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On the cofinalities of Boolean algebras and the ideal of null sets

Krzysztof Ciesielski and Janusz Pawlikowski

Abstract. We will show that if the cofinality of the ideal of Lebesgue measure zero sets is equal to ω_1 then there exists a Boolean algebra B of cardinality ω_1 which is not a union of strictly increasing ω-sequence of its subalgebras. This generalizes a result of Just and Koszmider who showed that it is consistent with ZFC+$\neg$CH that such an algebra exists.

1. Preliminaries

For an infinite Boolean algebra B its *cofinality* $\operatorname{cof}(B)$ is defined as the least infinite cardinal number κ such that B is a union of strictly increasing sequence of type κ of subalgebras of B; its *homomorphism type* $h(B)$ is the least cardinality of an infinite homomorphic image of B.

In [6] Koppelberg proved that

(a) $\omega \le \operatorname{cof}(B) \le h(B) \le \mathfrak{c}$, and
(b) if Martin's Axiom holds then $\operatorname{cof}(B) = \omega$ for every Boolean algebra B with $|B| < \mathfrak{c}$; in particular $h(B) \in \{\omega, \mathfrak{c}\}$ for every Boolean algebra B.

(See also [5] and [4].)

In [5] Just and Koszmider examined a question whether in (b) the assumption of Martin's Axiom is important. They gave a positive answer to it by showing that there exists a model of ZFC (obtained by adding Sacks reals side-by-side) in which there is a Boolean algebra B such that $|B| = \operatorname{cof}(B) = \omega_1 < \mathfrak{c}$. Clearly for this algebra we have also $h(B) = \omega_1 \notin \{\omega, \mathfrak{c}\}$ since $\operatorname{cof}(B) \le h(B) \le |B|$.

The goal of this paper is to prove the following theorem, in which $\mathcal{N}$ denotes the ideal of Lebesgue measure subset of $\mathbb{R}$ and $\operatorname{cof}(\mathcal{N})$ its cofinality, that is,

$$\operatorname{cof}(\mathcal{N}) = \min\{|\mathcal{B}| : \mathcal{B} \subset \mathcal{N} \text{ generates } \mathcal{N}\}.$$

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Theorem 1. $\mathrm{cof}(\mathcal{N}) = \omega_1$ *implies that there exists a Boolean algebra* B *of cardinality* ω_1 *such that* $\mathrm{cof}(B) = \omega_1$.

This generalizes the result of Just and Koszmider since in the model they worked with $\mathrm{cof}(\mathcal{N}) = \omega_1$ holds. However, $\mathrm{cof}(\mathcal{N}) = \omega_1$ holds in many other models as well. (For example, it follows from the Covering Property Axiom CPA as shown by the authors in [3].) Moreover, the argument presented here is considerably simpler than the one from [5].

Our set theoretic terminology is standard and follows that of [2]. In particular, $|X|$ stands for the cardinality of a set X and $\mathfrak{c} = |\mathbb{R}|$. In what follows we will use the following characterization of $\mathrm{cof}(\mathcal{N})$, in which $\mathcal{C}_H$ stands for the family of all subsets $\prod_{n<\omega} T_n$ of ω^ω such that $T_n \in [\omega]^{\leq n+1}$ for all $n < \omega$.

Proposition 2. (Bartoszyński [1, Thm. 2.3.9])

$$\mathrm{cof}(\mathcal{N}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{C}_H \ \& \ \bigcup \mathcal{F} = \omega^\omega\right\}.$$

2. The proof

The proof of Theorem 1 will be based on the following lemma.

Lemma 3. *If* $\mathrm{cof}(\mathcal{N}) = \omega_1$ *then for every infinite countable Boolean algebra* $\mathcal{A}$ *there exists a family* $\{a_n^\xi \in \mathcal{A} : n < \omega \ \& \ \xi < \omega_1\}$ *with the following properties.*

(i) $a_n^\xi \wedge a_m^\xi = \mathbf{0}$ *for every* $n < m < \omega$ *and* $\xi < \omega_1$.

(ii) *For every increasing sequence* $\langle \mathcal{A}_n : n < \omega \rangle$ *of proper subalgebras of* $\mathcal{A}$ *with* $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n$ *there exists a* $\xi < \omega_1$ *such that* $a_n^\xi \notin \mathcal{A}_n$ *for all* $n < \omega$.

Proof. In the argument that follows every sequence $\bar{\mathcal{A}} = \langle \mathcal{A}_n : n < \omega \rangle$ as in (ii) will be identified with a function $f_{\bar{\mathcal{A}}} = f \in \omega^{\mathcal{A}}$ for which $f^{-1}(n) = \mathcal{A}_n \setminus \bigcup_{i<n} \mathcal{A}_i$. We will denote the set of all such functions by X. Also, let $\{b_n : n < \omega\}$ be an enumeration of $\mathcal{A}$ and for each $n < \omega$ let B_n be a finite algebra generated by $\{b_i : i < n\}$. Thus, $\mathcal{A} = \bigcup_{i<\omega} B_i$.

Since $\mathrm{cof}(\mathcal{N}) = \omega_1$, the dominating number

$$\mathfrak{d} = \min\{|\mathcal{K}| : \mathcal{K} \subset \omega^\omega \ \& \ (\forall f \in \omega^\omega)(\exists g \in \mathcal{K})(\forall n < \omega)\ f(n) < g(n)\}$$

is equal to ω_1. (See e.g. [1].) So, there exists a dominating family $\mathcal{K} \subset \omega^\omega$ of cardinality ω_1. We can also assume that the sequences in $\mathcal{K}$ are strictly increasing and that for every $g \in \mathcal{K}$ function $\bar{g}$ defined by $\bar{g}(n) = \sum_{i\leq n} g(i)$ also belongs to $\mathcal{K}$.

Next note that for every $f \in X$ there are sequences $\bar{d} = \langle d_k : k < \omega \rangle \in \mathcal{K}$ and $\bar{r} = \langle r_k : k < \omega \rangle \in \mathcal{K}$ such that for every $k < \omega$

(a) $f(b) < r_k$ for all $b \in B_k$; and

(b) there are disjoint $b_0, \ldots, b_{2k} \in B_{d_k}$ with $r_{d_{k-1}} < f(b_0) < \cdots < f(b_{2k})$.

Indeed, the existence of $\bar{r}$ satisfying (a) follows directly from the definition of a dominating family. Moreover, since all algebras $\mathcal{A}_n = f^{-1}(\{0,\dots,n\})$ are proper, for every number $d<\omega$ there exist disjoint $b_0,\dots,b_{2d}\in\mathcal{A}$ such that $r_d < f(b_0) < \cdots < f(b_{2d})$. Let $h\in\omega^\omega$ be such that $b_0,\dots,b_{2d}\in B_{d+h(d)}$ for every $d<\omega$ and let $g\in\mathcal{K}$ be a function dominating h. Then $\bar{d}=\bar{g}$ is as required.

The above implies, in particular, that for every $f\in X$ there are $\bar{d},\bar{r}\in\mathcal{K}$ such that f satisfies (b) and the sequence $f^{\bar{r}} = \left\langle f\restriction (B_{d_k}\setminus B_{d_{k-1}}) : k<\omega\right\rangle$ belongs to

$$X(\bar{d},\bar{r}) = \prod_{k<\omega} (r_{d_k})^{B_{d_k}\setminus B_{d_{k-1}}}.$$

Now, since $\operatorname{cof}(\mathcal{N})=\omega_1$, by Proposition 2 (applied to $\prod_{k<\omega}\omega^{B_{d_k}\setminus B_{d_{k-1}}}$ in place of ω^ω) we can find an ω_1-covering of $X(\bar{d},\bar{r})$ by sets T of the form $\prod_{k<\omega}T_k$, where $T_k\in\left[\omega^{B_{d_k}\setminus B_{d_{k-1}}}\right]^{\le k+1}$ for all $k<\omega$. Since the total number of these sets T (for different $\bar{d},\bar{r}\in\mathcal{K}$) is equal to ω_1, to finish the proof it is enough to show that for any such T there is one sequence $\langle a_n : n<\omega\rangle$ satisfying (i) and such that (ii) holds for every for every $\bar{\mathcal{A}}=\langle\mathcal{A}_n : n<\omega\rangle$ for which $f^{\bar{r}}_{\bar{\mathcal{A}}}$ belongs to T and $f_{\bar{\mathcal{A}}}$ satisfies (b).

So, let T be as above and let T^* be the set of all functions $f_{\bar{\mathcal{A}}}$ satisfying condition (b) for which $f^{\bar{r}}_{\bar{\mathcal{A}}}\in T$. By induction on $k<\omega$ we will construct a sequence $\langle c_k\in B_{d_k}\setminus B_{d_{k-1}} : k<\omega\rangle$ such that

$$f(c_k) > r_{d_k} \ge k \text{ for every } k<\omega \text{ and } f\in T^*. \tag{$*$}$$

So fix a $k<\omega$ and let $\{f_i : i<k\}$ be such that

$$\{f_i\restriction B_{d_k}\setminus B_{d_{k-1}} : i<k\} = \{f\restriction B_{d_k}\setminus B_{d_{k-1}} : f\in T^*\}\subset T_k.$$

We show inductively that for every $m<k$

$$\text{there is a } c\in B_{d_k} \text{ such that } f_j(c) > r_{d_k} \text{ for all } j\le m. \tag{1}$$

So, fix an $m<k$ and let $a\in B_{d_k}$ such that $f_j(a^c)=f_j(a)>r_{d_k}$ for all $j<m$. If $f_m(a)>r_{d_k}$ then $c=a$ satisfies (1). Thus, assume that $f_m(a^c)=f_m(a)\le r_{d_k}$. By (b) we can find $b_0,\dots,b_{2k}\in B_{d_k}$ such that $r_{d_{k-1}}<f_m(b_0)<\cdots<f_m(b_{2k})$. By Pigeon Hole Principle we can find an $I\in[\{0,\dots,2k\}]^{k+1}$ and a $b\in\{a,a^c\}$ such that $f_m(b\wedge b_i)=f_m(b_i)$ for all $i\in I$. Without loss of generality we can assume that $I=\{0,\dots,k\}$ and $b\wedge b_i=b_i$ for all $i\le k$. Then

$$f_m(b^c\vee b_i) > r_{d_k} \quad \text{for all } i\le k.$$

Moreover, for every $j<m$ there is at most one $i_j\le k$ for which

$$f_j(b^c\vee b_{i_j}) \le r_{d_k}$$

since for different $i,i'\le k$ we have $f_j((b^c\vee b_i)\wedge(b^c\vee b_{i'})) = f_j(b^c) > r_{d_k}$. Thus, by Pigeon Hole Principle, there is an $i\le k$ such that $c=b^c\vee b_i$ satisfies (1). This finishes the proof of $(*)$.

Clearly the sequence $\langle c_k \in B_{d_k} \setminus B_{d_{k-1}} : k < \omega\rangle$ satisfies (ii) for every $\bar{\mathcal{A}}$ with $f_{\bar{\mathcal{A}}} \in T^*$. Thus, we need only to modify it to get also the condition (i).

To do it, use the fact that

$$r_{d_k} < f(c_k) < r_{d_{k+1}} \text{ for every } k < \omega \text{ and } f \in T^*$$

to construct the sequences: $\omega = I_0 \supset I_1 \supset \cdots$ of infinite subsets of ω, increasing $\langle k_j \in I_j : j < \omega\rangle$, and $\langle c^*_{k_j} \in \{c_{k_j}, c^c_{k_j}\} : j < \omega\rangle$ such that for every $j < \omega$

$$f(\bar{a}_{k_j} \wedge c_l) > r_{d_l} \text{ for every } f \in T^* \text{ and } l > k_j \text{ with } l \in I_j,$$

where $\bar{a}_{k_j} = c^*_{k_0} \wedge \cdots \wedge c^*_{k_j}$. Then the sequence $\langle \bar{a}_{k_j} : j < \omega\rangle$ is a strictly decreasing sequence satisfying (ii) and it is now easy to see that by putting $a_j = \bar{a}_{k_j} \wedge \bar{a}^c_{k_{j+1}}$ we obtain the desired sequence. □

Proof of Theorem 1. The algebra B we construct will be a subalgebra of the algebra $\mathcal{P}(\omega)$ of all subsets of ω. First, let $\mathcal{K} \subset \omega^\omega$ be a dominating family with $|\mathcal{K}| = \omega_1$ and fix a partition $\{D_k : k < \omega\}$ of ω into infinite subsets.

For every sequence $\bar{a} = \langle a_n : n < \omega\rangle$ of pairwise disjoint subsets of ω and $k < \omega$ put $a^*_k = \bigcup\{a_n : n \in D_k\}$. In addition, for every $h \in \mathcal{K}$ we put

$$a^h = \bigcup\{a_{n_h(k)} : k < \omega\}$$

where $n_h(k) = \min\{n \in D_k : n > \max\{h(k), k\}\}$. We also put

$$F(\bar{a}) = \{a^*_k : k < \omega\} \cup \{a^h : h \in \mathcal{K}\} \in [\mathcal{P}(\omega)]^{\leq\omega_1}.$$

Next, we will construct an increasing sequence $\langle B_\xi \in [\mathcal{P}(\omega)]^{\omega_1} : \xi \leq \omega_1\rangle$ of subalgebras of $\mathcal{P}(\omega)$ aiming for $B = B_{\omega_1}$. Thus, we choose B_0 as an arbitrary subalgebra of $\mathcal{P}(\omega)$ with $|B_0| = \omega_1$ and for limit ordinal numbers $\lambda \leq \omega_1$ we put $B_\lambda = \bigcup_{\xi<\lambda} B_\xi$. The algebra $B_{\xi+1}$ is formed from B_ξ in the following way.

Let $\{b_\eta : \eta < \omega_1\}$ be an enumeration of B_ξ and for $\eta < \omega$ let $\mathcal{A}^\xi_\eta$ be a subalgebra of B_ξ generated by $\{b_\zeta : \zeta < \eta\}$. For each such algebra we apply Lemma 3 to find the sequences $\bar{a}^\gamma = \langle a^\gamma_n : n < \omega\rangle$, $\gamma < \omega_1$, satisfying (i) and (ii) and let

$$G(\mathcal{A}^\xi_\eta) = \bigcup_{\gamma<\omega_1} F(\bar{a}^\gamma).$$

$B_{\xi+1}$ is defined as the algebra generated by $B_\xi \cup \bigcup_{\eta<\omega_1} G(\mathcal{A}^\xi_\eta)$. This finishes the construction of B.

Clearly, $|B| = \omega_1$. To prove that $\mathrm{cof}(B) = \omega_1$ it is enough to show that B is not a union of an increasing sequence $\bar{\mathcal{B}} = \langle B_n : n < \omega\rangle$ of proper subalgebras. So, by way of contradiction, assume that such a sequence $\bar{\mathcal{B}}$ exists. For every $n < \omega$ choose $b_n \in B \setminus B_n$ and find $\xi, \eta < \omega_1$ such that $\{b_n : n < \omega\} \subset \mathcal{A}^\xi_\eta$. Then the algebras $\mathcal{A}_n = B_n \cap \mathcal{A}^\xi_\eta$ form an increasing sequence of proper subalgebras of $\mathcal{A} = \mathcal{A}^\xi_\eta$. Thus, one of the sequences $\bar{a}^\gamma$ satisfies (ii) for $\bar{\mathcal{A}}$. So, if we put $\bar{a}^\gamma = \bar{a} = \langle a_n : n < \omega\rangle$ we conclude that $\{a^*_k : k < \omega\} \cup \{a^h : h \in \mathcal{K}\} \subset B$. Let $f(k) = \min\{n < \omega : a^*_k \in B_n\}$ and let $h \in \mathcal{K}$ be such that $f(k) < h(k)$ for all $k < \omega$.

The final contradiction is obtained by noticing that a^h cannot belong to any B_k. Indeed, if $a^h \in B_k$ for some k, then $a^h \cap a^*_k = a_{n_h(k)}$ belongs to $B_{\max\{f(k),k\}}$, since $a^*_k \in B_{f(k)}$. But $\max\{f(k), k\} \le \max\{h(k), k\} < n_h(k)$ so we get $a_{n_h(k)} \in B_{n_h(k)}$ contradicting the fact that $a_{n_h(k)}$ belongs to $\bar{\mathcal{A}} \setminus \mathcal{A}_{n_h(k)}$, which is disjoint with $B_{n_h(k)}$. □

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Krzysztof Ciesielski

Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA
e-mail: K_Cies@math.wvu.edu

Janusz Pawlikowski

Department of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

and

Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA
e-mail: pawlikow@math.uni.wroc.pl, pawlikow@math.wvu.edu