Real Analysis Exchange
Vol. 27(1), 2001/2002, pp. 579–598

Krzysztof Ciesielski,* Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310. e-mail: `K_Cies@math.wvu.edu`
web page: `http://www.math.wvu.edu/homepages/kcies`
Harvey Rosen, Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487. e-mail: `hrosen@gp.as.ua.edu`

TWO EXAMPLES CONCERNING EXTENDABLE AND ALMOST CONTINUOUS FUNCTIONS

Abstract

The main purpose of this paper is to describe two examples. The first is that of an almost continuous, Baire class two, non-extendable function $f\colon [0,1] \to [0,1]$ with a G_δ graph. This answers a question of Gibson [15]. The second example is that of a connectivity function $F\colon \mathbb{R}^2 \to \mathbb{R}$ with dense graph such that $F^{-1}(0)$ is contained in a countable union of straight lines. This easily implies the existence of an extendable function $f\colon \mathbb{R} \to \mathbb{R}$ with dense graph such that $f^{-1}(0)$ is countable. We also give a sufficient condition for a Darboux function $f\colon [0,1] \to [0,1]$ with a G_δ graph whose closure is bilaterally dense in itself to be quasi-continuous and extendable.

1 Definitions and Notation

Our terminology is standard and follows [6]. We consider only real-valued functions of one or more real variables. No distinction is made between a function and its graph. A restriction of a function $f\colon X \to Y$ to a set $A \subset X$ is denoted by $f \restriction A$. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the set of all real and rational numbers, respectively, while I will stand for the interval $[0,1]$. The closure of a set $A \subset \mathbb{R}^n$ is denoted by $\mathrm{cl}(A)$, its boundary by $\mathrm{bd}(A)$ and its diameter by

---

Key Words: extendable functions, peripherally continuous functions.
Mathematical Reviews subject classification: Primary: 26A15; Secondary: 54A25.
Received by the editors March 19, 1999

*The work of the first author was partially supported by NSF Cooperative Research Grant INT-9600548 with its Polish part financed by Polish Academy of Science PAN.

Papers authored or co-authored by a Contributing Editor are managed by a Managing Editor or one of the other Contributing Editors.

diam(A). The first coordinate projection of a set $A \subset \mathbb{R}^2$ will be denoted by pr(A).

The ordinal numbers will be identified with the sets of all their predecessors and cardinals with the initial ordinals. In particular $2 = \{0, 1\}$ and the first infinite ordinal ω number is equal to the set of all natural numbers $\{0, 1, 2, \dots\}$.

We will also use the following terminology [16]. For $X \subseteq \mathbb{R}^n$ a function $f\colon X \to \mathbb{R}$ is:

- *Darboux* if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset K of X;
- *almost continuous* (in the sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing f also contains a continuous function from X to $\mathbb{R}$ [27];
- *connectivity* function if the graph of $f \restriction Z$ is connected in $Z \times \mathbb{R}$ for any connected subset Z of X;
- *extendable* function if there is a connectivity function $F\colon X \times [0, 1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in X$;
- *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subset V$.

The classes of these functions are denoted by D, AC, Conn, Ext and PC, respectively, where the space X will be always clear from the context. Recall also (see e.g. [16] or [8]) that for the functions from $\mathbb{R}$ to $\mathbb{R}$ and from I into I we have the following strict inclusions

$$\mathrm{Ext} \subsetneq \mathrm{AC} \subsetneq \mathrm{Conn} \subsetneq \mathrm{D} \subsetneq \mathrm{PC} \tag{1}$$

while in the Baire class one all these classes coincide. (See Brown, Humke and Laczkovich [5].) On the other hand, for the classes of functions from $\mathbb{R}^n$, with $n > 1$, into $\mathbb{R}$ we have

$$\begin{tikzcd} & & \mathrm{AC} \\ \mathrm{Ext} = \mathrm{Conn} = \mathrm{PC} \arrow[r] & \mathrm{AC} \cap \mathrm{D} \arrow[ur] \arrow[dr] & \\ & & \mathrm{D} \end{tikzcd}$$

where arrows denote strict inclusions. (Equation Ext $=$ Conn on $\mathbb{R}^n$ is proved in [10].)

2 Almost Continuous Non-Extendable Function with G_δ Graph

As noted above the classes listed in (1) cannot be distinguished within the family $\mathcal{B}_1$ of Baire one functions. On the other hand, the classes of functions on $\mathbb{R}$ or I listed in (1) can be distinguished within the family $\mathcal{B}_2$ of Baire class two functions. (See e.g. [8, thm 1.2].) The next natural question is: for which classes $\mathcal{G}$ containing $\mathcal{B}_1$ but not $\mathcal{B}_2$ will the inclusions remain strict? For several classes $\mathcal{G}$ strictly between $\mathcal{B}_1$ and $\mathcal{B}_2$ this has been investigated by Brown [3] and Brown, Humke and Laczkovich [5].

In this section we will investigate the inclusions in (1) within the class $\mathcal{G} = \mathcal{G}_\delta \cap \mathcal{B}_2$, where $\mathcal{G}_\delta$ is the class of functions with G_δ graphs. Notice that $\mathcal{B}_1 \subset \mathcal{G}_\delta$ since for every pointwise limit f of continuous functions $f_n \colon I \to \mathbb{R}$ the graph of f is equal to $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{\langle x, y\rangle \colon |f_i(x) - y| < 2^{-n}\}$. Clearly also every function f from $\mathcal{G}_\delta$ is Borel, since for every open $U \subset \mathbb{R}$ the set $f^{-1}(U) = \mathrm{pr}(f \cap [I \times U]) = I \setminus \mathrm{pr}(f \cap [I \times (\mathbb{R} \setminus U)])$ is simultaneously analytic and coanalytic, and hence Borel. (See e.g. [23, p. 489] or [19, p. 89].) On the other hand there are functions in $\mathcal{G}_\delta$ of arbitrary high Borel class. This follows from the fact that every Borel set is a one-to-one continuous image of a closed subset of $\mathbb{R} \setminus \mathbb{Q}$ (see e.g. [19, 15.3, p. 89]) which is clearly G_δ in $\mathbb{R}$. Indeed, if $A \subset I$ is of the class at least α for some $\alpha < \omega_1$, take one-to-one continuous functions f_0 from a closed subset of $(0,1) \setminus \mathbb{Q}$ onto A and f_1 from a closed subset of $(2,3) \setminus \mathbb{Q}$ onto $I \setminus A$. Then $f = f_0^{-1} \cup f_1^{-1} \colon I \to \mathbb{R}$ has a G_δ graph and is of Borel class at least α, since $f^{-1}[(0,1)] = A$. Thus, if $\mathcal{B}_\alpha$ stands for the Baire class α functions, then we have the following relations for every $2 \leq \alpha < \omega_1$, where arrows denote strict inclusions.

$$\begin{tikzcd}
 & & \mathcal{B}_\alpha \arrow[dr] & \\
\mathcal{B}_1 \arrow[r] & \mathcal{G}_\delta \cap \mathcal{B}_2 \arrow[ur] \arrow[dr] & & \mathit{Borel} \\
 & & \mathcal{G}_\delta \arrow[ur] &
\end{tikzcd}$$

The properness of inclusions in (1) within the class $\mathcal{G}_\delta$ has been summarized by Brown in [4]. (See also [16, sec. 3].) In particular, he noticed that the inclusions AC $\subset$ Conn $\subset$ D $\subset$ PC remain proper within this class $\mathcal{G}_\delta$ leaving open the problem of properness of inclusion Ext $\subset$ AC in the $\mathcal{G}_\delta$ class. (This problem is stated explicitly by Gibson in [15, Question 4].) In fact the examples exhibiting properness of the inclusions AC $\subset$ Conn $\subset$ D $\subset$ PC within the class $\mathcal{G}_\delta \cap \mathcal{B}_2$ can be found in the literature. For AC $\subset$ Conn Jones and Thomas [18] (also see Thomas [28]) constructed a non almost continuous function $f \colon I \to I$ with a connected G_δ graph. This f is also $\mathcal{B}_2$, but it is not pointed out in the paper. Also Brown [3, remark 3] gives a bit stronger example and states explicitly that it is in $\mathcal{G}_\delta \cap \mathcal{B}_2 \cap \text{Conn}$ but not in AC. The properness of the inclusion $\text{Conn} \subset \text{D}$ within $\mathcal{G}_\delta \cap \mathcal{B}_2$ is stated explicitly by Brown in [3, example 1 and remark 3], using the example constructed by him in [1, example 2]. (Also, in [24] Miller gives an example of a Darboux function $f\colon I \to I$ having a G_δ graph but no fixed point and so the graph of f is not connected. This function is also $\mathcal{B}_2$, but it is not pointed out in the paper.) Concerning the properness of the inclusion $\text{D} \subset \text{PC}$ in $\mathcal{G}_\delta \cap \mathcal{B}_2$ Brown [4] states that there is one (not mentioning $\mathcal{B}_2$) and Miller [24] describes a peripherally continuous non Darboux function with G_δ graph, which turns out to be also in $\mathcal{B}_2$.

The main goal of this section is to describe an example justifying properness of $\text{Ext} \subset \text{AC}$ within $\mathcal{G}_\delta \cap \mathcal{B}_2$, thus answering a question of Gibson in [15, Question 4]. However, we will notice also that the small modifications of this example justify also properness of the remaining inclusions of (1) within $\mathcal{G}_\delta \cap \mathcal{B}_2$.

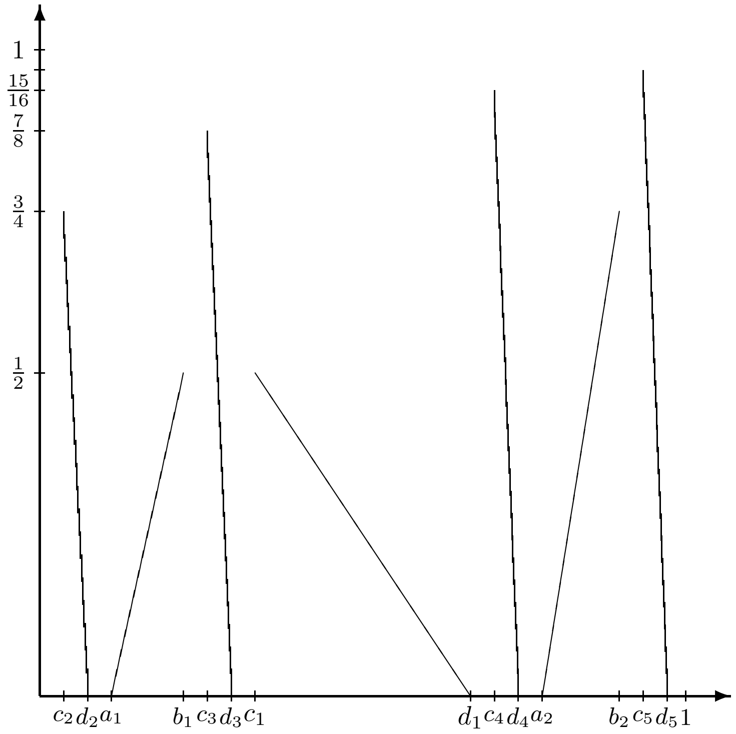

Figure 1: function $f \in \mathcal{G}_\delta \cap \mathcal{B}_2 \cap \text{AC} \setminus \text{Ext}$ from Example 2.1

Example 2.1. *There exists an almost continuous, non-extendable, Baire class two function $f\colon I \to I$ with G_δ graph.*

PROOF. The example is a slight modification of a function constructed by Ciesielski and Jastrzębski in [8, thm 3.1]. It also slightly simplifies the argument given there.

Let C be the Cantor ternary set in I and let $\mathcal{J}$ be the family of all component intervals of $I \setminus C$. For $i \in \{0,1\}$ let $\mathcal{J}_i$ be the family of all intervals $J \in \mathcal{J}$ of length 3^{-2n+i} for some integer n. Thus, $\{\mathcal{J}_0, \mathcal{J}_1\}$ is a partition of $\mathcal{J}$. Let $\{(a_n, b_n)\colon 0 < n < \omega\}$ and $\{(c_n, d_n)\colon 0 < n < \omega\}$ be the enumerations of $\mathcal{J}_0$ and $\mathcal{J}_1$, respectively. Define function $f\colon [0,1] \to [0,1]$ in the following way. (See Figure 1.)

- For every $0 < n < \omega$ we put $f(a_n) = f(d_n) = 0$, $f(b_n) = f(c_n) = 1 - 2^{-n}$ and extend it linearly on each interval $[a_n, b_n]$ and $[c_n, d_n]$. We refer to b_n and c_n as *upper endpoints for* f of $[a_n, b_n]$ and $[c_n, d_n]$, respectively.
- For all other x's we put $f(x) = 0$.

(Ciesielski and Jastrzębski define their function by assigning to $f(b_n)$ and $f(c_n)$ value 1 instead of $1 - 2^{-n}$. It does not have G_δ graph since $f \cap (I \times \{1\}) = \bigcup_{n=1}^{\infty}\{b_n, c_n\} \times \{1\}$ is not G_δ being countable dense in $C \times \{1\}$.) First note that the graph of f is a G_δ set as a union of three G_δ sets:

- $f \restriction (C \setminus \bigcup_{n=1}^{\infty}\{b_n, c_n\}) = (C \setminus \bigcup_{n=1}^{\infty}\{b_n, c_n\}) \times \{0\}$,
- $f \restriction (I \setminus C) = \bigcap_{n=1}^{\infty}\{\langle x, y\rangle \colon x \in I \setminus C \ \& \ |f(x) - y| < 2^{-n}\}$ and
- $f \restriction \bigcup_{n=1}^{\infty}\{b_n, c_n\} = \bigcup_{n=1}^{\infty}\{\langle b_n, 1 - 2^{-n}\rangle, \langle c_n, 1 - 2^{-n}\rangle\}$ which is discrete.

Also, since the first two restrictions are continuous, it is easy to see that the preimage $f^{-1}(U)$ of any open set $U \subset I$ is a union of a G_δ-set and an F_σ-set; so f is of Baire class two.

Next we will show that f is not extendable. By way of contradiction, assume that f is extendable; that is, that there is a connectivity function $F\colon I^2 \to I$ with $F(x, 0) = f(x)$ for all $x \in I$. Thus F is peripherally continuous. We will deduce from this that there exists an $x \in I$ such that $F(x, 0) = 1$, which implies $f(x) = 1$, contradicting our definition of f. For this we define inductively the sequences $\langle p_1, p_2, p_3, \ldots\rangle$ of upper endpoints for f and $\langle B_0, B_1, B_2, \ldots\rangle$ of closed connected subsets of I^2 such that for each $n < \omega$ we have

$$F[B_n] \subset (1 - 2^{-n+1}, 1], \quad B_n \cap B_{n+1} \neq \emptyset, \quad \operatorname{diam}(B_n) \leq 2^{-n},$$

and

$$\langle p_{n+1},0\rangle\in B_n,\quad F(p_{n+1},0)=f(p_{n+1})\geq 1-2^{-(n+1)}.$$

We will start with $B_0=[c_2,d_2]\times\{0\}$ and $p_1=c_2$. Clearly, the above conditions are satisfied. So assume that p_n and B_{n-1} satisfying the above are already constructed for some $n>0$. Then p_n is the upper endpoint for f of some interval $J_0\in\mathcal{J}$. We have to find B_n and p_{n+1}. So, choose an $\varepsilon>0$ less than the length of J_0 such that $\varepsilon<\min\{2^{-(n+1)},p_n,1-p_n,\operatorname{diam}(B_{n-1})\}$ and

($*$) if $J\in\mathcal{J}\setminus\{J_0\}$ is closer to p_n than ε, then it has the length at most $2^{-(n+1)}$ and $f(p)\geq 1-2^{-(n+1)}$ for the upper endpoint p for f of J.

Since $F(p_n,0)\geq 1-2^{-n}$ and F is peripherally continuous we can find an open neighborhood $W_n\subset[0,1]^2$ of $\langle p_n,0\rangle$ with diameter less than ε and such that $F[\operatorname{bd}(W_n)]\subset(1-2^{-n+1},1]$. Without loss of generality we can also assume that $\operatorname{bd}(W_n)$ is connected. (This is a standard argument. See e.g. [27].) Note that the choice of ε guarantees that $\operatorname{bd}(W_n)\cap B_{n-1}\neq\emptyset\neq\operatorname{bd}(W_n)\cap(I_0\times\{0\})$, where I_0 is a component of $I\setminus J_0$ containing p_n. This is so, since $\operatorname{bd}(W_n)$ disconnects $[0,1]^2$, while B_{n-1} and $I_0\times\{0\}$ are connected, containing $\langle p_n,0\rangle\in W_n$ and of diameter greater than $\varepsilon\geq\operatorname{diam}(\operatorname{bd}(W_n))$. Let $z\in I_0$ be such that $\langle z,0\rangle\in\operatorname{bd}(W_n)\cap(I_0\times\{0\})$. Since $F(z,0)\in F[\operatorname{bd}(W_n)]\subset(1-2^{-n+1},1]\subset(0,1]$, there exists a $J\in\mathcal{J}$ such that $z\in\operatorname{cl}(J)$. Let p_{n+1} be the upper endpoint for f of J. Then, by ($*$), $f(p_{n+1})\geq 1-2^{-(n+1)}$ and the distance of $\langle p_{n+1},0\rangle$ from $\langle z,0\rangle\in\operatorname{bd}(W_n)$ is at most $2^{-(n+1)}$. The set B_n is defined as a union of $\operatorname{bd}(W_n)$ and a closed segment joining $\langle z,0\rangle$ and $\langle p_{n+1},0\rangle$. It is easy to see that the inductive conditions are satisfied. This finishes the construction.

Now, to finish the argument notice that for every $n<\omega$ the set $B^n=\bigcup_{k=n}^{\infty}B_k$ is connected, has diameter at most $\sum_{k=n}^{\infty}2^{-k}=2^{-n+1}$ and contains $\langle p_{n+1},0\rangle$. In particular, there exists $p\in I$ with $p=\lim_n p_n$. We claim that $F(p,0)=1$. Indeed, if this is not the case, then there exists an $n<\omega$ such that $|F(p,0)-1|>2^{-n+2}$. Take $\varepsilon>0$ less than the diameter of B^n and let U be an open neighborhood of $\langle p,0\rangle$ of diameter less than ε and such that $F[\operatorname{bd}(U)]\subset(F(p,0)-2^{-n+1},F(p,0)+2^{-n+1})$. Then there exists a point $w\in B^n\cap\operatorname{bd}(U)$, since $U\cap B^n\neq\emptyset$, U has diameter less than $\operatorname{diam}(B^n)$ and B^n is connected. But then $F(w)$ belongs to both $(F(p,0)-2^{-n+1},F(p,0)+2^{-n+1})$ and $F[B^n]\subset(1-2^{-n+1},1]$, which is impossible, since these two sets are disjoint. So, $F(p,0)=1$. But this is impossible as well since then we would have $f(p)=F(p,0)=1$ and f does not attain 1. This contradiction finishes the proof that f is not extendable.

To show that f is almost continuous let G be an open subset of I^2 containing the graph of f. Notice that for every $x\in[0,1]$ there exists an interval (r_x,s_x) (for $x=0$ and $x=1$ we consider intervals $[r_x,s_x)$ and $(r_x,s_x]$, respectively) such that

- $x \in (r_x, s_x)$, $f(r_x) = f(s_x) = 0$ and
- there is a continuous function $g_x \colon [0,1] \to \mathbb{R}$ with $g_x \restriction [r_x, s_x] \subset G$ and such that $g_x(t) = 0$ for $t \notin (r_x, s_x)$.

Indeed, if $f(x) = 0$, then it is easy to find $r_x < s_x$ for which $g_x \equiv 0$ works. If $f(x) \neq 0$, then $x \in \mathrm{cl}(J)$ for some $J \in \mathcal{J}$. For the sake of simplicity assume that f is increasing on J; that is, that $J \in \mathcal{J}_0$, the other case being similar. Let $J = (a, b)$. Then $f(b) > 0$. Let $U \subset G$ be an open circular neighborhood of $\langle b, f(b)\rangle$. Then there exists an interval $J' = (c, d) \in \mathcal{J}_1$ with $b < c$ such that $f \restriction (c, d)$ intersects U. Let $u \in (c, d)$ be such that $\langle u, f(u)\rangle \in U$. Put $r_x = a$, $s_x = d$, define $g_x(b) = f(b)$, $g_x(u) = f(u)$, $g_x(0) = g_x(r_x) = g_x(s_x) = g_x(1) = 0$ and extend g_x linearly on each interval with these endpoints. Thus g_x has a "hat" shape. We have $g_x \restriction [r_x, s_x] \subset G$ since $g_x \restriction [b, u] \subset U \subset G$ and $g_x \restriction ([a, b] \cup [u, d]) = f \restriction ([a, b] \cup [u, d]) \subset G$. Now choose a finite subcover $\{[r_0, s_0), (r_1, s_1]\} \cup \{(r_{x_i}, s_{x_i})\colon i \leq n\}$, of the cover $\{[r_0, s_0), (r_1, s_1]\} \cup \{(r_x, s_x)\colon x \in (0, 1)\}$ of the interval $[0, 1]$. Then

$$g(x) = \max\{g_0(x), g_1(x), g_{x_0}(x), g_{x_1}(x), \ldots, g_{x_n}(x)\}$$

is continuous and $g \subset G$. This ends the proof that f is almost continuous. □

Note that as in [8, Proposition 3.4] one can show the following.

Remark 2.1. *If f is from Example 2.1, then there exists a characteristic function χ_B of a meager F_σ subset B of C such that $f_0 = f + \chi_B$ is extendable.*

The next example is a slight modification of the function from Example 2.1 and is a variant of examples of Jastrzębski [17] and Kellum [21]. In what follows we will use the notation from Example 2.1.

Example 2.2. *There exists a connectivity, non almost continuous, Baire class two function $f \colon I \to I$ with G_δ graph.*

PROOF. Define f as follows. (See Figure 2.)

- For every $0 < n < \omega$ we put $f(a_n) = f(c_n) = 0$, $f(b_n) = f(d_n) = 1 - 2^{-n}$ and extend it linearly on each interval $[a_n, b_n]$ and $[c_n, d_n]$. (Of course we could now reduce our enumeration of $\mathcal{J}$ to just one sequence, instead of two.)
- For all other x's we put $f(x) = 0$.

The proof that such f is in $\mathcal{G}_\delta \cap \mathcal{B}_2$ is identical to that for f from Example 2.1. The proof that f is connectivity is easy and is essentially the same as for the examples in [17, 21]. The fact that $f \notin \mathrm{AC}$ follows from [21, Lemma 1]. □

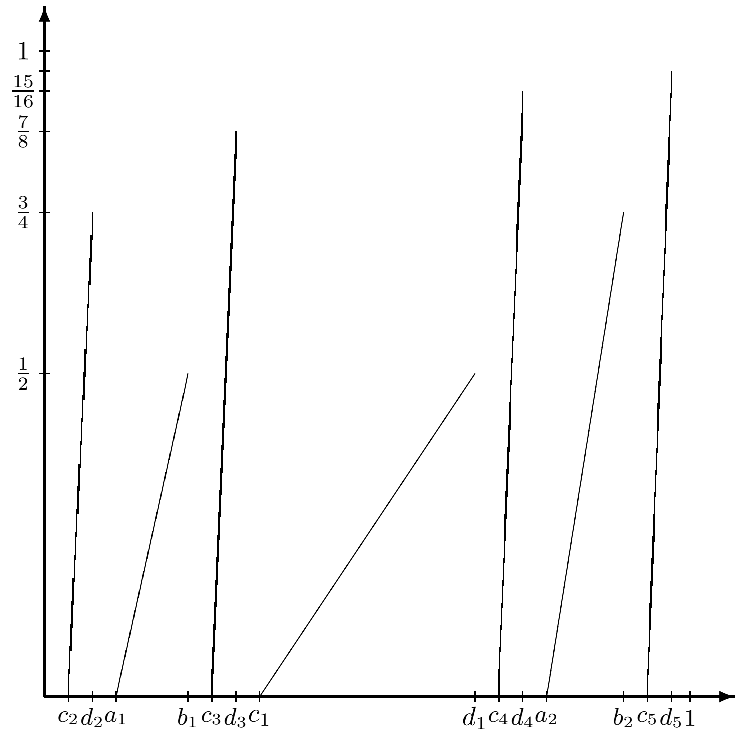

Figure 2: function $f \in \mathcal{G}_\delta \cap \mathcal{B}_2 \cap \text{Conn} \setminus \text{AC}$ from Example 2.2

Example 2.3. *There exists a Darboux, non connectivity, Baire class two function $f\colon I \to I$ with G_δ graph.*

Proof. This is a slight modification of function $k\colon I \to I$ constructed by Ciesielski and Kellum in [9]. Let C denote the Cantor middle two-fifths set:

$$C = \left\{ \sum_{n=1}^{\infty} \frac{i_n}{5^n} \colon i_n \in \{0,2,4\} \ \text{ for every } n \right\}.$$

Geometrically, C is obtained from I by first removing the pair of intervals (1/5,2/5) and (3/5,4/5) from I; then by removing similar pairs of intervals (the middle two fifths) from each of [0,1/5], [2/5,3/5] and [4/5,1], etc. Let $\{\langle (a_n,b_n),(c_n,d_n)\rangle \colon n<\omega\}$ be an enumeration of all such pairs (with $b_n < c_n$). Also, put $C^\circ = I \setminus \bigcup_{n<\omega}[a_n,b_n] \cup [c_n,d_n]$ and let Δ denote the diagonal $\{\langle x,x\rangle \colon x \in I\}$ in I^2. Define f as follows.

- For $n < \omega$ put $f(c_n) = 0$, choose $f(b_n) > d_n > b_n$ from $(1 - 2^{-n}, 1)$, pick $a_n < f(a_n) < f(d_n) < d_n$ and extend f linearly on the intervals $[a_n, b_n]$ and $[c_n, d_n]$. Note that $f[[a_n, b_n] \cup [c_n, d_n]] = [0, f(b_n)]$ and that $f \restriction \bigcup_{n<\omega} [a_n, b_n] \cup [c_n, d_n]$ is disjoint from Δ.
- Put $f(0) = 1/2$ and $f(x) = 0$ for $x \in C^\circ \setminus \{0\}$.

The proof that such an f is in $\mathcal{G}_\delta \cap \mathcal{B}_2$ is identical to that used in Example 2.1. The function f is not connectivity, since Δ separates its graph. It is Darboux, since $f[J] = [0, 1)$ for every interval J intersecting C°. □

Example 2.4. *There exists a peripherally continuous, non Darboux, Baire class two function $f\colon I \to I$ with G_δ graph.*

PROOF. We use here the notation from Example 2.3. Define f as follows.

- For $n < \omega$ put $f(a_n) = 0$, $f(b_n) = \frac{1}{2} - 2^{-n-3}$, $f(c_n) = \frac{1}{2} + 2^{-n-3}$, $f(d_n) = 1 - 2^{-n-3}$ and extend f linearly on intervals $[a_n, b_n]$ and $[c_n, d_n]$.
- Put $f(x) = 0$ for all $x \in C^\circ$.

The proof that such an f is in $\mathcal{G}_\delta \cap \mathcal{B}_2$ is essentially identical to that used in Example 2.1. The function f is in PC $\setminus$ D since $f[J] = [0, 1) \setminus \left\{\frac{1}{2}\right\}$ for every interval J intersecting C°. □

Corollary 2.1. *The inclusions*

$$\text{Ext} \subset \text{AC} \subset \text{Conn} \subset \text{D} \subset \text{PC}$$

remain strict for the Baire class two functions $f\colon I \to I$ with G_δ graphs.

3 An Extendable Function with Dense Graph and Countable Zero Level

In this section we concentrate on functions from $\mathbb{R}$ to $\mathbb{R}$, though essentially all presented results remain the same for functions from I to I. Our main result here is the following.

Theorem 3.1. *There exists an extendable function $f\colon \mathbb{R} \to \mathbb{R}$ with dense graph for which $f^{-1}(0)$ is countable.*

The motivation for searching for such an example comes from several directions. First of all in 1970 Brown [2] proved that if $\mathcal{F}$ is the class of connectivity functions from $\mathbb{R}$ to $\mathbb{R}$, then

(A) if $f \in \mathcal{F}$ has a dense graph, then every nowhere dense set N is *f-negligible for $\mathcal{F}$*; that is, any function $g: \mathbb{R} \to \mathbb{R}$ equal to f outside of N remains in $\mathcal{F}$;

(B) part (A) is false for countable sets N; that is, there exists an $f \in \mathcal{F}$ with a dense graph and a countable dense set $D \subset \mathbb{R}$ which is not f-negligible for $\mathcal{F}$.

The similar result for the class $\mathcal{F}$ of almost continuous functions has been proved in 1982 by Kellum [20]. Also for the class $\mathcal{F}$ of extendable functions in 1994 Rosen [25] proved part (A). However, so far there has been no example justifying (B) for the class Ext. The function f from Theorem 3.1 clearly justifies (B) for $\mathcal{F} = \text{Ext}$, since the countable zero level $D = f^{-1}(0)$ cannot be f-negligible for Ext. (Modifying f on D we can get a non Darboux function.)

The other motivation comes for the following theorem of Ciesielski and Rosłanowski [12, thm. 3.1], (which has been used in constructing an additive, almost continuous, non extendable function with the SCIVP property).

Proposition 3.2. *If $f: \mathbb{R} \to \mathbb{R}$ is extendable and has a dense graph, then f has the following (super SCIVP) property*

> *For every $x, y \in \mathbb{R}$ and for each perfect K between $f(x)$ and $f(y)$ there is a perfect C between x and y such that $f[C] \subset K$ and $f \restriction C$ is continuous* `strictly monotone`.

It seems natural to ask whether in the above the phrase "strictly monotone" can be replaced with "constant" or, more generally, the perfect set K be replaced by a singleton. We do not know the answer to the first of these questions. However the example from Theorem 3.1 is a counterexample for the second conjecture.

Theorem 3.1 is also related to the following open problem [16, Question 9.31]. *Is it true that $g: \mathbb{R} \to \mathbb{R}$ is a product of two extendable functions if and only if g has a zero in each subinterval in which it changes sign?* (This characterization is true if the class of extendable functions is replaced by the class of Darboux, connectivity, or almost continuous functions.) It gives the positive answer in the following particular case.

Corollary 3.3. *If $g: \mathbb{R} \to \mathbb{R}$ is such that $g^{-1}(0)$ is dense in $\mathbb{R}$, then g is a product of two extendable functions.*

PROOF. The proof is a slight modification of the fact that every function $g: \mathbb{R} \to \mathbb{R}$ is a sum of two extendable functions. (See [25] or [11].)

Let f be from Theorem 3.1. Then there exists a dense G_δ subset G of $\mathbb{R}$ which is f-negligible for Ext. (See [25, Theorem 1].) Also there exists a homeomorphism h_0 of $\mathbb{R}$ such that $G \cup h_0[G] = \mathbb{R}$. Moreover, $f_0 = f \circ h_0^{-1}$ is extendable and $h_0[G]$ is f_0-negligible for Ext.

Let h be a homeomorphism of $\mathbb{R}$ such that $h[f^{-1}(0) \cup f_0^{-1}(0)] \subset g^{-1}(0)$ and put $f_1 = f \circ h^{-1}$ and $f_2 = f_0 \circ h^{-1} = f \circ h_0^{-1} \circ h^{-1}$. Then, f_1 and f_2 are extendable and the sets $G_1 = h[G]$ and $G_2 = h[h_0[G]]$ are f_1- and f_2-negligible, respectively. Also, $Z = f_1^{-1}(0) \cup f_2^{-1}(0) \subset g^{-1}(0)$. Define

$$\hat{f}_1(x) = \begin{cases} \frac{g(x)}{f_2(x)} & \text{for } x \in G_1 \setminus Z \\ f_1(x) & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{f}_2(x) = \begin{cases} f_2(x) & \text{for } x \in G_1 \cup Z \\ \frac{g(x)}{f_1(x)} & \text{otherwise.} \end{cases}$$

Then $\hat{f}_1$ and $\hat{f}_2$ are well defined since f_1 and f_2 are not 0 outside Z. They are extendable, since they are respective modifications of f_1 and f_2 on negligible sets $G_1 \setminus Z \subset G_1$ and $\mathbb{R} \setminus (G_1 \cup Z) \subset G_2$. Also,

- for $x \in Z$ we have $\hat{f}_1(x)\hat{f}_2(x) = 0 = g(x)$,
- for $x \in G_1 \setminus Z$ we have $\hat{f}_1(x)\hat{f}_2(x) = \frac{g(x)}{f_2(x)}\, f_2(x) = g(x)$ and
- for $x \in \mathbb{R} \setminus (G_1 \cup Z)$ we have $\hat{f}_1(x)\hat{f}_2(x) = f_1(x)\, \frac{g(x)}{f_1(x)} = g(x)$.

Thus, $g = \hat{f}_1 \hat{f}_2$. □

Theorem 3.1 will be concluded from the following theorem. The construction is a modification of Ciesielski-Recław's construction from [11, Theorem 3.3]. However, the triangles in the triangulations are not equilateral, as in the generalization of [11, Theorem 3.3] presented in [13, prop. 2.3].

Theorem 3.4. *There exists a connectivity function $F\colon \mathbb{R}^2 \to \mathbb{R}$ with graph dense in $\mathbb{R}^3$ such that $F^{-1}(0)$ is contained in a countable union of straight lines in $\mathbb{R}^2$.*

Before we prove Theorem 3.4 let us notice how it implies Theorem 3.1.

Proof of Theorem 3.1. Let F be as in Theorem 3.4. Note that by rotating the domain $\mathbb{R}^2$ of F we obtain the same kind of example. Thus, without loss of generality we can assume the countable family $\mathcal{L}$ of lines with $F^{-1}(0) \subset \bigcup \mathcal{L}$ does not contain a line parallel to the x-axis. Then every line $L_y = \mathbb{R} \times \{y\}$ intersects $F^{-1}(0)$ in a countable set. Moreover, by examining the function F constructed below (or by an argument similar to that of Rosen from [25, Theorem 1]) there exists a dense G_δ subset G of $\mathbb{R}^2$ such that G is F-negligible for the class Conn. Thus, by the Kuratowski-Ulam theorem (a category analog of the Fubini theorem), there is $y \in \mathbb{R}$ such that $G \cap L_y$ is a dense G_δ subset of L_y. This implies that the graph of $F \restriction L_y$ is dense in $L_y \times \mathbb{R}$. It is easy to see that f defined by $f(x) = F(x, y)$ has all the desired properties. □

Proof of Theorem 3.4. The constructed function F will be peripherally continuous, so connectivity. (See e.g. [16, Theorem 8.1] or [7, p. 171].)

Basic Idea: By induction on $n < \omega$ we will construct a sequence $\langle S_n \colon n < \omega\rangle$ of triangular "grids," that is, triangulations of the plane. We will begin with S_0 formed with equilateral triangles of side length 1 as in Figure 3. Then, at the stage $n+1$, we will subdivide each triangle from S_n into finitely many pieces making sure that their sizes tend to zero. The grid S_n will be identified with the points on the edges of triangles forming it and we will assume that $S_n \subseteq S_{n+1}$ for all $n < \omega$. With each grid S_n we will associate a continuous function $f_n \colon S_n \to \mathbb{R}$ which is linear on each side of a triangle from S_n. Moreover, each f_{n+1} will be an extension of f_n. The function F will be defined as an extension of $\bigcup_{n<\omega} f_n$ such that $F^{-1}(0) \subset \bigcup_{n<\omega} S_n$. Thus, $F^{-1}(0)$ will be contained in a countable union of straight lines; those that contain sides of triangles from S_n's.

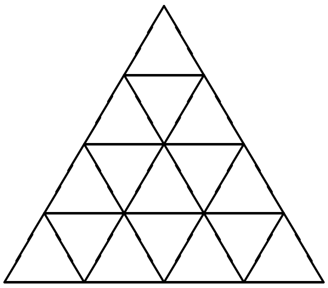

Figure 3: grid S_0

Terminology: In what follows a *triangle* will be identified with the set of points of its interior or its boundary. For a grid S we say that a *triangle T is from S* if the interior of T is equal to a component of $\mathbb{R}^2 \setminus S$.

For a triangle T its *basic partition* will be its split into ten triangles as in Figure 4. The central triangle $\hat{T}$ of Figure 4 will be referred to as *the middle quarter of* T. It is similar to T and its sides are parallel to the sides of T and of 1/4 of their respective lengths. Also, T and $\hat{T}$ have the same center and the vertices on the sides of T are at their centers. It is easy to see that the diameter of each of triangle from the basic partition of T is at most half of the diameter of T. Also $\hat{T} \cap \mathrm{bd}(T) = \emptyset$.

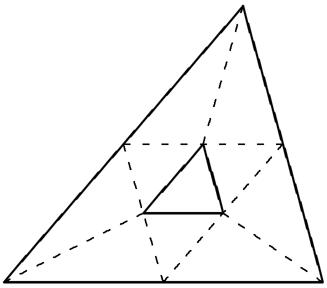

Figure 4: the middle quarter $\hat{T}$ of a triangle T and the basic partition of T

If a function f is defined on the three vertices of a triangle T, its *basic extension* is defined as the unique function $\hat{f}\colon T \to \mathbb{R}$ extending f whose graph is a subset of a plane in $\mathbb{R}^3$. Notice, that $\hat{f}$ is linear on each side of the triangle T and that $\hat{f}$ extends f even if the function f has already been defined on some side of T as long as f is linear on this side. In particular, if f is defined on a grid S and is linear on each side of every triangle from S, then $\hat{f}\colon \mathbb{R}^2 \to \mathbb{R}$ is defined in a natural way.

Inductive construction: We will inductively define two increasing sequences: $\langle S_n\colon n<\omega\rangle$ of triangular grids and $\langle f_n \in \mathbb{R}^{S_n}\colon n<\omega\rangle$ of continuous functions such that the following inductive conditions are satisfied for every $n<\omega$.

(i) For each triangle T from S_n function f_n on T is linear not identically 0 and $f_n[T]$ is a subset of one of the intervals: $[-2^n,-1]$, $[-1,0]$, $[0,1]$, or $[1,2^n]$.

(ii) The side length of each triangle from S_n is at most $1/2^n$.

(iii) The variation of f_n on each triangle from S_n is at most $1/2^n$.

(iv) If $n>0$, $k\in\{-1,1\}$, then for every triangle T from S_{n-1} for which $(k\cdot f_n)[T]\subset[0,\infty)$ the following holds.

(a) With every non-zero dyadic number $i/2^n \in [-2^n,2^n]$, where i belongs to $D_n=\{i\in\mathbb{Z}\colon -4^n\le i\le 4^n \ \&\ i\neq 0\}$, we associate a triangle $T_i\subseteq\hat{T}$ from S_n such that all triangles T_i are disjoint, $\hat{f}_n\left[\hat{T}\setminus\bigcup_{i\in D_n}T_i\right]=\{k\}$ and $\hat{f}_n\left[\hat{T}_i\right]=\{i/2^n\}$ for every $i\in D_n$.

(b) For every $x\in T\setminus\hat{T}$ either $\hat{f}_{n-1}(x)\le\hat{f}_n(x)\le k$ or $\hat{f}_{n-1}(x)\ge\hat{f}_n(x)\ge k$.

To start the induction define grid S_0 as in Figure 3 with all sides of length 1 and let $f_0\colon S_0 \to \mathbb{R}$ be identically 1. It is easy to see that conditions (i)–(iv) are satisfied for such a choice.

Next, assume that for some $n > 0$ we already have S_{n-1} and f_{n-1} satisfying (i)–(iv). We will define S_n and extend f_{n-1} to $f_n\colon S_n \to \mathbb{R}$ such that (i)–(iv) will still hold. Let T be a triangle from S_{n-1} and put $k = 1$ if $\hat{f}_{n-1}[T] \subset [0, \infty)$ and $k = -1$ for $\hat{f}_{n-1}[T] \subset (-\infty, 0]$. The S_n-triangulation of T will be a refinement of the basic triangulation of T. This will guarantee (ii).

Clearly f_n is already defined on $\mathrm{bd}(T)$. We define f_n on each vertex of the middle quarter $\hat{T}$ of T by assigning it the value k. Next, on each triangle T' from the basic partition of T except for $\hat{T}$ (see Figure 4) we extend f_n to $\hat{f}_n$ linearly. Such an extension is unique since f_n is already defined on each vertex of T'. Note that this and the choice of k guarantee (iv)(b). This also gives us (i) for any triangle from a subtriangulation of T'.

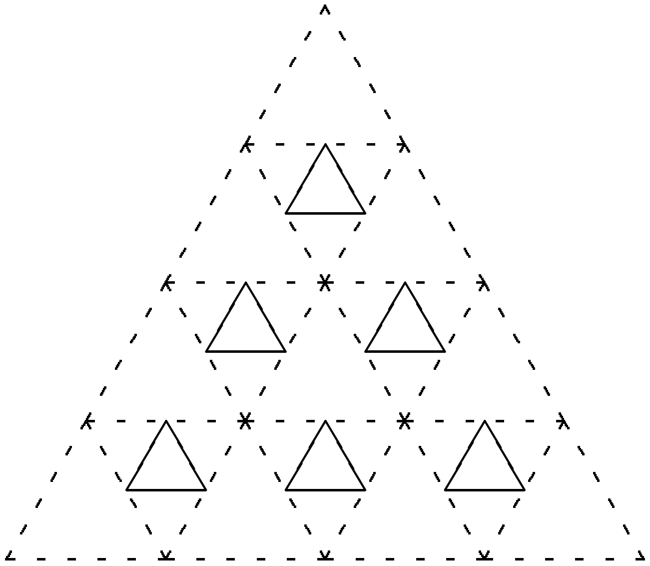

Figure 5: some triangles T_i of the grid of $\hat{T}$

Thus, $\hat{f}_n$ is defined so far on $T \setminus \hat{T}$. To extend $\hat{f}_n$ to $\hat{T}$ we proceed as follows. Partition $\hat{T}$ into a grid S such that S contains $2 \cdot 4^n$ disjoint triangles $\{T_i\colon i \in D_n\}$. (See Figure 5.) We extend $\hat{f}_n$ to $\hat{T} \setminus \bigcup_{i \in D_n} T_i$ by assigning it the constant value k, while on each $\hat{T}_i$ with $i \in D_n$, $\hat{f}_n$ is identically $i/2^n$. This guarantees the reminder of condition (iv) as well as (i) for the triangles considered so far. Finally, $\hat{f}_n$ is extended linearly on each triangle from the basic partition of T_i using the fact that it is already defined on each of its vertices. This finishes the construction of $\hat{f}_n$.

Now, to make sure that (iii) and the remaining part of (i) are satisfied we will refine the triangulations defined so far to the grid S_n. This will lead to the definition of f_n as $\hat{f}_n \restriction S_n$.

For (i) we proceed as follows. If k and $i \in D_n$ have the same sign, then $\hat{f}^n \restriction T_i$ does not attain value 0 and (i) is already satisfied by the triangles from the basic partition of T_i. In this case S_n on T_i will be formed from the triangles considered so far. On the other hand, if k and $i \in D_n$ have different signs, then $(\hat{f}^n \restriction T_i)^{-1}(0)$ is a boundary of a triangle T' similar to T_i. Then we subtriangulate the basic partition of T_i in such a way that every edge of T' is covered by the edges of this subtriangulation and that each triangle from the subtriangulation intersects at most one edge (excluding the end points) of T'. If, in addition, $(\hat{f}^n \restriction T_i)^{-1}(-k)$ is nonempty, then it is a boundary of a triangle (or the entire triangle) and we will make sure that this boundary is a subset of our subtriangulation.

Condition (iii) is easily guaranteed by refining the triangulation described so far. This finishes the inductive construction.

Now, F is defined on $S = \bigcup_{n<\omega} S_n$ as $f = \bigcup_{n<\omega} f_n$. We will extend it to $\mathbb{R}^2$ making sure that $F(x) \neq 0$ for every $x \in \mathbb{R}^2 \setminus S$. This will guarantee that $F^{-1}(0) \subset S$. Note also that, by (iv)(a), the graph of f is already dense in $\mathbb{R}^3$. So any extension F of f will have this property as well.

To extend F to $\mathbb{R}^2$ pick an $x \in \mathbb{R}^2 \setminus S$. Note that, by (i), $\hat{f}_n(x) \neq 0$ for every $n < \omega$. For every $n < \omega$ let T^x_n be the triangle from S_n containing x and let $N = \{n < \omega \colon x \in \hat{T}^x_n\}$. We consider two cases.

Case 1. The set N is infinite. If $N^k = \{n \in N \colon \hat{f}_n[\hat{T}^x_n] = \{k\}\}$ for $k \in \{-1, 1\}$, then $N = N^{-1} \cup N^1$ and so at least one of the sets N^{-1} and N^1 is infinite. Let $k = 1$ if N^1 is infinite and put $k = -1$ otherwise. In this case we define $F(x) = k$. Notice that this guarantees that F is peripherally continuous at x since for any $n \in N^k$

$$F[\mathrm{bd}(\hat{T}^x_n)] = f_n[\mathrm{bd}(\hat{T}^x_n)] = \{k\} = \{F(x)\},$$

x belongs to the interior of $\hat{T}^x_n$ and the diameter of $\hat{T}^x_n$ is at most $1/2^n$.

Case 2. The set N is finite. Fix an $m < \omega$ such that $N \subset \{0, \dots, m-2\}$. Then $x \in T^x_n \setminus \hat{T}^x_n$ for every $n \geq m$. So, if $k \in \{-1, 1\}$ is such that $(k \cdot f_m)[T^x_m] \subset [0, \infty)$, then, by (iv)(b), either $\hat{f}_m(x) \leq \hat{f}_{m+1}(x) \leq \hat{f}_{m+2}(x) \leq \cdots \leq k$ or $\hat{f}_m(x) \geq \hat{f}_{m+1}(x) \geq \hat{f}_{m+2}(x) \geq \cdots \geq k$. So the limit $L = \lim_{m\to\infty} \hat{f}_m(x)$ is well defined and not equal to 0. We put $F(x) = L$. This, together with (ii) and (iii), guarantees that F is peripherally continuous at x.

This finishes the construction of function F and the argument that F is peripherally continuous on $\mathbb{R}^2 \setminus S$. To see that F is peripherally continuous on S take an $x \in S$. Then, there exists a $k < \omega$ such that $x \in S_n$ for every $n \geq k$. For any such n let $\mathcal{T}_n$ be the set of all triangles from S_n to which x belongs. Then x belongs to the interior of the polygon $P_n = \bigcup \mathcal{T}_n$. Moreover, by (ii) and (iii), the variation on the boundary of P_n and the diameter of P_n are at most $2/2^n$. So, the sequence $\langle P_n \rangle$ guarantees that F is peripherally continuous at x. This finishes the proof of Theorem 3.4. $\square$

4 Quasi-Continuous Extendable Functions

To state the results of this section we need the following additional terminology and facts.

For $x \in I$ let $l_x = \{x\} \times I$ and for function $f\colon I \to I$ let $C(f)$ stand for the set of points of continuity of f. Recall that for a Darboux function $f\colon I \to I$ the set $\mathrm{cl}(f) \cap l_x$ is connected for every $x \in I$ and that $C(f)$ is a dense G_δ provided f has a G_δ graph. (See e.g. [18].) The function $f\colon I \to I$ is *quasi-continuous* if $f \restriction C(f)$ is dense in the graph of f. A function $f\colon I \to I$ is said to have a closure that is *bilaterally dense in itself* if $\mathrm{cl}(f \restriction (0,x)) \cap l_x = \mathrm{cl}(f \restriction (x,1)) \cap l_x$ for each $x \in (0,1)$.

Croft's function $f\colon I \to I$ from [14] is Darboux, upper semicontinuous (hence of Baire class one) and 0 almost everywhere, but is not identically 0. It follows that the closure of f is bilaterally dense in itself. However, since it is not quasi-continuous, Croft's function does not satisfy the second category condition in the following result. But a function like Kellum and Garrett's [22, Example 1] does satisfy it and the rest of the hypothesis.

Theorem 4.1. *Suppose $f\colon I \to I$ is a Darboux function with a G_δ graph whose closure is bilaterally dense in itself. Also suppose for every point $\langle x, f(x) \rangle$ of f there exists an open neighborhood $W \subset I^2$ of $\langle x, f(x) \rangle$ such that if B is closed and nowhere dense in* $\mathrm{pr}(f \cap W)$*, then* $\mathrm{pr}^{-1}(B) \cap (f \cap W)$ *is nowhere dense in $f \cap W$. Then f is quasi-continuous and extendable.*

PROOF. Let $A = \{x \in I\colon \mathrm{cl}(f) \cap l_x \text{ is nondegenerate}\}$. Assume f is not quasi-continuous at some point $x_0 \in I$. Then there exists a rectangular open neighborhood $W = I_1 \times I_2 \subset I^2$ of $\langle x_0, f(x_0) \rangle$ that obeys the second category condition of the hypothesis and that contains no point of $f \restriction C(f)$. Note that $\mathrm{pr}(f \cap W) = \{x \in A \cap I_1\colon \langle x, f(x) \rangle \in W\}$. According to the Alexandroff theorem, the G_δ subset $f \cap W$ of I^2 is homeomorphic to a complete metric space. Therefore, by hypothesis, $\mathrm{pr}(f \cap W)$ is of second category in itself.

Observe that if $x \in \mathrm{pr}(f \cap W)$, then $\mathrm{cl}(f) \cap l_x$ is a nondegenerate interval meeting $I^2 \setminus W$ and so $\mathrm{cl}(f \cap W) \cap l_x$ contains a nondegenerate interval containing $\langle x, f(x)\rangle$. To show $f \cap W$ is nowhere dense in $\mathrm{pr}(f \cap W) \times I$ let $[a,b] \times [c,d]$ be a rectangle in I^2 that meets $\mathrm{pr}(f \cap W) \times I$ and let $G_1 \supset G_2 \supset \cdots$ be open subsets of I^2 such that $f \cap W = (\bigcap_{n=1}^{\infty} G_n) \cap (\mathrm{pr}(f \cap W) \times I)$. For all positive integers n and rational numbers r and s with $c \leq r < s \leq d$ let $H(n,r,s)$ be the set of all $x \in \mathrm{pr}(f \cap W) \cap [a,b]$ for which some component of $l_x \setminus G_n$ meets both $I \times \{r\}$ and $I \times \{s\}$. Each $H(n,r,s)$ is closed in $\mathrm{pr}(f \cap W) \cap [a,b]$ and each point of $\mathrm{pr}(f \cap W) \cap [a,b]$ belongs to some $H(n,r,s)$. Therefore there exist n, r, s and an interval (u,v) such that $H(n,r,s) \supset \mathrm{pr}(f \cap W) \cap (u,v) \neq \emptyset$. Since the sets $f \cap W$ and $[\mathrm{pr}(f \cap W) \cap (u,v)] \times (r,s)$ are disjoint, $f \cap W$ is nowhere dense in $\mathrm{pr}(f \cap W) \times I$.

For each positive integer n let Q_n be the set of all $x \in \mathrm{pr}(f \cap W)$ for which $\mathrm{cl}(f \cap W) \cap l_x$ contains an interval containing $\langle x, f(x)\rangle$ of length at least $\frac{1}{n}$. Each Q_n is closed in $\mathrm{pr}(f \cap W)$. Since $f \cap W$ is nowhere dense in $\mathrm{pr}(f \cap W) \times I$, each Q_n is nowhere dense in $\mathrm{pr}(f \cap W)$. Because $\mathrm{pr}(f \cap W)$ is of second category and $\mathrm{pr}(f \cap W) = \bigcup_{n=1}^{\infty} Q_n$, this a contradiction. By [26, Theorem 1] a Darboux quasi-continuous function f whose closure is bilaterally dense in itself is extendable. □

References

[1] J. B. Brown, *Nowhere dense Darboux graphs*, Colloq. Math. **20** (1969), 243–247.

[2] J. B. Brown, *Negligible sets for real connectivity functions*, Proc. Amer. Math. Soc. **24** (1970), 263–269.

[3] J. B. Brown, *Almost continuous Darboux functions and Reed's pointwise convergence criteria*, Fund. Math. **86** (1974), 1–7.

[4] J. B. Brown, *Measurable Darboux functions*, Real Anal. Exchange **15** (1989–90), 65–66.

[5] J. B. Brown, P. Humke and M. Laczkovich, *Measurable Darboux functions*, Proc. Amer. Math. Soc. **102** (1988), 603–610.

[6] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts **39**, Cambridge Univ. Press 1997.

[7] K. Ciesielski, *Set theoretic real analysis*, J. Appl. Anal. **3**(2) (1997), 143–190. (Preprint* available.[1])

---

[1] Preprints marked by $\star$ are available in electronic form from *Set Theoretic Analysis Web Page:* http://www.math.wvu.edu/homepages/kcies/STA/STA.html

[8] K. Ciesielski, J. Jastrzębski, *Darboux–like functions within the classes of Baire one, Baire two, and additive functions*, Topology Appl., in print. (Preprint* available.)

[9] K. Ciesielski, K. R. Kellum, *Compositions of Darboux and connectivity functions*, Real Anal. Exchange **24**(2) (1998–99), 599–605. (Preprint* available.)

[10] K. Ciesielski, T. Natkaniec, J. Wojciechowski, *Extending connectivity functions on* $\mathbb{R}^n$, Topology Appl., to appear. (Preprint* available.)

[11] K. Ciesielski, I. Recław, *Cardinal invariants concerning extendable and peripherally continuous functions*, Real Anal. Exchange **21** (1995–96), 459–472. (Preprint* available.)

[12] K. Ciesielski, A. Rosłanowski, *Two examples concerning almost continuous functions*, Topology Appl., in print. (Preprint* available.)

[13] K. Ciesielski, J. Wojciechowski, *Sums of connectivity functions on* $\mathbb{R}^n$, Proc. London Math. Soc. **76**(2) (1998), 406–426. (Preprint* available.)

[14] H. T. Croft, *A note on a Darboux continuous function*, London Math. Soc. **38** (1963), 9–10.

[15] R. G. Gibson, *Darboux functions with a perfect road*, Real Anal. Exchange **15** (1989–90), 582–591.

[16] R. G. Gibson, T. Natkaniec, *Darboux like functions*, Real Anal. Exchange **22**(2) (1996–97), 492–533. (Preprint* available.)

[17] J. Jastrzębski, *An answer to a question of R. G. Gibson and F. Roush*, Real Anal. Exchange **15** (1989–90), 340–341.

[18] F. B. Jones, E. S. Thomas, Jr., *Connected* G_δ *graphs*, Duke Math. J. **33** (1966), 341–345.

[19] A. S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1994.

[20] K. R. Kellum, *Almost continuity and connectivity—sometimes it's as easy to prove a stronger result*, Real Anal. Exchange **8** (1982–83), 244–252.

[21] K. R. Kellum, *Compositions of Darboux-like functions*, Real Anal. Exchange **23**(1) (1997–98), 211–216.

[22] K. R. Kellum, B. D. Garrett, *Almost continuous real functions*, Proc. Amer. Math. Soc. **33** (1972), 181–184.

[23] K. Kuratowski, *Topology,* Vol. I, Academic Press, New York, NY, 1968.

[24] M. H. Miller, Jr., *Discontinuous G_δ graphs*, Studies in Topology, Academic Press, Inc. (1975), 383–392.

[25] H. Rosen, *Limits and sums of extendable connectivity functions*, Real Anal. Exchange **20**(1) (1994–95), 183–191.

[26] H. Rosen, *Darboux quasicontinuous functions*, Real Anal. Exchange **22**(2) (1998–99), 631–639.

[27] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.

[28] E. S. Thomas, Jr., *Almost continuous functions and functions of Baire class 1*, Topology Seminar Wisconsin, 1965, Princeton Univ. Press (1966), 125–128.