Real Analysis Exchange
Vol. (), , pp. 581–592

Marek Balcerzak*, Institute of Mathematics, Łódź Technical University, al. Politechniki 11, I-2, 90-924 Łódź, Poland, and Faculty of Mathematics, University of Łódź, ul. Banacha 22, 90-238 Łódź, Poland. e-mail: mbalce@krysia.uni.lodz.pl

Artur Bartoszewicz, Institute of Mathematics, Łódź Technical University, al. Politechniki 11, I-2, 90-924 Łódź, Poland e-mail: arturbar@ck-sg.p.lodz.pl

Krzysztof Ciesielski†, Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA e-mail: K_Cies@math.wvu.edu http://www.math.wvu.edu/~kcies

ON MARCZEWSKI-BURSTIN REPRESENTATIONS OF CERTAIN ALGEBRAS OF SETS

Abstract

We show that the Generalized Continuum Hypothesis GCH (its appropriate part) implies that many natural algebras on $\mathbb{R}$, including the algebra $\mathcal{B}$ of Borel sets and the interval algebra Σ, are outer Marczewski-Burstin representable by families of non-Borel sets. Also we construct, assuming again an appropriate part of GCH, that there are algebras on $\mathbb{R}$ which are not MB-representable. We prove that some algebras (including $\mathcal{B}$ and Σ) are not inner MB-representable. We give examples of algebras which are inner and outer MB-representable, or are inner but not outer MB-representable.

1 Introduction

Our set theoretic notation is standard and follows that from [Ci].

Key Words: Borel sets, ultrafilters, Continuum Hypothesis, MB-representation.
Mathematical Reviews subject classification: Primary 03E35; Secondary 28A05
Received by the editors January 19, 2000
*This work was partially supported by NSF Cooperative Research Grant INT-9600548 and its Polish part financed by KBN.

†Papers authored or co-authored by a Contributing Editor are managed by a Managing Editor or one of the other Contributing Editors.

For a fixed non-empty set X and a family $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ define, following the idea of Burstin and Marczewski,

$$\mathcal{S}(\mathcal{F}) = \{A \subset X \colon (\forall T \in \mathcal{F})(\exists W \in \mathcal{F})(W \subset T \cap A \text{ or } W \subset T \cap A^c)\}$$

and

$$\mathcal{S}_0(\mathcal{F}) = \{A \subset X \colon (\forall T \in \mathcal{F})(\exists W \in \mathcal{F})(W \subset T \cap A^c)\}.$$

Then $\mathcal{S}(\mathcal{F})$ constitutes an algebra of sets and $\mathcal{S}_0(\mathcal{F})$ is an ideal of subsets of X. (See [BBRW].) We will always assume that the whole space X is in an algebra; usually such a family is called a field of sets. Burstin in [Bu] proved that the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ is of the form $\mathcal{S}(\mathcal{F})$ for $\mathcal{F}$ being the family of perfect subsets of $\mathbb{R}$ of positive measure. It can be also shown that, for the same $\mathcal{F}$, the family $\mathcal{S}_0(\mathcal{F})$ consists of Lebesgue null sets. (See [Re] or [BET].) On the other hand, if $\mathcal{F}$ is the family of all perfect subsets of $\mathbb{R}$ then $\mathcal{S}(\mathcal{F})$ and $\mathcal{S}_0(\mathcal{F})$ constitute a quite different pair of a σ-algebra and a σ-ideal on $\mathbb{R}$ which were introduced by Marczewski in [Ma].

If a given algebra $\mathcal{A}$ (an ideal $\mathcal{I}$, respectively) on a set X can be represented as $\mathcal{S}(\mathcal{F})$ (respectively, as $\mathcal{S}_0(\mathcal{F})$) for some family $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$, we say that it is *Marczewski-Burstin representable* (or, briefly, *MB-representable*) by $\mathcal{F}$. If additionally, $\mathcal{F} \subset \mathcal{A}$ (respectively, $\mathcal{F} \cap \mathcal{A} = \emptyset$), we say that $\mathcal{A}$ is inner (outer) MB-representable by $\mathcal{F}$. Similarly, for $\mathcal{I} \subset \mathcal{A}$ we say that the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ is MB-representable if $\mathcal{A} = \mathcal{S}(\mathcal{F})$ and $\mathcal{I} = \mathcal{S}_0(\mathcal{F})$ for the same family $\mathcal{F}$. This notion is most often considered when $\mathcal{I}$ is the ideal

$$\mathrm{H}(\mathcal{A}) = \{A \subset X \colon (\forall B \subset A)(B \in \mathcal{A})\}$$

of sets which hereditarily belong to $\mathcal{A}$.

Systematic studies of MB-representations of algebras and ideals were initiated in [Re], [BET], and [BBRW]. For instance, in [BET] it is proved that the algebra of sets in $\mathbb{R}$ with the Baire property is inner MB-representable by a family of Borel sets, and in [BBRW] it is shown that the *interval algebra* Σ generated by intervals $[a, b)$ with $a < b$, is MB-representable by a family of Borel sets. Some necessary conditions for MB-representability of a pair $\langle \mathcal{A}, \mathcal{I} \rangle$ by a family of Borel sets are given in [BET] and [ET]. Until now, however, the following basic questions about MB-representability (see [BBRW]) were without an answer: *Is every algebra of sets MB-representable? What about some basic algebras, like the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$? Is it MB-representable, and if so, is it inner (outer) MB-representable?*

In this note we show that, assuming appropriate set theoretical assumptions (which follow from the Generalized Continuum Hypothesis GCH), there are algebras (on $\mathbb{R}$ and other infinite sets) which are not MB-representable.

We also show, under similar set theoretical assumptions, that many "natural" algebras, including the algebras $\mathcal{B}$ and Σ, are outer MB-representable in some strong manner. It has to be pointed out here that our representation families $\mathcal{F} \subset \mathcal{P}(X)$, unlike those studied in the earlier papers, are not nice in a sense that they are not connected with the Borel structure of a space. The same is true for our example of algebras which are not MB-representable. Moreover, these facts are not proved in ZFC. On the other hand we prove that the algebras $\mathcal{B}$ and Σ are not inner MB-representable. In Section 1 we show a simple criterion for an inner MB-representable algebra to be outer MB-representable. We apply it to some classical σ-algebras on $\mathbb{R}$.

2 Algebras Which Are Inner and Outer MB-representable

Recall that the algebras of Lebesgue measurable sets, of sets with the Baire property, and of Marczewski s-sets are inner MB-representable. We shall prove that they are also outer MB-representable. To this end we propose some general scheme.

We need the following fact which easily results from the definition of $\mathcal{S}(\mathcal{F})$ and $\mathcal{S}_0(\mathcal{F})$. (See [BBRW].)

Fact 1. *For $\mathcal{F}_0, \mathcal{F}_1 \subset \mathcal{P}(X) \setminus \{\emptyset\}$ assume that*

$$(\forall i \in \{0,1\})(\forall A \in \mathcal{F}_i)(\exists B \in \mathcal{F}_{1-i})(B \subset A).$$

(We thus say that $\mathcal{F}_0, \mathcal{F}_1$ are mutually coinitial.) Then $S(\mathcal{F}_0) = S(\mathcal{F}_1)$ and $S_0(\mathcal{F}_0) = S_0(\mathcal{F}_1)$.

Proposition 2. *Assume that an algebra $\mathcal{A}$ on X is inner MB-representable by a family $\mathcal{F} \subset \mathcal{A}$ with the following properties:*

(a) $(\forall F \in \mathcal{F})(\exists F_1, F_2 \in \mathcal{F})(F_1 \cup F_2 \subset F \ \&\ F_1 \cap F_2 = \emptyset)$;

(b) $(\exists B \subset X)(\forall F \in \mathcal{F}) B \cap F \notin \mathcal{A}$.

Then $\mathcal{A}$ is outer MB-representable by the family

$$\mathcal{F}_B = \{F_1 \cup (F_2 \cap B) \colon F_1, F_2 \in \mathcal{F} \ \&\ F_1 \cap F_2 = \emptyset\},$$

where B is a set realizing (b).

Proof. From (a) and the definition of $\mathcal{F}_B$ it follows that $\mathcal{F}$ and $\mathcal{F}_B$ are mutually coinitial. So $\mathcal{S}(\mathcal{F}) = S(\mathcal{F}_B)$ by Fact 1. Condition (b) implies that $\mathcal{F}_B \cap \mathcal{A} = \emptyset$. □

Corollary 3. *The following algebras on $\mathbb{R}$ are outer MB-representable:*

- *the algebra of Lebesgue measurable sets,*
- *the algebra of sets with the Baire property, and*
- *the algebra of Marczewski measurable sets.*

Proof. We use Proposition 2. For any algebra listed in the assertion, the role of $\mathcal{F}$ is played by: the perfect sets of positive measure, the comeager G_δ sets in nonempty open sets (see [BET]), and by all perfect sets, respectively. In every case, $\mathcal{F}$ satisfies conditions (a) and (b) where B in (b) stands for a Bernstein set. □

3 Algebras Which Are Strongly Outer MB-representable

Let $\mathcal{A}$ be an algebra on X. If, for each family $\mathcal{C} \subset \mathcal{P}(X)$ with $\mathcal{A} \subset \mathcal{C}$ and $|\mathcal{C}| = |\mathcal{A}|$, there is an $\mathcal{F} \subset \mathcal{P}(X) \setminus \mathcal{C}$ such that $\mathcal{A} = \mathcal{S}(\mathcal{F})$, we say that $\mathcal{A}$ is *strongly outer MB-representable*. If additionally, $\mathrm{H}(\mathcal{A}) = S_0(\mathcal{F})$, we say that the pair $\langle \mathcal{A}, \mathrm{H}(\mathcal{A})\rangle$ is strongly outer MB-representable.

Let X be an infinite set of cardinality κ. The following is the main theorem of this section.

Theorem 4. *Let $\mathcal{A}$ be an algebra of subsets of X such that $[X]^{<\kappa} \subset \mathcal{A}$. If $2^\kappa = \kappa^+$ and $|\mathcal{A}| < 2^\kappa$ then the pair $\langle \mathcal{A}, H(\mathcal{A})\rangle$ is strongly outer MB-representable.*

From this theorem we immediately obtain the following corollary.

Corollary 5. *If $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ then the pair $\langle \mathcal{B}, [\mathbb{R}]^{\leq\omega}\rangle$ is strongly outer MB-representable.*

In the sequel we will use the following fact which is well known (see e.g. [Wr1, Lemma 2]). However we provide its easy proof for the reader's convenience.

Fact 6. *For every algebra $\mathcal{A}$ on X and $Z \in \mathcal{P}(X)\setminus\mathcal{A}$ there exists an ultrafilter $\mathcal{U}_Z$ in $\mathcal{A}$ such that $U \cap Z \notin \mathcal{A}$ for every $U \in \mathcal{U}_Z$.*

Proof. Observe that the family $\mathcal{G} = \{E \in \mathcal{A}\colon Z \setminus E \in \mathcal{A}\}$ is a filter in the algebra $\mathcal{A}$. Consider an ultrafilter $\mathcal{U}_Z \supset \mathcal{G}$ in $\mathcal{A}$. Then $\mathcal{U}_Z$ is as desired. □

Proof of Theorem 4. To construct family $\mathcal{F}$ let $\{Z_\xi\colon \xi < \kappa^+\}$ be an enumeration of $\mathcal{P}(X) \setminus \mathcal{A}$. For each $\xi < \kappa^+$ use Fact 6 to choose an ultrafilter $\mathcal{U}_\xi = \mathcal{U}_{Z_\xi}$ in $\mathcal{A}$ for which

$$U \cap Z_\xi \notin \mathcal{A} \text{ for each } U \in \mathcal{U}_\xi. \tag{1}$$

Fix a family $\mathcal{C} \subset \mathcal{P}(X)$ with $\mathcal{A} \subset \mathcal{C}$ and $|\mathcal{C}| = |\mathcal{A}|$. By induction on $\xi < \kappa^+$ we construct a sequence $\langle D_\xi \subset X : \xi < \kappa^+ \rangle$ of "very independent sets" in a sense that

$$|D_\xi \cap Y| = \kappa = |D_\xi^c \cap Y| \tag{2}$$

for each set $Y \subset X$ of cardinality κ which belongs to the algebra $\mathcal{K}_\xi$ of sets generated by the family

$$\{D_\zeta : \zeta < \xi\} \cup \{Z_\zeta : \zeta \leq \xi\} \cup \mathcal{C}. \tag{3}$$

Such D_ξ can be chosen by an easy diagonal argument (another transfinite induction) since $|\mathcal{K}_\xi| = \kappa$.

$$\mathcal{F} = \bigcup_{\xi<\kappa^+} \{U \cap D_\xi : U \in \mathcal{U}_\xi\}.$$

Note that by (2) and (3) we clearly have $\mathcal{F} \cap \mathcal{C} = \emptyset$. The remaining properties of $\mathcal{F}$ will be shown in the following three steps.

Step 1. If $Z = Z_\xi \in \mathcal{P}(X) \setminus \mathcal{A}$ then $Z \notin \mathcal{S}(\mathcal{F})$.

To see this let $T = D_\xi$. Then $T = X \cap D_\xi \in \mathcal{F}$. We shall prove that $W \not\subset T \cap Z$ and $W \not\subset T \cap Z^c$ for all $W \in \mathcal{F}$. Thus let $W \in \mathcal{F}$, say $W = U \cap D_\eta$ for some $U \in \mathcal{U}_\eta$, $\eta < \kappa^+$. Consider three cases:

- If $\eta < \xi$ then $W \not\subset T$ since, by (2), $W \cap T^c = (U \cap D_\eta) \cap D_\xi^c \neq \emptyset$.
- If $\eta > \xi$ then once again we have $W \not\subset T$ since condition (2) implies that $W \cap T^c = (U \cap D_\eta) \cap D_\xi^c = D_\eta \cap (U \cap D_\xi^c) \neq \emptyset$.
- If $\eta = \xi$ then by (1) we have $U \cap Z \notin \mathcal{A}$. So, $|U \cap Z| = |U \cap Z^c| = \kappa$. Consequently, by (2), $|D_\xi \cap (U \cap Z)| = |D_\xi \cap (U \cap Z^c)| = \kappa$. Thus we have $W \not\subset T \cap Z$ since

  $$W \not\subset T \cap Z \iff U \cap D_\xi \not\subset D_\xi \cap Z \iff U \cap D_\xi \cap Z^c \neq \emptyset,$$

  and also $W \not\subset T \cap Z^c$ since

  $$W \not\subset T \cap Z^c \iff U \cap D_\xi \not\subset D_\xi \cap Z^c \iff U \cap D_\xi \cap Z \neq \emptyset.$$

This completes Step 1.

Step 2. If $V \in \mathcal{A}$ then $V \in \mathcal{S}(\mathcal{F})$.

Let $T \in \mathcal{F}$, say $T = U \cap D_\xi$ where $\xi < \kappa^+$ and $U \in \mathcal{U}_\xi$. Since $\mathcal{U}_\xi$ is an ultrafilter in $\mathcal{A}$, we have either $V \in \mathcal{U}_\xi$ or $V \notin \mathcal{U}_\xi$. If $V \in \mathcal{U}_\xi$ then $U \cap V \in \mathcal{U}_\xi$

and thus for $W = (U \cap V) \cap D_\xi = T \cap V$ we have $W \in \mathcal{F}$. If $V \notin \mathcal{U}_\xi$ then $V^c \in \mathcal{U}_\xi$ and thus for $W = (U \cap V^c) \cap D_\xi = T \cap V^c$ we have $W \in \mathcal{F}$. Hence in the both cases $V \in \mathcal{S}(\mathcal{F})$. Step 2 has been completed.

Step 3. $\mathcal{S}_0(\mathcal{F}) = \mathrm{H}(\mathcal{A})$.

We clearly have

$$\mathcal{S}_0(\mathcal{F}) \subset \mathrm{H}(\mathcal{S}(\mathcal{F})) = \mathrm{H}(\mathcal{A}).$$

To show that $\mathrm{H}(\mathcal{A}) \subset \mathcal{S}_0(\mathcal{F})$ consider $V \in \mathrm{H}(\mathcal{A})$ and let $T = U \cap D_\xi \in \mathcal{F}$ where $\xi < \kappa^+$ and $U \in \mathcal{U}_\xi$. Since $V \in \mathrm{H}(\mathcal{A})$, by (1) we have $V \notin \mathcal{U}_\xi$. So $V^c \in \mathcal{U}_\xi$ and $W = (U \cap V^c) \cap D_\xi = T \cap V^c$ belongs to $\mathcal{F}$. Hence $V \in \mathcal{S}_0(\mathcal{F})$. This finishes the proof of Theorem 4. □

Remark. It is worth to point out here that we do not need a full strength of the assumption $2^\kappa = \kappa^+$ to prove Theorem 4. In fact there are models of ZFC in which $2^\kappa > \kappa^+$ and we can find sets D_ξ satisfying (2) for any family of less than 2^κ-many sets of cardinality κ. In such models the proof presented above remains valid.

The structural assumptions on $\mathcal{A}$ in Theorem 4 were that $|\mathcal{A}| < 2^\kappa$ and $[X]^{<\kappa} \subset \mathcal{A}$. Although the example presented in the next section clearly violates both of these assumptions, it is worth to mention that the second assumption can be modified with resulting statement still being true. This is stated in the next theorem.

Theorem 7. *Let $\mathcal{A}$ be an algebra of subsets of X such that $\mathcal{A} \cap [X]^{<\kappa} = \{\emptyset\}$. If $2^\kappa = \kappa^+$ and $|\mathcal{A}| < 2^\kappa$ then the pair $\langle \mathcal{A}, H(\mathcal{A})\rangle = \langle \mathcal{A}, \{\emptyset\}\rangle$ is strongly outer MB-representable.*

Sketch of proof. Put $\overline{\mathcal{A}} = \{A \cup M \colon A \in \mathcal{A} \ \&\ M \in [X]^{<\kappa}\}$, where $\mathcal{A}$ is as above, and notice that the following version of Fact 6 remains true:

> For every $Z \in \mathcal{P}(X) \setminus \overline{\mathcal{A}}$ there exists an ultrafilter $\mathcal{U}_Z$ in $\mathcal{A}$ such that $U \cap Z \notin \overline{\mathcal{A}}$ for every $U \in \mathcal{U}_Z$.

Indeed, similarly as in Fact 6, it is enough to show that if $\mathcal{V}$ is a maximal filter in $\mathcal{A}$ such that $V \cap Z \notin \overline{\mathcal{A}}$ for each $V \in \mathcal{V}$ then $\mathcal{V}$ is an ultrafilter in $\mathcal{A}$. But if $\mathcal{V}$ is not an ultrafilter in $\mathcal{A}$ then there are $V_0, V_1 \in \mathcal{V}$ such that $V_0 \cap A \cap Z \in \overline{\mathcal{A}}$ and $V_1 \cap A^c \cap Z \in \overline{\mathcal{A}}$. Hence there are $A_0, A_1 \in \mathcal{A}$ and $M_0, M_1 \in [X]^{<\kappa}$ such that

$$V_0 \cap V_1 \cap A \cap Z = A_0 \cup M_0 \in \mathcal{A} \text{ and } V_0 \cap V_1 \cap A^c \cap Z = A_1 \cup M_1 \in \mathcal{A}$$

which implies that $V_0 \cap V_1 \cap Z = (A_0 \cup A_1) \cup (M_0 \cup M_1) \in \overline{\mathcal{A}}$ where $V_0 \cap V_1 \in \mathcal{V}$, a contradiction.

Then proceed as in the proof above listing as sets Z_ξ only the sets from $\mathcal{P}(X) \setminus \overline{\mathcal{A}}$. This will clearly results with $\mathcal{F} \cap \mathcal{C} = \emptyset$, $\mathcal{A} \subset \mathcal{S}(\mathcal{F})$, and with $(\mathcal{P}(X) \setminus \overline{\mathcal{A}}) \cap \mathcal{S}(\mathcal{F}) = \emptyset$. To finish the proof it is enough to notice that if $Z \in \overline{\mathcal{A}} \setminus \mathcal{A}$ then $Z^c \notin \overline{\mathcal{A}}$, as $\mathcal{A} \cap [X]^{<\kappa} = \{\emptyset\}$, and so $Z^c \notin \mathcal{S}(\mathcal{F})$. Thus we have also $Z \notin \mathcal{S}(\mathcal{F})$. □

It was proved in [BBRW] that the interval algebra Σ on $\mathbb{R}$ is outer representable by a family of Borel sets. If we use Theorem 7 with $\mathcal{A} = \Sigma$, we obtain the following

Corollary 8. *If $2^{\mathfrak{c}} = \mathfrak{c}^+$ then the interval algebra Σ on $\mathbb{R}$ is strongly outer MB-representable. In particular, it is outer MB-representable by a family of non-Borel sets.*

4 Algebras Which Are Not MB-representable

The key step towards constructing an algebra which is not MB-representable is the following fact.

Proposition 9. *Let X be an infinite set of cardinality κ and let $\mathcal{A}$ be an algebra on X having the following properties:*

(i) $\mathcal{A} \cap [X]^{<\kappa} = \{\emptyset\}$;

(ii) *for every $E \in [X]^{<\kappa}$ and $x \notin E$ there exists an $A \in \mathcal{A}$ such that $x \in A \subset X \setminus E$;*

(iii) *for every $Z \in [X]^{\kappa}$ and $x \notin Z$ there exists an $A \in \mathcal{A} \setminus \{\emptyset\}$ such that either $|A \cap Z^c| < \kappa$ or $x \in A$ and $|A \cap Z| < \kappa$.*

If $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ is such that $\mathcal{A} \subset \mathcal{S}(\mathcal{F})$ then $\mathcal{S}(\mathcal{F})$ contains a singleton. In particular algebra $\mathcal{A}$ is not MB-representable.

Proof. Let $\mathcal{A}$ and $\mathcal{F}$ be as in the assumptions. If $\{x\} \in \mathcal{S}(\mathcal{F})$ for every $x \in X$ then there is nothing to prove. So assume that there exists an $x \in X$ for which $\{x\} \notin \mathcal{S}(\mathcal{F})$. This means that there exists a $Z \in \mathcal{F}$ for which neither $W \subset Z \cap \{x\}$ nor $W \subset Z \setminus \{x\}$ for every $W \in \mathcal{F}$. Thus $x \in Z$ and

$$x \in W \quad \text{for every } W \in \mathcal{F} \text{ with } W \subset Z. \tag{4}$$

Next note that

$$\text{there is no } A \in \mathcal{A} \text{ containing } x \text{ with } |A \cap Z| < \kappa. \tag{5}$$

Indeed, if there is such an A then, by (ii) used with $E = A \cap Z \setminus \{x\}$ we can find an $A_1 \in \mathcal{A}$ with $A \cap A_1 \cap Z = \{x\}$. Since $A \cap A_1 \in \mathcal{A} \subset \mathcal{S}(\mathcal{F})$, there exists a

$W \in \mathcal{F}$ such that either $W \subset Z \cap (A \cap A_1) = \{x\}$ or $W \subset Z \setminus (A \cap A_1) = Z \setminus \{x\}$. But, by (4), the second case is impossible. Thus, $\{x\} = W \in \mathcal{F}$ and so $\{x\} \in \mathcal{S}(\mathcal{F})$, contradicting our choice of x.

Note that the condition (5) works also for Z replaced with $Z' = Z \setminus \{x\}$, which implies that $|Z'| = \kappa$. Thus, applying (iii) to Z' and our x, we conclude that there is an $A \in \mathcal{A} \setminus \{\emptyset\}$ such that $|A \cap (Z')^c| < \kappa$. For this A we have also $|A \cap Z^c| < \kappa$. Now, using (ii) if necessary to decrease A, we can additionally assume that $A \cap Z^c = \emptyset$ and $x \notin A$. (Indeed, pick an $x_0 \in A \cap Z$, $x_0 \neq x$, and let $E = (A \cap Z^c) \cup \{x\}$. Then $x_0 \notin E$ and, by (ii), there exists an $A^* \in \mathcal{A}$ such that $x_0 \in A^* \subset X \setminus E$. Then $\hat{A} = A \cap A^* \ni x_0$ is as required.) So $A \subset Z \setminus \{x\}$. Thus, by (4), A contains no $W \in \mathcal{F}$. Since $A \in \mathcal{S}(\mathcal{F})$, this implies that $A \in \mathcal{S}_0(\mathcal{F})$. So $\{a\} \in \mathcal{S}(\mathcal{F})$ for every $a \in A$. □

Proposition 10. *If $2^\kappa = \kappa^+$ and $|X| = \kappa$ then there exists an algebra $\mathcal{A}$ on X satisfying conditions (i)–(iii) from Proposition 9.*

Proof. Let $\{\langle Z_\xi, x_\xi\rangle : \xi < \kappa^+\}$ be an enumeration of all pairs $\langle Z, x\rangle \in \mathcal{P}(X) \times X$ with $x \notin Z$. Similarly as in the proof of Theorem 4, by induction on $\xi < \kappa^+$, we construct a sequence $\langle D_\xi \subset X : \xi < \kappa^+\rangle$ such that

$$|D_\xi \cap A| = \kappa = |D_\xi^c \cap A| \tag{6}$$

for every non-empty set A which belongs to the algebra $\mathcal{L}_\xi$ of sets generated by the family $\{D_\zeta : \zeta < \xi\}$. In addition, if there is no $A \in \mathcal{L}_\xi \setminus \{\emptyset\}$ with $|A \setminus Z_\xi| < \kappa$ then we will additionally require that $z_\xi \in D_\xi \subset X \setminus Z_\xi$.

Such D_ξ can be chosen by an easy diagonal argument since $|\mathcal{L}_\xi| \leq \kappa$ and sets D_ξ and D_ξ^c need to intersect all sets $A \in \mathcal{L}_\xi \setminus \{\emptyset\}$ and, if additional requirement is claimed, they need also to intersect all sets $A \setminus Z_\xi \in [X \setminus Z_\xi]^\kappa$ for $A \in \mathcal{L}_\xi \setminus \{\emptyset\}$. Let $\mathcal{A}$ denote the algebra generated by all sets $\{D_\xi : \xi < \kappa^+\}$. Then $\mathcal{A}$ has all the desired properties.

Indeed, (i) is obvious.

To see (ii) let $E \in [X]^{<\kappa}, x \notin E$, and take an $\xi < \kappa^+$ with $\langle Z_\xi, x_\xi\rangle = \langle E, x\rangle$. Then from $|Z_\xi| < \kappa$ and (6) it follows that $|A \setminus Z_\xi| = \kappa$ for all $A \in \mathcal{L}_\xi \setminus \{\emptyset\}$. So $x = z_\xi \in D_\xi \subset X \setminus Z_\xi = X \setminus E$ and $A = D_\xi \in \mathcal{A}$ is as desired.

To see (iii) let $E \in [X]^\kappa, x \notin E$, and take a $\xi < \kappa^+$ such that $\langle Z_\xi, x_\xi\rangle = \langle Z, x\rangle$. If there exists an $A \in \mathcal{L}_\xi \setminus \{\emptyset\}$ such that $|A \cap Z^c| = |A \setminus Z_\xi| < \kappa$ then (iii) holds. Otherwise $x = z_\xi \in D_\xi \subset X \setminus Z_\xi = X \setminus Z$ and $A = D_\xi \in \mathcal{A}$ is as desired since $A \cap Z = \emptyset$. □

Corollary 11. *If $2^\kappa = \kappa^+$ and $|X| = \kappa$ then there exists an algebra $\mathcal{A}$ on X which is not MB-representable.*

It is also worth to notice that, as in the case of Theorem 4, Corollary 11 remains valid also in some models with $2^\kappa > \kappa^+$. However, it is also worth to note that if $\mathcal{A}$ is an example as in Proposition 10 and $X \subset Y$, then the algebra $\mathcal{A}_Y$ on Y generated by $\mathcal{A}$ still is not MB-representable. Thus if X is such that there exists an infinite $\kappa \leq |X|$ with $2^\kappa = \kappa^+$ then there exists an algebra $\mathcal{A}$ on X which is not MB-representable.

5 Algebras Which Are Not Inner MB-representable

Now, we shall prove that the algebras Σ and $\mathcal{B}$ are not inner MB-representable.

Proposition 12. (S. Wroński [Wr2]) *The interval algebra Σ is not inner MB-representable.*

Proof. Suppose that $\Sigma = S(\mathcal{F})$ for some $\mathcal{F} \subset \Sigma$. Let $\mathcal{G}$ stand for the family of all intervals $[a, b)$ with $a < b$. Evidently, Σ and $\mathcal{G}$ are mutually coinitial, so $S(\Sigma) = S(\mathcal{G})$ by Fact 1. Since $S(\mathcal{G})$ contains singletons, we have $S(\mathcal{G}) \setminus \Sigma \neq \emptyset$. Thus, by Fact 1, $\mathcal{G}$ and $\mathcal{F}$ cannot be mutually coinitial, and since $\mathcal{F} \subset \Sigma$, it follows that there is a $A \in \mathcal{G}$ such that $P \setminus A \neq \emptyset$ for each $P \in \mathcal{F}$. Let $A = [a, b)$. To obtain a contradiction we shall show that $\{a\} \in S(\mathcal{F})$. Let $P \in \mathcal{F}$. If $a \notin P$ then obviously $P \subset P \cap \{a\}^c$. Let $a \in P$. Since $A \in S(\mathcal{F})$ and since we cannot find a $Q \in \mathcal{F}$ such that $Q \subset P \cap A$, there is a $Q \subset P \cap A^c$, so $Q \subset P \cap \{a\}^c$. Consequently $\{a\} \in \mathcal{S}(\mathcal{F}) \subset \mathcal{S}_0(\mathcal{F})$. □

Theorem 13. *Let X be an infinite set of cardinality κ. Let $\mathcal{A}$ be an algebra on X such that:*

(I) $H(\mathcal{A}) \subset [X]^{<\kappa}$;

(II) $\mathcal{A} \cap [X]^{<\kappa} \subset H(\mathcal{A})$;

(III) *for $\mathcal{A}^* = \mathcal{A} \setminus [X]^{<\kappa}$ we have $S(\mathcal{A}^*) \setminus \mathcal{A} \neq \emptyset$.*

Then $\mathcal{A}$ is not inner MB-representable.

Proof. Suppose that $\mathcal{A} = S(\mathcal{F})$ for some $\mathcal{F} \subset \mathcal{A}$. Put $\mathcal{F}^* = \mathcal{F} \setminus [X]^{<\kappa}$. First we shall prove that

$$(\forall B \in \mathcal{A}^*)(\exists F \in \mathcal{F}^*)\ F \subset B. \tag{7}$$

Suppose it is not the case and let $B \in \mathcal{A}^*$ witness that (7) is false. We have $|B| = \kappa$ and so, $B \notin \mathcal{S}_0(\mathcal{F})$ since, by (I),

$$\mathcal{S}_0(\mathcal{F}) \subset \mathrm{H}(\mathcal{S}(\mathcal{F})) = \mathrm{H}(\mathcal{A}) \subset [X]^{<\kappa}. \tag{8}$$

From $B \notin \mathcal{S}_0(\mathcal{F})$ it follows that there are sets $Q \in \mathcal{F}$, $Q \subset B$. Since (7) is false, we have $|Q| < \kappa$ for each set $Q \in \mathcal{F}$ contained in B. We shall show that $B \in \mathrm{H}(\mathcal{S}(\mathcal{F}))$ which yields a contradiction since $|B| = \kappa$ and $\mathrm{H}(\mathcal{S}(\mathcal{F})) \subset [X]^{<\kappa}$ by (I). Let $Z \subset B$ and $P \in \mathcal{F}$. We have to find $T \in \mathcal{F}$ such that either $T \subset P \cap Z^c$ or $T \subset P \cap Z$.

Since $B \in \mathcal{A} = \mathcal{S}(\mathcal{F})$, there is a $Q \in \mathcal{F}$ such that either $Q \subset P \cap B^c$ or $Q \subset P \cap B$. If $Q \subset P \cap B^c$ then $T = Q$ is as desired. So, assume that $Q \subset P \cap B$. Then $|Q| < \kappa$ by our supposition. Thus, by (II), we have $Q \cap Z \in \mathcal{A} = \mathcal{S}(\mathcal{F})$ and so, there is a $T \in \mathcal{F}$ such that either $T \subset Q \cap (Q \cap Z) \subset P \cap Z$ or $T \subset Q \cap (Q \cap Z)^c \subset P \cap Z^c$. Consequently, $Z \in \mathcal{S}(\mathcal{F})$ and thus $B \in \mathrm{H}(\mathcal{S}(\mathcal{F}))$ as desired.

By Fact 1, condition (7) together with an obvious inclusion $\mathcal{F}^* \subset \mathcal{A}^*$ imply that $S(\mathcal{F}^*) = S(\mathcal{A}^*)$.

Next, we shall show that $S(\mathcal{F}^*) \subset S(\mathcal{F})$. Assume that $B \in S(\mathcal{F}^*)$. Let $P \in \mathcal{F}$. Consider two cases:

- if $P \in \mathcal{F}^*$ then $B \in S(\mathcal{F}^*)$ implies that we can find a $Q \in \mathcal{F}^*$ (hence $Q \in \mathcal{F}$) with $Q \subset P \cap B$ or $Q \subset P \cap B^c$;
- if $|P| < \kappa$ then, by (II), $P \cap B \in \mathcal{A} = \mathcal{S}(\mathcal{F})$, so there is a $Q \in \mathcal{F}$ with $Q \subset P \cap (P \cap B)$ or $Q \subset P \cap (P \cap B)^c = P \cap B^c$.

Hence $B \in \mathcal{S}(\mathcal{F})$.

Finally, we have $S(\mathcal{A}^*) = S(\mathcal{F}^*) \subset \mathcal{S}(\mathcal{F}) = \mathcal{A}$ which contradicts (III). □

From the result by Elaloui-Talibi [ET, Thm. 1.1] it follows that there is no $\mathcal{F} \subset \mathcal{B}$ with $\mathcal{B} = \mathcal{S}(\mathcal{F})$ and $[\mathbb{R}]^{\leq\omega} = \mathcal{S}_0(\mathcal{F})$. We can derive a bit more

Corollary 14. *The algebra $\mathcal{B}$ is not inner MB-representable.*

PROOF. It suffices to check (III). Here $\mathcal{B}^*$ is the family of uncountable Borel sets. Thus $\mathcal{B}^*$ and the family of perfect sets are mutually coinitial. So, by Fact 1, $S(\mathcal{B}^*)$ is exactly the algebra of classical Marczewski (s)-sets. Since there is a non-Borel (s_0)-set [Mi], therefore it belongs to $S(\mathcal{B}^*) \setminus \mathcal{B}$. □

Example. Let us observe that the condition (III) in Theorem 13 is essential. It was shown in [BBRW] that if $\mathcal{I}$ is an ideal in $\mathcal{P}(X)$ then, for the dual filter $\mathcal{F}_\mathcal{I} = \{E^c \colon E \in \mathcal{I}\}$ and the algebra $\mathcal{A}_\mathcal{I} = \mathcal{I} \cup \mathcal{F}_\mathcal{I}$, we have $\mathcal{A}_\mathcal{I} = S(\mathcal{F}_\mathcal{I})$ and $\mathcal{I} = S_0(\mathcal{F}_\mathcal{I})$. Hence $\mathcal{A}_\mathcal{I}$ is inner MB-representable. Let us consider a special case. Assume that λ is an infinite cardinal with $\lambda \leq \kappa = |X|$, and put $\mathcal{I} = [X]^{<\lambda}$, $\mathcal{A} = \mathcal{A}_\mathcal{I}$. Then conditions (I) and (II), stated in Theorem 13, are obviously satisfied. However, $\mathcal{A}$ is inner MB-representable, and (III) is false since $\mathcal{A}^* = \mathcal{F}_\mathcal{I}$ and so $\mathcal{A} = S(\mathcal{F}_\mathcal{I}) = S(\mathcal{A}^*)$. Finally, note that $\mathcal{A}$ is not outer MB-representable. Indeed, suppose that $\mathcal{A} = S(\mathcal{G})$ and $\mathcal{G} \cap \mathcal{A} = \emptyset$. Thus

$|E| \geq \lambda$ and $|E^c| \geq \lambda$ for all $E \in \mathcal{G}$. Since for each $A \in \mathcal{F}_{\mathcal{I}}$ there is an $E \in \mathcal{G}$ such that $E \subset A$, therefore, by Fact 1, from $S(\mathcal{F}_{\mathcal{I}}) = \mathcal{A} = S(\mathcal{G})$ it follows that for each $E \in \mathcal{G}$ there is an $A \in \mathcal{F}_{\mathcal{I}}$ such that $A \subset E$. This however is impossible.

Acknowledgements. We would like to thank S. Wroński for a fruitful discussion. He has allowed us to include Proposition 12 in our paper.

References

[BBRW] M. Balcerzak, A. Bartoszewicz, J. Rzepecka, S. Wroński, *Marczewski fields and ideals*, Real Anal. Exchange, this issue.

[BET] J. B. Brown, H. Elalaoui-Talibi, *Marczewski-Burstin like characterizations of σ-algebras, ideals, and measurable functions*, Colloq. Math. **82** (1999), 227–286.

[Bu] C. Burstin, *Eigenschaften messbarer und nichtmessbarer Mengen*, Sitzungsber. Kaiserlichen Akad. Wiss. Math.-Natur. Kl. Abteilung IIa, **123** (1914), 1525–1551.

[Ci] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Stud. Texts **39**, Cambridge Univ. Press 1997.

[ET] H. Elalaoui-Talibi, *On Marczewski-Burstin like characterizations of certain σ-algebras and σ-ideals*, Real Anal. Exchange, to appear.

[Ma] E. Marczewski (Szpilrajn), *Sur un classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. **24** (1935), 17–34.

[Mi] A. Miller, *Special subsets of the real line*, in: Handbook of Set Theoretic Topology, (K. Kunen and J.E. Vaughan, eds.), Elsevier 1984.

[Re] P. Reardon, *Ramsey, Lebesgue and Marczewski sets and the Baire property*, Fund. Math. **149** (1996), 191–203.

[Wr1] S. Wroński, *On proper subuniverses of a Boolean algebra*, Acta Univ. Lodz. Folia Math. **9** (1997), 69–76.

[Wr2] S. Wroński, private communication.