth and Shelah (unpublished).

Theorem 4.17. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that* $\mathfrak{i} = \omega_1$.

The proof of the theorem is based on the following lemma. We say that a family $\mathcal{W} \subset [\omega]^\omega$ *separates points* provided for every $k < \omega$ there are $U, V \in \mathcal{W}$ such that $k \in U \setminus V$.

Lemma 4.18. *For every countable independent family* $\mathcal{W} \subset [\omega]^\omega$ *separating points and a prism* P *in* $[\omega]^\omega$ *there exist* $W \in [\omega]^\omega$ *and a subprism* Q *of* P *such that* $\mathcal{W} \cup \{W\}$ *is independent but* $\mathcal{W} \cup \{W, x\}$ *is not independent for every* $x \in Q$.

Proof. Let $\mathcal{W} = \{W_i : i < \omega\}$ and let $\varphi : \omega \to 2^\omega$ be a Marczewski function for $\mathcal{W}$, that is, for $i, k < \omega$

$$\varphi(k)(i) = \begin{cases} 1 & \text{for } k \in W_i \\ 0 & \text{for } k \notin W_i. \end{cases}$$

Note that φ is one-to-one, since $\mathcal{W}$ separates points. Notice also that for every $k, n < \omega$ and $\tau \in 2^n$

$$k \in \bigcap_{i<n} W_i^{\tau(i)} \Leftrightarrow (\forall i < n)\, k \in W_i^{\tau(i)} \Leftrightarrow (\forall i < n)\, \varphi(k)(i) = \tau(i) \Leftrightarrow \tau \subset \varphi(k).$$

Now, if $[\tau] = \{t \in 2^\omega : \tau \subset t\}$ then sets $\{[\tau] : \tau \in 2^{<\omega}\}$ form a base for 2^ω and

$$k \in \bigcap_{i<n} W_i^{\tau(i)} \Leftrightarrow \varphi(k) \in [\tau]. \tag{4.17}$$

Thus, independence of $\mathcal{W}$ implies that $\varphi[\omega]$ is dense in 2^ω, so it is homeomorphic to the set $\mathbb{Q}$ of rational numbers. Note also that from (4.17) it follows immediately that

(a) if $W \subset \omega$ is such that $\varphi[W]$ and $\varphi[\omega] \setminus \varphi[W]$ are dense then $\mathcal{W} \cup \{W\}$ is independent;
(b) if $W, x \subset \omega$ are such that for some $\tau \in 2^{<\omega}$ either $\varphi[x \cap W] \cap [\tau] = \emptyset$ or $\varphi[W] \cap [\tau] \subset \varphi[x]$ then $\mathcal{W} \cup \{W, x\}$ is not independent.

Let $\mathcal{I}$ be the ideal of nowhere dense subsets of $\varphi[\omega]$. Then, by Fact 4.6, $\mathcal{I}$ is weakly selective, since $\varphi[\omega]$ is homeomorphic to $\mathbb{Q}$. So, identifying $\varphi[\omega]$ with ω and applying Lemma 4.9(b), we can find a subprism Q of P and a $V \in [\omega]^\omega \setminus \mathcal{I}$ such that either

- $x \cap V = \emptyset$ for every $x \in Q$, or else
- $V \subset x$ for every $x \in Q$.

Since $V \notin \mathcal{I}$, there exists a $\tau \in 2^{<\omega}$ such that $\varphi[V]$ is dense in $[\tau]$. Trimming V, if necessary, we can assume that $\varphi[V] \subset [\tau]$ and that $\varphi[\omega \setminus V]$ is also dense in $[\tau]$. Now let $W \supset V$ be such that $\varphi[W] \cap [\tau] = \varphi[V]$ and both $\varphi[W]$ and $\varphi[\omega \setminus W]$ are dense in $\varphi[\omega]$. Then, by (a) and (b), $\mathcal{W} \cup \{W\}$ is independent while $\mathcal{W} \cup \{W, x\}$ is not independent for every $x \in Q$. □

Proof of Theorem 4.17. For a countable independent family $\mathcal{W} \subset [\omega]^\omega$ separating points and an $f \in \mathcal{F}_{\text{prism}}([\omega]^\omega)$ define $P = \text{range}(f)$ and let $W(\mathcal{W}, P) \in [\omega]^\omega$ and a subprism $Q(\mathcal{W}, P)$ of P be as in Lemma 4.18. If $f \in \mathcal{C}_{\text{prism}}([\omega]^\omega)$ and $P = \text{range}(f) = \{x\}$ then we put $Q(\mathcal{W}, P) = P$ and define $W(\mathcal{W}, P)$ as an arbitrary W such that $\mathcal{W} \cup \{W\}$ is independent while $\mathcal{W} \cup \{W, x\}$ is not.

Let $\mathcal{A}_0 \subset [\omega]^\omega$ be an arbitrary countable independent family separating points and consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(\mathcal{A}_0 \cup \{W_\eta \colon \eta < \xi\}, P_\xi),$$

where sets W_η are defined inductively by $W_\eta = W(\mathcal{A}_0 \cup \{W_\zeta \colon \zeta < \eta\}, P_\eta)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{cube}}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S in which and Player II loses, that is, $[\omega]^\omega = \bigcup_{\xi<\omega_1} Q_\xi$.

Now, notice that the family $\mathcal{J} = \mathcal{A}_0 \cup \{W_\xi \colon \xi < \omega_1\}$ is a maximal independent family. It is clear that $\mathcal{J}$ is independent, since every set W_ξ was chosen so that $\mathcal{A}_0 \cup \{W_\zeta \colon \zeta \le \xi\}$ is independent. To see that $\mathcal{J}$ is maximal it is enough to note that every $x \in [\omega]^\omega$ belongs to a Q_ξ for some $\xi < \omega_1$, and so $\mathcal{A}_0 \cup \{W_\zeta \colon \zeta \le \xi\} \cup \{x\}$ is not independent. □

By Theorem 4.17 we see that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies the existence of an independent family of size ω_1. Next, answering a question of Michael Hrušák [17] we show that such a family can be simultaneously a splitting family. This is similar in flavor to Theorem 3.4. In the proof we will use the following lemma.

Lemma 4.19. *For every countable family $\mathcal{V} \subset [\omega]^\omega$ and a perfect set P in $[\omega]^\omega$ there exists a $W_1 \in [\omega]^\omega$ such that $\mathcal{V} \cup \{W_1\}$ is independent and W_1 splits every $A \in P$.*

Proof. We follow the argument from [11, p. 121] that $\mathfrak{s} \le \mathfrak{d}$.

For every $A \in [\omega]^\omega$ let b_A be a strictly increasing bijection from ω onto A. Then $b \colon [\omega]^\omega \to \omega^\omega$ defined by $b(A) = b_A$ is continuous. In particular $b[P] = \{b_A \colon A \in P\}$ is compact, so there exists a strictly increasing $f \in \omega^\omega$ such that $b_A(n) < f(n)$ for every $A \in P$ and $n < \omega$. For $n < \omega$ let f^n denote the n-fold composition of f and let $S_n = \{m < \omega \colon f^n(0) \le m < f^{n+1}(0)\}$. Then $f^n(0) \le b_A(f^n(0)) < f(f^n(0)) = f^{n+1}(0)$ for every $A \in P$ and $n < \omega$. In particular, for every $A \in P$

$$S_n \cap A \neq \emptyset.$$

So, if $T \subset \omega$ be infinite and co-infinite and $W_1 = \bigcup_{n\in T} S_n$ then W_1 splits every $A \in P$. Thus, it is enough to take infinite and co-infinite $T \subset \omega$ such that $\mathcal{V} \cup \{W_1\}$ is independent. □

Theorem 4.20. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *implies that there exists a family $\mathcal{F} \subset [\omega]^\omega$ of cardinality ω_1 which is simultaneously independent and splitting.*

Proof. The proof is just a slight modification of that for Theorem 4.17. (Compare also Theorem 3.4.)

For a countable independent family $\mathcal{W} \subset [\omega]^\omega$ separating points and an $f \in \mathcal{F}_{\rm prism}([\omega]^\omega)$ put $P = \mathrm{range}(f)$ and let $W_0 \in [\omega]^\omega$ and a subprism Q of P be as in Lemma 4.18. Let W_1 be as in Lemma 4.19 used with $P = Q$ and $\mathcal{V} = \mathcal{W} \cup \{W_0\}$. We put $Q(\mathcal{W}, P) = Q_1$ and $\mathcal{W}(\mathcal{W}, P) = \{W_0, W_1\}$.

If $f \in \mathcal{C}_{\rm cube}([\omega]^\omega)$ and $P = \mathrm{range}(f) = \{x\}$ then we put $Q(\mathcal{W}, P) = P$ and $\mathcal{W}(\mathcal{W}, P) = \{W_0, W_1\}$, where W_0 and W_1 are such that $\mathcal{W} \cup \{W_0, W_1\}$ is independent and W_1 splits $P = \{x\}$.

Let $\mathcal{A}_0 \subset [\omega]^\omega$ be an arbitrary countable independent family separating points and consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(\mathcal{A}_0 \cup \bigcup\{\mathcal{W}_\eta \colon \eta < \xi\}, P_\xi),$$

where $\mathcal{W}_\eta$'s are defined inductively by $\mathcal{W}_\eta = \mathcal{W}(\mathcal{A}_0 \cup \bigcup\{\mathcal{W}_\eta \colon \eta < \xi\}, P_\eta)$.

By $\mathrm{CPA}^{\rm game}_{\rm prism}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S in which and Player II loses, that is, $[\omega]^\omega = \bigcup_{\xi<\omega_1} Q_\xi$. Then the family $\mathcal{F} = \mathcal{A}_0 \cup \bigcup\{\mathcal{W}_\xi \colon \xi < \omega_1\}$ is independent and splitting. □

4.5. Crowded ultrafilters on $\mathbb{Q}$.

Let $\mathrm{Perf}(\mathbb{Q})$ stand for the family of all closed subsets A of $\mathbb{Q}$ without isolated points, that is, such that their closures $\mathrm{cl}_{\mathbb{R}}(A)$ in $\mathbb{R}$ are perfect sets. Recall that an ideal $\mathcal{I}$ on $\mathbb{Q}$ of is *crowded* provided $\mathcal{I}^+ = \mathcal{P}(\mathbb{Q}) \setminus \mathcal{I}$ is generated by the sets from $\mathrm{Perf}(\mathbb{Q})$. Crowded ultrafilters were studied by several authors (see e.g. [12, 10]) in connection with the reminder $\beta\,\mathbb{Q} \setminus \mathbb{Q}$ of the Čech-Stone compactification $\beta\,\mathbb{Q}$ of $\mathbb{Q}$.

In what follows we will also use the following simple fact, in which a non-scattered subset of $\mathbb{Q}$ is understood as a set containing a subset dense in itself.

Fact 4.21. *Every non-scattered set $B \subset \mathbb{Q}$ contains a subset from* $\mathrm{Perf}(\mathbb{Q})$.

Proof. Since B is non-scattered, decreasing it, if necessary, we can assume that B is dense in itself. Let $\{k_n \le n \colon n < \omega\}$ be an enumeration of ω with infinitely many repetitions and let $\mathbb{Q} \setminus B = \{a_n \colon n < \omega\}$. By induction construct a sequence $\langle\langle p_n, U_n\rangle \in B \cap \mathcal{P}(\mathbb{Q}) \colon n < \omega\rangle$ such that $p_n \in B \setminus \bigcup_{i<n} U_i$ and $|p_n - p_{k_n}| < 2^{-n}$ while $U_n \ni a_n$ is a clopen subset of $\mathbb{Q} \setminus \{p_i \colon i \le n\}$. Then $\mathbb{Q} \setminus \bigcup_{n<\omega} U_n \subset B$ is as desired. □

The following theorem answers in positive a question of M. Hrušák [17] on whether there exists a crowded ultrafilter in the iterated perfect set model.

Theorem 4.22. $\mathrm{CPA}_{\mathrm{prism}}^{\mathrm{game}}$ *implies there exists a non-principal ultrafilter on* $\mathbb{Q}$ *which is crowded.*

Fix a $p \in \mathbb{R} \setminus \mathbb{Q}$ and for a family $\mathcal{D} \subset \mathcal{P}(\mathbb{Q})$ let $F(\mathcal{D})$ denote a filter on $\mathbb{Q}$ generated by the family $\mathcal{D} \cup \{I_n \cap \mathbb{Q} \colon n < \omega\}$, where $I_n = [p - 2^{-n}, p + 2^{-n}]$. The proof of the theorem is based on the following lemma, in which $[\mathbb{Q}]^\omega$ is considered with the same topology as $[\omega]^\omega$ upon natural identification.

Lemma 4.23. *Let* $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ *be a countable family such that* $F(\mathcal{D})$ *is non-trivial. Then for every prism* P *in* $[\mathbb{Q}]^\omega$ *there exist a subprism* Q *of* P *and a* $Z \in \mathrm{Perf}(\mathbb{Q})$ *such that* $F(\mathcal{D} \cup \{Z\})$ *is non-trivial and either*

(i) $Z \cap x = \emptyset$ *for every* $x \in Q$*, or else*
(ii) $Z \subset x$ *for every* $x \in Q$*.*

Proof. In what follows we will identify $[\mathbb{Q}]^\omega$ with $2^{\mathbb{Q}}$, the identification mapping given by the characteristic function. Thus, we will consider P as a prism in $2^{\mathbb{Q}}$. Fix an $f \in \mathcal{F}_{\mathrm{prism}}(2^{\mathbb{Q}})$ from $\mathfrak{C}^\alpha$ onto P.

Let $\{D_n \in \mathrm{Perf}(\mathbb{Q}) \colon n < \omega\}$ be a cofinal sequence in $F(\mathcal{D})$ with a property that $D_{n+1} \subset D_n \subset I_n$ for every $n < \omega$. Choosing a subsequence, if necessary, we can find disjoint intervals J_n such that $K_n = D_n \cap J_n \in \mathrm{Perf}(\mathbb{Q})$.

First we will show that there exist a sequence $\langle B_n \subset K_n \colon n < \omega\rangle$ of non-scattered sets and a subprism P_0 of P such that

$$g \restriction B_n \text{ is constant for every } g \in P_0 \text{ and } n < \omega. \tag{4.18}$$

For each $n < \omega$ let $\mathcal{D}_n$ be the collection of all pairwise disjoint families $\mathcal{E} \in [\mathbb{P}_\alpha]^{<\omega}$ for which there exists a non-scattered set $B_n \subset K_n$ with the property that

$$g \restriction B_n \text{ is constant for every } g \in f\left[\bigcup \mathcal{E}\right].$$

To see that the families $\mathcal{D}_n$ satisfy condition (†) from Lemma 4.4 it is enough to notice that for every non-scattered set $B \subset \mathbb{Q}$ and every prism P_1 there is a subprism Q_1 of P_1 and a non-scattered subset B' of B such that $g \restriction B'$ is constant for every $g \in Q_1$. But B contains a subset W homeomorphic to $\mathbb{Q}$. So, by Fact 4.6, the ideal $\mathcal{I}$ of nowhere dense subsets of W is weakly selective. So, applying Lemma 4.9 to this ideal and the prism P_1 we can find a $B' \in \mathcal{I}^+$, which clearly is not scattered, and a Q_1 as desired.

Thus, using Lemma 4.4, we can find a sequence $\langle \mathcal{E}_n \in \mathcal{D}_n \colon n < \omega\rangle$ satisfying the assumptions of Lemma 4.3. Let $E_0 = \bigcap_{n<\omega} \bigcup \mathcal{E}_n \in \mathbb{P}_\alpha$. It is easy to see that the sets B_n witnessing $\mathcal{E}_n \in \mathcal{D}_n$ and $Q_0 = f[E_0]$ satisfy (4.18). Notice also that by Fact 4.21 we can assume that $B_n \in \mathrm{Perf}(\mathbb{Q})$ for every $n < \omega$.

Now let A be a selector from the family $\{B_n \colon n < \omega\}$. Then $A \in \mathcal{I}^+$, where $\mathcal{I}$ is the ideal of finite subsets of $\mathbb{Q}$. Applying Lemma 4.9(a) to A and

P_0 we can find $i < 2$, $S \in [A]^\omega$, and a subprism Q of P_0 such that $g \restriction S$ is constant equal to i for every $g \in Q$. Put $Z = \bigcup_{n\in S} B_n$. Then, by (4.18), $g \restriction Z$ is constant for every $g \in Q$. Finally, note that $Z \in \mathrm{Perf}(\mathbb{Q})$ since Z is closed, as $B_n \to p \notin \mathbb{Q}$. □

Proof of Theorem 4.22. For a prism P in $[\mathbb{Q}]^\omega$ and a countable family $\mathcal{D} \subset \mathrm{Perf}(\mathbb{Q})$ for which $F(\mathcal{D})$ is non-trivial let $Z(\mathcal{D}, P) \in \mathrm{Perf}(\mathbb{Q})$ and a subprism $Q(\mathcal{D}, P)$ of P be as in Lemma 4.23. Consider the following strategy S for Player II:

$$S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P_\xi) = Q(\{Z_\eta \colon \eta < \xi\}, P_\xi),$$

where sets Z_η are defined inductively by $Z_\eta = Z(\{Z_\zeta \colon \zeta < \eta\}, P_\eta)$.

By $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ strategy S is not a winning strategy for Player II. So there exists a game $\langle\langle P_\xi, Q_\xi\rangle \colon \xi < \omega_1\rangle$ played according to S in which Player II loses, that is, $[\mathbb{Q}]^\omega = \bigcup_{\xi<\omega_1} Q_\xi$.

Now, let $\mathcal{F} = F(\{Z_\xi \colon \xi < \omega_1\})$. Then clearly $\mathcal{F}$ is a crowded non-principal filter. To see that it is maximal, take an $x \in [\mathbb{Q}]^\omega$. Then there is a $\xi < \omega_1$ such that $x \in Q_\xi$. Then either $Z_\xi \cap x = \emptyset$ or $Z_\xi \subset x$. Thus, either x or its complement belong to $\mathcal{F}$. □

Note also that similarly as for Remarks 4.10 and 4.16 we can argue that there are many non-principal crowded ultrafilters.

Remark 4.24. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies that there are ω_2-many different crowded non-principal ultrafilters.

The existence of 2^{ω_1} different such ultrafilters follows also from a slightly stronger version of $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$. This can be found in [8].

The construction of crowded ultrafilters is quite similar to that of selective ultrafilters and of nonselective P-points. This similarity suggests that it may be possible to construct a crowded ultrafilter which is also selective. This, however, cannot be done:

Proposition 4.25. *There is no non-principal crowded ultrafilter on $\mathbb{Q}$ which is also a P-point.*

Proof. Let $\mathcal{F}$ be a non-principal crowded ultrafilter on $\mathbb{Q}$ and let $\{x\} = \bigcap_{F\in\mathcal{F}} \mathrm{cl}_{\mathbb{R}}(F) \in \mathbb{R} \setminus \mathbb{Q}$. Then $I_n = (x - 2^{-n}, x + 2^{-n}) \cap \mathbb{Q}$ belongs to $\mathcal{F}$ for every $n < \omega$. Let $\mathcal{P} = \{I_n \setminus I_{n+1} \colon n < \omega\} \cup \{\mathbb{Q} \setminus I_0\}$. Then $\mathcal{P}$ is a partition of ω disjoint with $\mathcal{F}$. It is also easy to see that if $F \subset \omega$ is such that $|F \cap P| < \omega$ for every $P \in \mathcal{P}$ then $F \notin \mathcal{F}$. □

It is also not difficult to show that no non-principal crowded ultrafilter on $\mathbb{Q}$ can be a Q-point.

It is worth mentioning that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ implies also the existence of many other kinds of ultrafilters, like those constructed in [16]. In fact, many constructions that are done under CH can be carried out also under $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$. However, this always needs some combinatorial lemma, such as Lemma 4.9, which allows replacing points with prisms.

5. Proof of fusion Lemma 4.4 and of consistency of $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$

Proof of Lemma 4.4. For $s \in 2^{A_k}$ and $j < 2$ let $s\hat{\ }j$ stand for the function $s \cup \{\langle\langle\beta_k, n_k\rangle, j\rangle\} \in 2^{A_{k+1}}$.

Let $\{s_i \colon i < 2^{k+1}\}$ be an enumeration of $2^{A_{k+1}}$. By induction on $i < 2^{k+1}$ we will construct a sequence $\langle x_{s_i} \in \mathfrak{C}^\alpha \colon i < 2^{k+1}\rangle$ such that for every $i < 2^{k+1}$

(a) if $s_i = s\hat{\ }j$, where $s \in 2^{A_k}$ and $j < 2$, then $x_{s_i} \in E_s$,
(b) if $m < i$ and $\beta = \max\{\bar\beta \colon s_i \restriction (\bar\beta \times \omega) = s_m \restriction (\bar\beta \times \omega)\}$ then

$$x_{s_i} \restriction \beta = x_{s_m} \restriction \beta \text{ and } x_{s_i}(\beta) \neq x_{s_m}(\beta).$$

The point x_{s_0} is chosen arbitrarily from $E_{s_0 \restriction A_k}$. To make an inductive step, if for some $0 < i \le 2^{k+1}$ points $\{x_{s_m} \colon m < i\}$ are already constructed choose an $\bar m < i$ for which β as in (b) is maximal. Notice that by the inductive assumption and the condition (ag) we have $x_{s_{\bar m}} \restriction \beta \in \pi_\beta[E_{s_{\bar m} \restriction A_k}] = \pi_\beta[E_{s_i \restriction A_k}]$. So we can choose an $x_{s_i} \in E_{s_i \restriction A_k}$ extending $x_{s_{\bar m}} \restriction \beta$ and such that $x_{s_i}(\beta) \neq x_{s_m}(\beta)$ for all $m < i$. It is easy to see that such x_{s_i} satisfies (a) and the condition (b) for $m = \bar m$. For other $m < i$ condition (b) follows from the maximality of β and the assumption that $\mathcal{E}_k$ satisfies (ag) and (sp).

Conditions (a) and (b) imply that $\mathcal{E}'_{k+1} = \left\{\{x_s\} \colon s \in 2^{A_{k+1}}\right\}$ satisfy (A) except for being a subset of $\mathbb{P}_\alpha$. Let $\varepsilon \in (0, 2^{-(k+1)}]$ be small enough that for every $m < i < 2^{k+1}$ and β as in (b) we have $\pi_{\beta+1}[B_\alpha(x_{s_i}, \varepsilon)] \cap \pi_{\beta+1}[B_\alpha(x_{s_m}, \varepsilon)] = \emptyset$. For $s \in 2^{A_k}$ and $j < 2$ define

$$E_{s\hat{\ }j} = E_s \cap B_\alpha(x_{s\hat{\ }j}, \varepsilon).$$

Then $\mathcal{E}_{k+1} = \left\{E_s \colon s \in 2^{A_{k+1}}\right\}$ is a subset of $\mathbb{P}_\alpha$ by (4.9). Conditions (i) and (ii) are clear from the construction, while (ag) for $\mathcal{E}_{k+1}$ follows from (b) and (4.7). Property (sp) holds by (b) and the choice of ε, since (sp) was true for $\mathcal{E}'_{k+1}$. We have completed the proof of (A).

To prove condition (B), fix an enumeration $\{s_i \colon i < 2^k\}$ of 2^{A_k} and define $\gamma = \max\{\beta_0, \ldots, \beta_k\} < \alpha$. Also for $i, m < 2^k$ put $E^{-1}_{s_i} = E_{s_i}$ and

$$\beta^m_i = \max\{\beta \le \gamma \colon s_i \restriction (\beta \times \omega) = s_m \restriction (\beta \times \omega)\}.$$

By induction we will construct the sequences $\langle\{E^m_{s_i}\in\mathbb{P}_\alpha\colon i<2^k\}\colon m<2^k\rangle$ and $\langle P_m\in\mathbb{P}_\alpha\colon m<2^k\rangle$ such that for every $j,m<2^k$

(a) $\mathcal{E}^m=\{E^m_{s_i}\colon i<2^k\}$ satisfies (ag),

(b) $E^m_{s_j}\subset E^{m-1}_{s_j}$ and if $x\in E^{m-1}_{s_j}$ and $\pi_\gamma(x)\in\pi_\gamma[E^m_{s_j}]$ then $x\in E^m_{s_j}$,

(c) $\pi_\gamma[P_m]=\pi_\gamma[E^m_{s_m}]$,

(d) $P_m\subset E^{m-1}_{s_m}$ and $\{P_i\colon i\le m\}\in\mathcal{D}$.

So, assume that for some $m<2^k$ the sequence $\langle P_i\colon i<m\rangle$ and the family $\mathcal{E}^{m-1}$ satisfying (ag) are already constructed. Notice that, by (b), sets in $\mathcal{E}^{m-1}$ are pairwise disjoint, since this was the case for $\mathcal{E}^{-1}=\mathcal{E}_k$. So, by condition (†) applied to $\mathcal{E}=\{P_i\colon i<m\}$, we can choose $P_m\in\mathbb{P}_\alpha\cap\mathcal{P}(E^{m-1}_{s_m})$ such that $\{P_m\}\cup\{P_i\colon i<m\}\in\mathcal{D}$. This guarantees (d).

Next, for $i<2^k$ define

$$E^m_{s_i}=E^{m-1}_{s_i}\cap\pi^{-1}_{\beta^m_i}(\pi_{\beta^m_i}[P_m])=\left\{x\in E^{m-1}_{s_i}\colon x\restriction\beta^m_i\in\pi_{\beta^m_i}[P_m]\right\}$$

and notice that $\pi_{\beta^m_i}[P_m]\subset\pi_{\beta^m_i}[E^{m-1}_{s_m}]=\pi_{\beta^m_i}[E^{m-1}_{s_i}]$. So, by (4.11), $E^m_{s_i}\in\mathbb{P}_\alpha$. Also, the definition ensures (b) since $\beta^m_i\le\gamma$.

Note that, by the inductive assumption (a), for all $i<2^k$ we have

$$\pi_{\beta^m_i}[E^m_{s_i}]=\pi_{\beta^m_i}[E^{m-1}_{s_i}]\cap\pi_{\beta^m_i}[P_m]=\pi_{\beta^m_i}[E^{m-1}_{s_m}]\cap\pi_{\beta^m_i}[P_m]=\pi_{\beta^m_i}[P_m].$$

Since $\beta^m_m=\gamma$, this implies (c). To prove (a) pick $\beta<\alpha$ and different $i,j<2^k$ such that $s_i\restriction(\beta\times\omega)=s_j\restriction(\beta\times\omega)$. If $\beta\le\beta^m_i$ then also $\beta\le\beta^m_j$ and $\pi_\beta[E^m_{s_i}]=\pi_\beta[P_m]=\pi_\beta[E^m_{s_j}]$. So, assume that $\beta>\beta^m_i$ and $\beta>\beta^m_j$. Then $\beta^m_i=\beta^m_j$ and

$$\begin{aligned}\pi_\beta[E^m_{s_i}] &= \left\{\pi_\beta(x)\colon x\in E^{m-1}_{s_i}\ \&\ x\restriction\beta^m_i\in\pi_{\beta^m_i}[P_m]\right\}\\ &= \left\{\pi_\beta(x)\colon x\in E^{m-1}_{s_j}\ \&\ x\restriction\beta^m_j\in\pi_{\beta^m_j}[P_m]\right\}\\ &= \pi_\beta[E^m_{s_j}].\end{aligned}$$

So $\mathcal{E}^m$ satisfies (a). This finishes the construction.

Notice that by the maximality of γ and the properties (a) and (c) the family $\mathcal{E}'_k=\{P_m\colon m<2^k\}$ satisfies (ag). Since it is a refinement of $\mathcal{E}_k$ it also satisfies (sp). So (B) is proved.

To find $\mathcal{E}_{k+1}$ as in (C) first take an $\mathcal{E}'_{k+1}$ satisfying (A) and then use (B) to find its refinement $\mathcal{E}'_{k+1}\in\mathcal{D}$ satisfying (ag) and (sp). □

In what follows we will show that $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ holds in the generic extension $V[G]$ of a model V of ZFC+CH, where G is a V-generic filter over $\mathbb{S}_{\omega_2}$, the ω_2 countable support iteration of Sacks forcing. We will use here terminology from [1].

Let $\mathbb{P}=\langle\mathrm{Perf}(\mathfrak{C}),\subset\rangle$. Recall that perfect set (Sacks) forcing $\mathbb{S}$ is usually defined as the set of all trees $T(P)=\{x\restriction n\in 2^{<\omega}\colon x\in P\ \&\ n<\omega\}$, where

$P \in \mathbb{P}$, and is ordered by inclusion, that is, $s \in \mathbb{S}$ is stronger than $t \in \mathbb{S}$, $s \le t$, if $s \subset t$. It is important to realize that

$$P \subset Q \quad \text{if and only if} \quad T(P) \subset T(Q),$$

so $T\colon \mathbb{P} \to \mathbb{S}$ establishes isomorphism between forcings $\mathbb{P}$ and $\mathbb{S}$. Also if for $s \in \mathbb{S}$ we define $\lim(s) = \{x \in 2^\omega\colon \forall n < \omega\ (x \restriction n \in s)\}$ then $\lim\colon \mathbb{S} \to \mathbb{P}$ is the inverse of T.

Perfect set forcing is usually represented as $\mathbb{S}$ rather that in its more natural form $\mathbb{P}$ since the conditions in $\mathbb{S}$ are absolute, unlike those in $\mathbb{P}$. However, in light of our axiom, it is important to think of this forcing in terms of $\mathbb{P}$.

Recall also that for a countable $A \subset \omega_2$ we defined $\Phi_{\text{prism}}(A)$ as the family of all projection-keeping homeomorphisms $f\colon \mathfrak{C}^A \to \mathfrak{C}^A$. For $A \subset \alpha \le \omega_2$ define

$$\mathbb{P}^\alpha_A = \{[\text{range}(f)]_\alpha\colon f \in \Phi_{\text{prism}}(A)\},$$

where $[E]_\alpha = \{g \in \mathfrak{C}^\alpha\colon g \restriction A \in E\}$ for every $E \subset \mathfrak{C}^A$. Also, we put

$$\mathbb{P}_\alpha = \bigcup \left\{\mathbb{P}^\alpha_A\colon A \in [\alpha]^{\le\omega}\right\}$$

and order it by inclusion. (Thus, for a countable α we have two different definitions of $\mathbb{P}_\alpha$. However, it is not difficult to see that they describe the same family.)

It is known that forcing $\mathbb{P}_\alpha$ is equivalent to $\mathbb{S}_\alpha$, a countable support iteration of $\mathbb{S}$ of length α. This fact is stated explicitly by Kanovei in [19], though it was also used, in less explicit form, in earlier papers of Miller [23] and Steprāns [25]. More precisely, in [23] and [25] the authors consider the family $\mathbb{S}^D_\alpha \subset \mathbb{P}_\alpha$ of *determined conditions* in $\mathbb{P}_\alpha$, which form a dense subset of $\mathbb{P}_\alpha$, and notice that $\mathbb{S}^D_\alpha$ is equivalent to $\mathbb{S}_\alpha$. This fact is most precisely described by the following fact, whose explicit proof can be found in [8].

Fact 5.1. *$\mathbb{S}_\alpha$ is order isomorphic to $\mathbb{S}^D_\alpha$ for every $0 < \alpha \le \omega_2$. In particular, forcings $\mathbb{S}_\alpha$ and $\mathbb{P}_\alpha$ are equivalent.*

In the proof of the consistency of $\text{CPA}^{\text{game}}_{\text{prism}}$ we will use the following proposition, which is of interest by its own. In its statement the symbol $\text{CPA}^{\text{game}}_{\text{prism}}[X]$ stands for $\text{CPA}^{\text{game}}_{\text{prism}}$ for a fixed Polish space X.

Proposition 5.2. *For any Polish space X axiom $\text{CPA}^{\text{game}}_{\text{prism}}[X]$ implies the full axiom $\text{CPA}^{\text{game}}_{\text{prism}}$.*

Proof. Let X be a Polish space. First notice the following two facts.

(F1) If Y is a Polish subspace of X then $\text{CPA}^{\text{game}}_{\text{prism}}[X]$ implies $\text{CPA}^{\text{game}}_{\text{prism}}[Y]$.

To see it, by way of contradiction assume that Player II has a winning strategy S in $\mathrm{GAME}_{\mathrm{prism}}(Y)$. For each prism P in X let Q_P be its subprism such that either $Q_P \cap Y = \emptyset$ or $Q_P \subset Y$. Such a subprism can be found by Claim 2.3 since Y is a G_δ subset of X. Define a strategy $\bar{S}$ for Player II in the game $\mathrm{GAME}_{\mathrm{prism}}(X)$ by putting

$$\bar{S}(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P) = S(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi \ \&\ Q_{P_\eta} \subset Y\rangle, Q_P)$$

provided $Q_P \subset Y$, and $\bar{S}(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P) = Q_P$ otherwise. It is easy to see that $\bar{S}$ is a winning strategy for Player II in $\mathrm{GAME}_{\mathrm{prism}}(X)$, contradicting $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[X]$. So (F1) is proved.

(F2) If a Polish space Y is a $1-1$ continuous image of X then $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[X]$ implies $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[Y]$.

Indeed, let f be a continuous bijection from X onto Y and by way of contradiction assume that Player II has a winning strategy S in $\mathrm{GAME}_{\mathrm{prism}}(Y)$. Define a strategy $\bar{S}$ for Player II in $\mathrm{GAME}_{\mathrm{prism}}(X)$ by putting

$$\bar{S}(\langle\langle P_\eta, Q_\eta\rangle \colon \eta < \xi\rangle, P) = \bar{S}(\langle\langle f[P_\eta], f[Q_\eta]\rangle \colon \eta < \xi\rangle, f[P]).$$

(Here if h is a coordinate function for a prism P then prism $f[P]$ is considered with a coordinate system $h \circ f$.) It is easy to see that $\bar{S}$ is a winning strategy for Player II in $\mathrm{GAME}_{\mathrm{prism}}(X)$, contradicting $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[X]$. So (F2) is proved.

To finish the proof take a Polish space X for which $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[X]$ holds and recall that the Baire space ω^ω is homeomorphic to a subspace of X (since X contains a copy of $\mathfrak{C}$ and $\mathfrak{C}$ contains a copy of ω^ω). Thus, by (F1), $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[Z]$ holds for an arbitrary Polish subspace Z of ω^ω. Now, if Y is an arbitrary Polish space then there exists a closed subset F of ω^ω such that Y is a one-to-one continuous image of F. (See e.g. [20, Theorem 7.9].) So, by (F2), $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[Y]$ holds as well. □

Theorem 5.3. $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ *holds in the iterated perfect set model. In particular, it is consistent with ZFC set theory.*

Proof. Start with a model V of ZFC+CH and let $V[G]$ be a generic extension of V with respect to forcing $\mathbb{P}_{\omega_2}$. By Fact 5.1 forcing $\mathbb{P}_{\omega_2}$ is equivalent to $\mathbb{S}_{\omega_2}$, so it preserves cardinals and $\mathfrak{c} = \omega_2$ in $V[G]$. For $\alpha \leq \omega_2$ let $G_\alpha = G \restriction \alpha$. Then G_α is V-generic over $\mathbb{P}_\alpha$. By Proposition 5.2 it is enough to prove only $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}[X]$ for $X = \mathfrak{C}$.

Let $\{c_\alpha \colon \alpha < \omega_2\}$ be an enumeration, in $V[G]$, of $\mathfrak{C}$ such that for every α-th element $\omega_1\,\alpha$ of Γ, $\alpha > 0$, we have:

- $\{c_\xi \colon \xi < \omega_1\,\alpha\} = \mathfrak{C} \cap V[G_\alpha]$;
- $c_{\omega_1\,\alpha}$ is the Sacks generic real in $V[G_{\alpha+1}]$ over $V[G_\alpha]$.

We will show that this sequence satisfies $\mathrm{CPA}^{\mathrm{game}}_{\mathrm{prism}}$ in $V[G]$.

So let S be a strategy for Player II. Thus, S is a function from a subset of $D = \bigcup_{\xi<\omega_1} \left(\mathcal{F}^*_{\mathrm{prism}}(X) \times \mathcal{F}^*_{\mathrm{prism}}(X)\right)^{\xi} \times \mathcal{F}^*_{\mathrm{prism}}(X)$ into $\mathcal{F}^*_{\mathrm{prism}}(X)$. (Here each prism is considered with its explicit coordinate function from $\mathcal{F}^*_{\mathrm{prism}}(X)$.) Since $\mathbb{P}_{\omega_2}$ is ω_2-cc and satisfies axiom A, there is an $\alpha \in \Gamma$ such that $\omega_1\,\alpha = \alpha$ and[4]

$$S \cap V[G_\alpha] = S \cap [(D \times \mathcal{F}^*_{\mathrm{prism}}(X)) \cap V[G_\alpha]] \in V[G_\alpha]. \tag{5.1}$$

Since the quotient forcing $\mathbb{P}_{\omega_2}/\mathbb{P}_\alpha$ is equivalent to $\mathbb{P}_{\omega_2}$ we can assume that $\alpha = 0$, that is, that $V[G_\alpha]$ is our ground model V.

Let $\langle\langle f_\xi, g_\xi\rangle \colon \xi < \omega_1\rangle$ be a game played according to the strategy S in which Player I plays in such a way that $\{f_\xi \colon \xi < \omega_1\} = \mathcal{F}^*_{\mathrm{prism}}(X) \cap V$. Then $\mathcal{G} = \{g_\xi \colon \xi < \omega_1\} \in V$ is $\mathcal{F}^*_{\mathrm{prism}}$-dense. It is enough to show that

$$X \setminus V \subset \bigcup_{g_\xi \in \mathcal{F}_{\mathrm{prism}}} \mathrm{range}(g_\xi).$$

So, take an $r \in X \setminus V$. Then there exists a $\mathbb{P}_{\omega_2}$-name τ for r such that

$$\mathbb{P}_{\omega_2} \Vdash \tau \in X \setminus V.$$

We can also choose τ such that it is a $\mathbb{P}(A)$-name for some $A \in [\omega_2]^{\le\omega}$ with $0 \in A$. Then all the information on r is coded by $G_A = G \restriction A$. Therefore $r \in V[\{c_{\omega_1\,\xi} \colon \xi \in A\}]$. Assume that A has an order type α. Clearly $\alpha < \omega_1$ and $\mathbb{P}(A)$ is isomorphic to $\mathbb{P}_\alpha$. Applying this isomorphism we can assume that τ is a $\mathbb{P}_\alpha$-name for r and $\mathbb{P}_\alpha \Vdash \tau \in X \setminus V$. Picking the smallest α with this property, we can also assume that for every $\beta < \alpha$ we have

$$\mathbb{P}_\alpha \Vdash \tau \in X \setminus V[G_\beta].$$

Now, for any such a name τ and any $R \in \mathbb{P}_\alpha$ there exist $P \in \mathbb{P}_\alpha$, $P \subset R$, and a continuous injection function $f \colon P \to X$ (so $f \in \mathcal{F}_{\mathrm{prism}}(X) \cap V$) which "reads τ continuously" in the sense that

$$Q \Vdash \tau \in f[Q] \tag{5.2}$$

for every $Q \subset P$, $Q \in \mathbb{P}_\alpha$. (See [25, Lemma 3.1] or [23, Lemma 6, p. 580]. This also can be deduced from [8, Lemma 3.2.2].) So, the set

$$D = \{Q \in \mathbb{P}_\alpha \colon (\exists \xi < \omega_1)\ \ Q = \mathrm{dom}(g_\xi)\ \&\ Q \Vdash \tau \in g_\xi[Q]\} \in V$$

is dense in $\mathbb{P}_\alpha$. (For $R \in \mathbb{P}_\alpha$ take f as in (5.2), find $\xi < \omega_1$ with $f = f_\xi$, and notice that $Q = \mathrm{dom}(g_\xi)$ justifies the density of D.)

[4]Formally no $f \in \mathcal{F}^*_{\mathrm{prism}}(X) \cap V[G_{\omega_2}]$ belongs to $V[G_\alpha]$ with $\alpha < \omega_2$. However in this proof the expression "$f \in \mathcal{F}^*_{\mathrm{prism}}(X) \cap V[G_\alpha]$" will be understood as "$\mathrm{Code}(f)$ belongs to $V[G_\alpha]$", where for $\mathrm{dom}(f) = P \subset \mathfrak{C}^\alpha$ with $P = \mathrm{range}(g)$, $g \in \Phi_{\mathrm{prism}}(\alpha)$, and $D_\alpha \in V$ being a fixed countable dense subset of $\mathfrak{C}^\alpha$ we define $\mathrm{Code}(f) = f \restriction g[D_\alpha]$.

Take $Q \in D \cap G_{\omega_1}$ and $\xi < \omega_1$ such that $Q = \mathrm{dom}(g_\xi)$. Then there is a $z \in Q$ such that $g_\xi(z) = r$. This finishes the proof. □

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